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The modified general iterative methods for asymptotically nonexpansive semigroups in Banach spaces

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Abstract
In this paper, we introduce the modified general iterative methods for finding a common fixed point of asymptotically nonexpansive semigroups, which is a unique solution of some variational inequality. We prove the strong convergence theorems of such iterative scheme in a real Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ and uniform normal structure. The main result extends various results existing in the current literature.

1 Introduction

Let E be a normed linear space, K be a nonempty, closed and convex subset of E. Let T be a self-mapping on K. Then T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \text{for all } x, y \in K \text{ and } n \geq 1.$$

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [1] as an important generalization of the class of nonexpansive maps (*i.e.*, mapping $T : K \to K$ such that $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in K$).

A mapping T is said to be *uniformly L-Lipschitzian*, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \text{for all } x, y \in K \text{ and } n \geq 1.$$

It is clear that every asymptotically nonexpansive is uniformly L-Lipschitzian with a constant $L = \sup_{n\geq 1} k_n \geq 1$. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

A self-mapping $f : K \to K$ is a contraction on K if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \text{for all } x, y \in K. \tag{1.1}$$

We use Π_K to denote the collection of all contractions on K. That is,

$$\Pi_K = \{f : f \text{ is a contraction on } K\}.$$

A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of K into itself is called an *asymptotically nonexpansive semigroup* on K if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in K$;

(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in K$ $\|T^n(t)x - T^n(t)y\| \leq k_n\|x-y\|$, $\forall t \geq 0$, $\forall n \geq 1$;

(iv) for all $x \in K$, the mapping $t \mapsto T(t)x$ is continuous.

An asymptotically nonexpansive semigroup $\mathcal{S}$ is called *nonexpansive semigroup* if $k_n = 1$ for all $n \geq 1$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$F(\mathcal{S}) := \{x \in K : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t\geq 0} F(T(t)).$$

A gauge function φ is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Let E^* be the dual space of E. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated to a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E.$$

In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$ for all $x \neq 0$ (see [2]). Set

$$\Phi(t) = \int_0^t \varphi(\tau)\,d\tau, \quad \forall t \geq 0,$$

then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in E,$$

where ∂ denotes the subdifferential in the sense of convex analysis. Furthermore, Φ is a continuous convex and strictly increasing function on $[0,\infty)$ (see [3]).

In a Banach space E having duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* [4] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|) \tag{1.2}$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \quad \alpha \in [0,1], \beta \in [-1,1], \tag{1.3}$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.2) reduces to

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{1.4}$$

Let $u \in C$. Then, for each $t \in (0,1)$ and for a nonexpansive map T, there exists a unique point $x_t \in C$ satisfying the following condition:

$$x_t = (1-t)Tx_t + tu,$$

since the mapping $G_t(x) = (1-t)Tx + tu$ is a contraction. When H is a Hilbert space and T is a self-map, Browder [5] showed that $\{x_t\}$ converges strongly to an element of $F(T)$, which is nearest to u as $t \to 0^+$. This result was extended to various more general Banach space by Morales and Jung [6], Takahashi and Ueda [7], Reich [8] and a host of other authors. Many authors (see, *e.g.* [9, 10]) have also shown convergence of the path

$$x_n = (1-\alpha_n)T^n x_n + \alpha_n u$$

in Banach spaces for asymptotically nonexpansive mapping self-map T under some conditions on α_n. In 2009, motivated and inspired by Moudafi [11], Shahzad and Udomene [12] introduced and studied the iterative procedures for the approximation of common fixed points of asymptotically nonexpansive mappings in a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure.

Let $\mathcal{S}$ be a nonexpasive semigroup on K. In 2002, Suzuki [13] introduced, in Hilbert space, the implicit iteration

$$u_n = (1-\alpha_n)T(t_n)u_n + \alpha_n u, \quad u \in K, n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ is a sequence of positive real numbers. Under certain restrictions to the sequence $\{\alpha_n\}$ and $\{t_n\}$, Suzuki proved strong convergence of (1.5) to a member of $F(\mathcal{S})$ nearest to u. In 2005, Xu [14] extended Suzuki [13]'s result from Hilbert space to a uniformly convex Banach space having a weakly continuous duality map j_φ with gauge function φ. In 2009, Chang *et al.* [15] introduced the following implicit and explicit schemes for an asymptotically nonexpansive semigroup:

$$y_n = (1-\alpha_n)T^n(t_n)y_n + \alpha_n u, \quad u \in K, n \geq 1, \tag{1.6}$$

and

$$x_{n+1} = (1-\beta_n)T^n(t_n)x_n + \beta_n u, \quad u \in K, n \geq 0, \tag{1.7}$$

where $\alpha_n, \beta_n \in (0,1)$ and $t_n \in \mathbb{R}^+$ in a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure. Suppose, in addition, that $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0,\infty)$. Then the $\{x_n\}$ and $\{y_n\}$ converge strongly to a point of $F(\mathcal{S})$.

Very recently, motivated and inspired by Moudafi [11], Cholumjiak and Suantai [16] studied the following implicit and explicit viscosity methods:

$$y_n = \alpha_n f(y_n) + (1-\alpha_n)T(t_n)y_n, \quad n \geq 1, \tag{1.8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 0. \tag{1.9}$$

They obtained the strong convergence theorems in the frame work of a real reflexive strictly convex Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ. For more related results, see [17–19].

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.10}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2009, motivated and inspired by Marino and Xu [20], Li *et al.* [21] introduced the following general iterative procedures for the approximation of common fixed points of a nonexpansive semigroup $\{T(s) : s \geq 0\}$ on a nonempty, closed and convex subset K in a Hilbert space:

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n\, ds, \quad n \geq 1, \tag{1.11}$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\, ds, \quad n \geq 0, \tag{1.12}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences in $[0,1]$ and $(0,\infty)$, respectively, A is a strongly positive bounded linear operator on C and f is a contraction on C. And their convergence theorems can be proved under some appropriate control conditions on parameter $\{\alpha_n\}$ and $\{t_n\}$. Furthermore, by using these results, they obtained two mean ergodic theorems for nonexpansive mappings in a Hilbert space. Many authors extended the Li *et al.* [21]'s results in direction of algorithms and spaces (see [22–27]).

In this paper, inspired and motivated by Chang *et al.* [15], Cholamjiak and Suantai [16], Li, Li and Su [21], Wangkeeree and Wangkeeree [24] and Wangkeeree *et al.* [4], we introduce the following iterative approximation methods (1.13) and (1.14) for the class of strongly continuous semigroup of asymptotically nonexpansive mappings $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$:

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)T^n(t_n)y_n, \quad n \geq 1, \tag{1.13}$$

and

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n A)T^n(t_n)x_n, \quad n \geq 0, \tag{1.14}$$

where A is a strongly positive bounded linear operator on K and f is a contraction on K. The strong convergence theorems of the iterative approximation methods (1.13) and (1.14) in a real Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping j_φ are studied. Moreover, we study the strong convergence results of the following two iterative approximation methods (1.15) and (1.16):

$$w_{n+1} = \beta_n \gamma f\big(T^n(t_n)w_n\big) + (I - \beta_n A)T^n(t_n)w_n, \quad n \geq 0, \tag{1.15}$$

and

$$z_{n+1} = T^n(t_n)\big(\beta_n\gamma f(z_n) + (I - \beta_n A)T^n(t_n)z_n\big), \quad n \geq 0. \tag{1.16}$$

2 Preliminaries

Throughout this paper, let E be a real Banach space and E^* be its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence. A Banach space E is said to *uniformly convex* if, for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [28]). Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. In this case, the norm of E is said to be *Gâteaux differentiable*. The space E is said to have a *uniformly Gâteaux differentiable* if for each $y \in U$, the limit attained uniformly for $x \in U$. The space E is said to have a *Fréchet differentiable* if for each $x \in U$, the limit attained uniformly for $y \in U$ and *uniformly Fréchet differentiable* if, the limit attained uniformly for $x, y \in U$. It is well known that (uniformly) Gâteaux differentiable of the norm of E implies (uniformly) Fréchet differentiable.

The following Lemma can be found in [16].

Lemma 2.1 [16, Lemma 2.6] *Let E be a Banach space which has a uniformly Gâteaux differentiable norm and admits the duality mapping J_φ, then J_φ is uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E.*

The next lemma is an immediate consequence of the subdifferential inequality can be found in [3].

Lemma 2.2 [3] *Assume that a Banach space E which admits a duality mapping J_φ with gauge φ. For all $x, y \in E$, the following inequality holds*:

$$\Phi\big(\|x + y\|\big) \leq \Phi\big(\|x\|\big) + \big\langle y, j_\varphi(x + y)\big\rangle, \quad j_\varphi(x + y) \in J_\varphi(x + y).$$

Let K be a nonempty, bounded, closed and convex subset of a Banach space E. The diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, denote $r(x, K) = \sup\{\|x - y\| : x, y \in K\}$ and denote by $r(K) := \inf\{r(x, K) : x \in K\}$ *the chebyshev radius of K relative to itself. The normal structure coefficient $N(E)$ of E* is defined by

$$N(E) := \inf\left\{\frac{d(K)}{r(K)} : K \text{ is a bounded, closed and convex of } E \text{ with } d(K) > 0\right\}.$$

A Banach space E is said to have *uniform normal structure* if $N(E) > 1$. It is known that every Banach space with a uniform normal structure is reflexive. Every uniformly convex and uniformly smooth Banach spaces have uniform normal structure.

The following existence theorem of an asymptotically nonexpansive mapping is useful tools for our proof.

Theorem 2.3 [9, Theorem 1] *Suppose E is a Banach space with uniformly normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is a uniformly k-Lipschitzian mapping with $k < N(X)^{1/2}$. Suppose also there exists a nonempty, bounded, closed and convex subset K^* of K with the following property (P):*

$$x \in K^* \quad \textit{implies} \quad \omega_w(x) \subset K^*,$$

where $\omega_w(x)$ is the weak ω-limit set if T at x, i.e., the set

$$\Big\{ y \in E : y = weak - \lim_{j\to\infty} T^{n_j} x \text{ for some } n_j \to \infty \Big\}.$$

Then T has a fixed point in K^*.

In order to prove our main result, we need the following lemmas and definitions.

Let l^∞ be the Banach space of all bounded real-valued sequences. Let **LIM** be a continuous linear functional on l^∞ satisfying $\|\mathbf{LIM}\| = 1 = \mathbf{LIM}(1)$. Then we know that **LIM** is mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \le \mathbf{LIM}(a) \le \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we shall use $\mathbf{LIM}_n(a_n)$ instead of $\mathbf{LIM}(a)$. A mean **LIM** on $\mathbb{N}$ is called a Banach limit if

$$\mathbf{LIM}_n(a_n) = \mathbf{LIM}_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed-point theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \mathbf{LIM}_n(a_n) \le \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$.

Subsequently, the following result was showed in [16].

Proposition 2.4 [16, Proposition 3.2] *Let K be a nonempty, closed and convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping J_φ. Suppose that $\{x_n\}$ is a bounded sequence of K and let $\mathbf{LIM}_n$ be a Banach limit and $z \in E$. Then*

$$\mathbf{LIM}_n \Phi\big(\|x_n - z\|\big) = \inf_{y\in K} \mathbf{LIM}_n \Phi\big(\|x_n - y\|\big),$$

if and only if

$$\mathbf{LIM}_n \big\langle y - z, j_\varphi(x_n - z)\big\rangle \le 0, \quad \forall y \in K.$$

The next valuable lemma is proved for applying our main results.

Lemma 2.5 [4, Lemma 3.1] *Assume that a Banach space E which admits a duality mapping J_φ with gauge φ. Let A be a strong positive linear bounded operator on E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\bar{\gamma})$.*

In the following, we also need the following lemma that can be found in the existing literature [29].

Lemma 2.6 [29, Lemma 2.1] *Let $\{a_n\}$ be a sequence of nonnegative real number satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\beta_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subseteq (0,1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\{a_n\}$ converges to zero, as $n \to \infty$.

Lemma 2.7 [3] *Let C be a nonempty, closed and convex subset of a reflexive Banach space E and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function such that $f(x_n) \to \infty$ as $\|x_n\| \to \infty$. Then there exists $x_0 \in D(f)$ such that $f(x_0) = \inf_{x\in C} f(x)$.*

3 Main theorem

Theorem 3.1 *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E such that $K \pm K \subset K$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator on K with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$. Then the following hold*:

(i) *If $\frac{k_n - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha$, $\forall n \geq 1$, then there exists a sequence $\{y_n\} \subset E$ defined by*

$$y_n = \alpha_n\gamma f(y_n) + (I - \alpha_n A)T^n(t_n)y_n, \quad n \geq 1. \tag{3.1}$$

(ii) *Suppose, in addition, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0, \infty)$ and the real sequence $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$.*

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\big\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\big\rangle \leq 0, \quad \forall z \in F(\mathcal{S}). \tag{3.2}$$

Proof We first show that the uniqueness of a solution of the variational inequality (3.2). Suppose both $\tilde{x} \in F(\mathcal{S})$ and $x^* \in F(\mathcal{S})$ are solutions to (3.2), then

$$\big\langle (A - \gamma f)\tilde{x}, J_\varphi\big(\tilde{x} - x^*\big)\big\rangle \leq 0 \tag{3.3}$$

and

$$\big\langle (A - \gamma f)x^*, J_\varphi\big(x^* - \tilde{x}\big)\big\rangle \leq 0. \tag{3.4}$$

Adding (3.3) and (3.4), we obtain

$$\langle (A-\gamma f)\tilde{x}-(A-\gamma f)x^*, J_\varphi(\tilde{x}-x^*)\rangle \le 0. \tag{3.5}$$

Noticing that for any $x, y \in K$,

$$\begin{aligned}
&\langle (A-\gamma f)x-(A-\gamma f)y, J_\varphi(x-y)\rangle \\
&\quad = \langle A(x-y), J_\varphi(x-y)\rangle - \gamma\langle f(x)-f(y), J_\varphi(x-y)\rangle \\
&\quad \ge \bar{\gamma}\|x-y\|\varphi(\|x-y\|) - \gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\| \\
&\quad \ge \bar{\gamma}\Phi(\|x-y\|) - \gamma\alpha\Phi(\|x-y\|) \\
&\quad = (\bar{\gamma}-\gamma\alpha)\Phi(\|x-y\|) \\
&\quad \ge (\bar{\gamma}\varphi(1)-\gamma\alpha)\Phi(\|x-y\|) \ge 0.
\end{aligned} \tag{3.6}$$

Therefore, $\tilde{x} = x^*$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of (3.2). Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$. For each integer $n \ge 1$, define a mapping $G_n : K \to K$ by

$$G_n(y) = \alpha_n\gamma f(y) + (I-\alpha_n A)T^n(t_n)y, \quad \forall y \in K.$$

We shall show that G_n is a contraction mapping. For any $x, y \in K$,

$$\begin{aligned}
\|G_n(x)-G_n(y)\| &= \|\alpha_n\gamma f(x) + (I-\alpha_n A)T^n(t_n)x - \alpha_n\gamma f(y) - (I-\alpha_n A)T^n(t_n)y\,ds\| \\
&\le \|\alpha_n\gamma(f(x)-f(y))\| + \|(I-\alpha_n A)(T^n(t_n)x - T^n(t_n)y)\| \\
&\le \alpha_n\gamma\alpha\|x-y\| + \varphi(1)(1-\alpha_n\bar{\gamma})k_n\|x-y\| \\
&= (\alpha_n\gamma\alpha + \varphi(1)(1-\alpha_n\bar{\gamma})k_n)\|x-y\| \\
&\le (k_n - \alpha_n\gamma\alpha + \varphi(1)\alpha_n\bar{\gamma}k_n)\|x-y\| \\
&\le (k_n - \alpha_n(\varphi(1)\bar{\gamma}k_n - \gamma\alpha))\|x-y\|.
\end{aligned}$$

Since $0 < \frac{k_n-1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha$, we have

$$0 < \frac{k_n-1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha \le \varphi(1)\bar{\gamma}k_n - \gamma\alpha.$$

It then follows that $0 < (k_n - \alpha_n(\varphi(1)\bar{\gamma}k_n - \gamma\alpha)) < 1$. We have G_n is a contraction map with coefficient $(k_n - \alpha_n(\varphi(1)\bar{\gamma}k_n - \gamma\alpha))$. Then, for each $n \ge 1$, there exists a unique $y_n \in K$ such that $G_n(y_n) = y_n$, that is,

$$y_n = \alpha_n\gamma f(y_n) + (I-\alpha_n A)T^n(t_n)y_n, \quad n \ge 1.$$

Hence, (i) is proved.

(ii) Define $\mu : K \to \mathbb{R}$ by

$$\mu(y) = \mathbf{LIM}_n\Phi(\|y_n - y\|), \quad y \in K,$$

where $\mathbf{LIM}_n$ is a Banach limit on l^∞. Since μ is continuous and convex and $g(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, by Lemma 2.7, g attains its infimum over E. Let $u \in K$ be such that

$$\mathbf{LIM}_n \Phi(\|y_n - u\|) = \inf_{y \in E} \mathbf{LIM}_n \Phi(\|y_n - y\|). \tag{3.7}$$

Let

$$K^* := \Big\{ z \in E : \mu(z) = \inf_{y \in K} \mu(y) \Big\}.$$

We have that K^* is a nonempty, bounded, closed and convex subset of K and also has the property (P), indeed, if $x \in K^*$ and $w \in \omega_w(x)$, *i.e.* $w = \text{weak} - \lim_{j\to\infty} T^{m_j} x$ as $j \to \infty$. Notice that, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0,\infty)$, by induction we can prove that for all $m \geq 1$

$$\lim_{n\to\infty} \big\|y_n - T^m(t)y_n\big\| = 0 \quad \text{uniformly in } t \in [0,\infty). \tag{3.8}$$

From (3.8) and weakly lower semicontinuous of μ, and for each $h \geq 0$, we have that

$$\begin{aligned}
\mu(w) &\leq \liminf_{j\to\infty} \mu\big(T^{m_j}(h)x\big) \leq \limsup_{m\to\infty} \mu\big(T^m(h)x\big) \\
&= \limsup_{m\to\infty} \mathbf{LIM}_n \Phi\big(\big\|y_n - T^m(h)x\big\|\big) \\
&\leq \limsup_{m\to\infty} \big[\mathbf{LIM}_n \Phi\big(\big\|y_n - T^m(h)y_n\big\| + \big\|T^m(h)y_n - T^m(h)x\big\|\big)\big] \\
&= \limsup_{m\to\infty} \mathbf{LIM}_n \Phi\big(\big\|T^m(h)y_n - T^m(h)x\big\|\big) \\
&\leq \limsup_{m\to\infty} \mathbf{LIM}_n \Phi\big(k_m \|y_n - x\|\big) \\
&= \mathbf{LIM}_n \Phi\big(\|y_n - x\|\big) \\
&= \mu(x) = \inf_{y\in K} \mu(y),
\end{aligned}$$

which implies that K^* satisfies the property (P). By Theorem 2.3, there exists a element $z \in K$ such that $z \in F(\mathcal{S}) \cap K^*$.

Since $K \pm K \subset K$, we have $z + \gamma f(z) - Az \in K$. By Proposition 2.4,

$$\mathbf{LIM}_n \big\langle z + \gamma f(z) - Az - z, J_\varphi(y_n - z)\big\rangle \leq 0,$$

it implies that

$$\mathbf{LIM}_n \big\langle \gamma f(z) - Az, J_\varphi(y_n - z)\big\rangle \leq 0. \tag{3.9}$$

In fact, since $\Phi(t) = \int_0^t \varphi(\tau)\,d\tau$, $\forall t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)\,d\tau = k\int_0^t \varphi(kx)\,dx \leq k\int_0^t \varphi(x)\,dx = k\Phi(t).$$

It follows from Lemma 2.2 that

$$\begin{aligned}\Phi(\|y_n - z\|) &= \Phi\big(\big\|(I-\alpha_n A)T^n(t_n)y_n - (I-\alpha_n A)z \\ &\quad + \alpha_n\big(\gamma f(y_n) - \gamma f(z) + \gamma f(z) - Az\big)\big\|\big) \\ &\le \Phi\big(\big\|(I-\alpha_n A)T^n(t_n)y_n - (I-\alpha_n A)z + \alpha_n\gamma\big(f(y_n) - f(z)\big)\big\|\big) \\ &\quad + \alpha_n\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\ &\le \Phi\big(\varphi(1)(1-\alpha_n\bar{\gamma})\big\|T^n(t_n)y_n - T^n(t_n)z\big\| + \alpha_n\gamma\alpha\|y_n - z\|\big) \\ &\quad + \alpha_n\gamma\langle f(z) - f(z), J_\varphi(y_n - z)\rangle \\ &\le \Phi\big(\varphi(1)(1-\alpha_n\bar{\gamma})(k_n)\|y_n - z\| + \alpha_n\gamma\alpha\|y_n - z\|\big) \\ &\quad + \alpha_n\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\ &\le \Phi\big(\big[\varphi(1)(1-\alpha_n\bar{\gamma})(k_n) + \alpha_n\gamma\alpha\big]\|y_n - z\|\big) + \alpha_n\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\ &\le \big[\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha\big]\Phi\big(\|y_n - z\|\big) + \alpha_n\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle.\end{aligned}$$

This implies that

$$\Phi(\|y_n - z\|) \le \frac{1}{1 - \varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha}\alpha_n\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle,$$

also

$$\Phi(\|y_n - z\|) \le \frac{1}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1-\alpha_n\bar{\gamma})d_n}\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle,$$

where $d_n = \frac{k_n - 1}{\alpha_n}$. Thus

$$\begin{aligned}\mathbf{LIM}_n\Phi(\|y_n - z\|) &\le \mathbf{LIM}_n\left(\frac{1}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1-\alpha_n\bar{\gamma})d_n}\langle\gamma f(z) - Az, J_\varphi(y_n - z)\rangle\right) \\ &\le 0,\end{aligned}$$

and hence

$$\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)\mathbf{LIM}_n\Phi\big(\|y_n - z\|\big) \le 0.$$

Since $\varphi(1)\bar{\gamma} > \gamma\alpha$, $\mathbf{LIM}_n\Phi(\|y_n - z\|) = 0$, and then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \to z$ as $j \to \infty$, we shall denoted by $\{y_j\}$.

Next, we prove that z solves the variational inequality (3.2). From (3.1), we have

$$(A - \gamma f)y_n = -\frac{1}{\alpha_n}(I - \alpha_n A)\big(I - T^n(t_n)\big)y_n.$$

On the other hand, note for all $x, y \in K$,

$$\begin{aligned}&\big\langle\big(I - T^n(t_n)\big)x - \big(I - T^n(t_n)\big)y, J_\varphi(x - y)\big\rangle \\ &\quad = \langle x - y, J_\varphi(x - y)\rangle - \big\langle T^n(t_n)x - T^n(t_n)y, J_\varphi(x - y)\big\rangle \\ &\quad = \|x - y\|\varphi\big(\|x - y\|\big) - \big\langle T^n(t_n)x - T^n(t_n)y, J_\varphi(x - y)\big\rangle\end{aligned}$$

$$\begin{aligned}&\geq \Phi\big(\|x-y\|\big) - k_n\|x-y\|\varphi\big(\|x-y\|\big)\\&\geq \Phi\big(\|x-y\|\big) - k_n\Phi\big(\|x-y\|\big)\\&= (1-k_n)\Phi\big(\|x-y\|\big).\end{aligned}$$

For $p \in F(\mathcal{S})$, we have

$$\begin{aligned}\big\langle (A-\gamma f)y_n, J_\varphi(y_n-p)\big\rangle &= -\frac{1}{\alpha_n}\big\langle (I-\alpha_n A)(I-T^n(t_n)y_n, J_\varphi(y_n-p)\big\rangle\\&= -\frac{1}{\alpha_n}\big\langle \big(I-T^n(t_n)\big)y_n - \big(I-T^n(t_n)\big)p, J_\varphi(y_n-p)\big\rangle\\&\quad + \big\langle A\big(I-T^n(t_n)\big)y_n, J_\varphi(y_n-p)\big\rangle\\&\leq \frac{k_n-1}{\alpha_n}\Phi\big(\|x-y\|\big) + \big\langle A\big(I-T^n(t_n)\big)y_n, J_\varphi(y_n-p)\big\rangle\\&\leq \frac{k_n-1}{\alpha_n}\Phi\big(\|x-y\|\big) + \|A\|\big\|y_n - T^n(t_n)y_n\big\|M,\end{aligned}$$

where $M \geq \sup_{n\geq 1}\varphi(\|y_n-p\|)$. Replacing y_n with y_{n_j} and letting $j \to \infty$, note that $\|y_n - T^n(t_n)y_n\| \to 0$ and $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$, we have that

$$\big\langle (A-\gamma f)z, J_\varphi(z-p)\big\rangle \leq 0, \quad \forall p \in F(T).$$

That is, $z \in F(\mathcal{S})$ is a solution of (3.2). Then $z = \tilde{x}$. In summary, we have that each cluster point of $\{y_n\}$ converges strongly to $\tilde{x}$ as $n \to \infty$. This completes the proof. □

If $A \equiv I$, the identity mapping on K, and $\gamma = 1$, then Theorem 3.1 reduces to the following corollary.

Corollary 3.2 *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$. Then the following hold*:

(i) *If $\frac{k_n-1}{\alpha_n} < 1-\alpha$, $\forall n \geq 1$, then there exists a sequence $\{y_n\} \subset K$ defined by*

$$y_n = \alpha_n f(y_n) + (1-\alpha_n)T^n(t_n)y_n, \quad n \geq 1. \tag{3.10}$$

(ii) *Suppose, in addition, $\lim_{n\to\infty}\|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0,\infty)$ and the real sequences $\{\alpha_n\}$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\frac{k_n-1}{\alpha_n} = 0$.*

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\big\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\big\rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{3.11}$$

If $f \equiv u$, the constant mapping on K, then Corollary 3.2 reduces to the following corollary.

Corollary 3.3 *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$, $t_n > 0$. Then the following hold*:

(i) *If $\frac{k_n-1}{\alpha_n} < 1$, $\forall n \geq 1$, then there exists a sequence $\{y_n\} \subset K$ defined by*

$$y_n = \alpha_n u + (1 - \alpha_n) T^n(t_n) y_n, \quad n \geq 1. \tag{3.12}$$

(ii) *Suppose, in addition, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0, \infty)$ and the real sequences $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$.*

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$, which solves the variational inequality:

$$\langle \tilde{x} - u, J_\varphi(\tilde{x} - z) \rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{3.13}$$

Next, we present the convergence theorem for the explicit scheme.

Theorem 3.4 *Let E be a real Banach space with uniform normal structure, which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E such that $K \pm K \subset K$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0, 1)$, A a strongly positive bounded linear operator on K with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1$, $t_n \geq 0$,*

(C1) $\lim_{n\to\infty} \beta_n = 0$;
(C2) $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$;
(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $x_0 \in K$, let the sequences $\{x_n\}$ be defined by

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n A) T^n(t_n) x_n \, ds, \quad n \geq 0. \tag{3.14}$$

Suppose, in addition, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ uniformly in $t \in [0, \infty)$. Then $\{x_n\}$ converge strongly as $n \to \infty$ to the same point $\tilde{x}$ in $F(\mathcal{S})$, which solves the variational inequality (3.2).

Proof By Theorem 3.1, there exists a unique solution $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (3.2) and $y_m \to \tilde{x}$ as $m \to \infty$. Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) \rangle \leq 0. \tag{3.15}$$

For all $m \geq 1$, $n \geq 1$, we have

$$\begin{aligned} y_m - x_n &= \alpha_m \gamma f(y_m) + (I - \alpha_m A) T^m(t_m) y_m - x_n \\ &= \alpha_m \big(\gamma f(y_m) - A y_m\big) + \big(T^m(t_m) y_m - T^m(t_m) x_n\big) \\ &\quad + \big(T^m(t_m) x_n - x_n\big) + \alpha_m \big(A y_m - A T^m(t_m) y_m\big). \end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
\Phi\big(\|y_m - x_n\|\big) &= \Phi\Big(\big\|(I-\alpha_m A)T^m(t_m)y_m - (I-\alpha_m A)x_n \\
&\quad + \alpha_m\big(\gamma f(y_m) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - A\tilde{x} + A\tilde{x} - Ax_n\big)\big\|\Big) \\
&\le \Phi\Big(\big\|(I-\alpha_m A)T^m(t_m)x_n - (I-\alpha_m A)x_n \\
&\quad + \alpha_m\gamma\big(f(x_m) - f(\tilde{x})\big) + A\tilde{x} - Ax_n\big\|\Big) + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(y_n - \tilde{x})\big\rangle \\
&\le \Phi\big(\varphi(1)(1-\alpha_n\bar{\gamma})\big\|T^n(t_n)x_n - T^n(t_n)\tilde{x}\big\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\big) \\
&\quad + \alpha_n\gamma\big\langle f(\tilde{x}) - f(\tilde{x}), J_\varphi(y_n - \tilde{x})\big\rangle \\
&\le \Phi\big(\varphi(1)(1-\alpha_n\bar{\gamma})k_n\|x_n - \tilde{x}\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\big) \\
&\quad + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\big\rangle \\
&\le \Phi\big(\big[\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha\big]\|x_n - \tilde{x}\|\big) \\
&\quad + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\big\rangle \\
&\le \big[\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha\big]\Phi\big(\|x_n - \tilde{x}\|\big) \\
&\quad + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\big\rangle \\
&\le \big[\big(1 - \varphi(1)\alpha_n\bar{\gamma}\big)(k_n - 1) + 1 - \alpha_n\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)\big]\Phi\big(\|x_n - \tilde{x}\|\big) \\
&\quad + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\big\rangle \\
&\le \big(1 - \alpha_n\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)\big)\Phi\big(\|x_n - \tilde{x}\|\big) + \big(1 - \varphi(1)\alpha_n\bar{\gamma}\big)(k_n - 1)M \\
&\quad + \alpha_n\big\langle\gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\big\rangle. \qquad (3.16)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|y_m - x_n\|\varphi\big(\|y_m - x_n\|\big) \\
&\quad = \big\langle\alpha_m\big(\gamma f(y_m) - Ay_m\big) + \big(T^m(t_m)y_m - T^m(t_m)x_n\big) \\
&\qquad + \big(T^m(t_m)x_n - x_n\big) + \alpha_m\big(Ay_m - AT^m(t_m)y_m\big), J_\varphi(y_m - x_n)\big\rangle \\
&\quad = \alpha_m\big\langle\gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\big\rangle + \big\langle T^m(t_m)y_m - T^m(t_m)x_n, J_\varphi(y_m - x_n)\big\rangle \\
&\qquad + \big\langle T^m(t_m)x_n - x_n, J_\varphi(y_m - x_n)\big\rangle + \alpha_m\big\langle Ay_m - AT^m(t_m)y_m, J_\varphi(y_m - x_n)\big\rangle \\
&\quad \le \alpha_m\big\langle\gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\big\rangle + \big\|T^m(t_m)y_m - T^m(t_m)x_n\big\|\varphi\big(\|y_m - x_n\|\big) \\
&\qquad + \big\|T^m(t_m)x_n - x_n\big\|\varphi\big(\|y_m - x_n\|\big) + \alpha_m\big\|Ay_m - AT^m(t_m)y_m\big\|\varphi\big(\|y_m - x_n\|\big) \\
&\quad \le \alpha_m\big\langle\gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\big\rangle + k_m\|y_m - x_n\|\varphi\big(\|y_m - x_n\|\big) \\
&\qquad + \big\|T^m(t_m)x_n - x_n\big\|\varphi\big(\|y_m - x_n\|\big) + \alpha_m\big\|A\big(y_m - T^m(t_m)y_m\big)\big\|\varphi\big(\|y_m - x_n\|\big).
\end{aligned}$$

Since K is bounded, so that $\{x_n\}$ and $\{y_m\}$ are all bounded, and hence

$$\begin{aligned}
&\big\langle\gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\big\rangle \\
&\quad \le \frac{k_m - 1}{\alpha_m}M^2 + \frac{\|T^m(t_m)x_n - x_n\|}{\alpha_m}M + \big\|A\big(y_m - T^m(t_m)y_m\big)\big\|M, \qquad (3.17)
\end{aligned}$$

where M is a constant satisfying $M \geq \sup_{n,m\in\mathbb{N}} \varphi(\|x_n - y_m\|)$. By our hypothesis, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$, uniformly in $t \in [0,\infty)$. By induction, we can prove that for all $m \geq 1$

$$\lim_{n\to\infty} \|x_n - T^m(t)x_n\| = 0, \quad \text{uniformly in } t \in [0,\infty).$$

Hence for all $m \geq 1$, we have

$$\lim_{n\to\infty} \|x_n - T^m(t_m)x_n\| = 0, \quad \text{as } n \to \infty. \tag{3.18}$$

Therefore, taking upper limit as $n \to \infty$ in (3.17), we have

$$\begin{aligned}&\limsup_{n\to\infty} \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle \\ &\quad \leq \limsup_{n\to\infty} \frac{k_m - 1}{\alpha_m} M^2 + \limsup_{n\to\infty} \|A(y_m - T^m(t_m)y_m)\| M.\end{aligned} \tag{3.19}$$

Since K is bounded, it follows from (C1) that

$$\|y_m - T^m(t_m)y_n\| = \alpha_n \|\gamma f(y_m) + AT^m(t_m)y_m\| \to 0 \quad \text{as } m \to \infty. \tag{3.20}$$

And then, taking upper limit as $m \to \infty$ in (3.19), by (C3) and (3.20), we get

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle \leq 0. \tag{3.21}$$

On the other hand, since $\lim_{m\to\infty} y_m = \tilde{x}$ due to the fact the duality mapping J_φ is norm-to-weak* uniformly continuous on bounded subset of E, it implies that

$$\begin{aligned}&\left|\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle - \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle\right| \\ &\quad = \left|\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) - J_\varphi(x_n - y_m)\rangle + \langle \gamma f(\tilde{x}) - \gamma f(y_m) + Ay_m - A\tilde{x}, J_\varphi(x_n - y_m)\rangle\right| \\ &\quad \leq \left|\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) - J_\varphi(x_n - y_m)\rangle\right| \\ &\qquad + \left(\|\gamma f(\tilde{x}) - \gamma f(y_m)\| + \|A(y_m - \tilde{x})\|\right)\varphi\left(\|x_n - y_m\|\right) \to 0, \quad \text{as } m \to \infty.\end{aligned}$$

Therefore, for any given $\varepsilon > 0$, there exists a positive number N such that for all $m \geq N$

$$\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \leq \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle + \varepsilon.$$

It follows from (3.21) that

$$\begin{aligned}&\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &\quad = \limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &\quad \leq \limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle + \varepsilon \\ &\quad \leq \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \le 0. \tag{3.22}$$

Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\begin{aligned}
\Phi(\|x_{n+1} - \tilde{x}\|) &= \Phi\big(\big\|(I - \beta_n A)T^n(t_n)x_n - (I - \beta_n A)\tilde{x} \\
&\quad + \beta_n\big(\gamma f(x_n) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - A(\tilde{x})\big)\big\|\big) \\
&\le \Phi\big(\big\|(I - \beta_n A)T^n(t_n)x_n - (I - \beta_n A)\tilde{x} + \beta_n\gamma\big(f(x_n) - f(\tilde{x})\big)\big\|\big) \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \\
&\le \Phi\big(\varphi(1)(1 - \beta_n\bar{\gamma})\big\|T^n(t_n)x_n - T^n(t_n)\tilde{x}\big\| + \beta_n\gamma\alpha\|x_n - \tilde{x}\|\big) \\
&\quad + \beta_n\gamma\langle f(\tilde{x}) - f(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \\
&\le \Phi\big(\varphi(1)(1 - \beta_n\bar{\gamma})k_n\|x_n - \tilde{x}\| + \beta_n\gamma\alpha\|x_n - \tilde{x}\|\big) \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\le \Phi\big(\big[\varphi(1)(1 - \beta_n\bar{\gamma})k_n + \beta_n\gamma\alpha\big]\|x_n - \tilde{x}\|\big) \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\le \big[\varphi(1)(1 - \beta_n\bar{\gamma})k_n + \beta_n\gamma\alpha\big]\Phi\big(\|x_n - \tilde{x}\|\big) \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\le \big[\big(1 - \varphi(1)\beta_n\bar{\gamma}\big)(k_n - 1) + 1 - \beta_n\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)\big]\Phi\big(\|x_n - \tilde{x}\|\big) \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\le \big(1 - \beta_n\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)\big)\Phi\big(\|x_n - \tilde{x}\|\big) + \big(1 - \varphi(1)\beta_n\bar{\gamma}\big)(k_n - 1)M'' \\
&\quad + \beta_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle,
\end{aligned} \tag{3.23}$$

where $M'' > 0$ such that $\sup_{n\ge 1}\Phi(\|x_n - \tilde{x}\|) \le M''$. Put

$$s_n = \beta_n\big(\varphi(1)\bar{\gamma} - \gamma\alpha\big)$$

and

$$\sigma_n = \left(\frac{1 - \varphi(1)\beta_n\bar{\gamma}}{\varphi(1)\bar{\gamma} - \gamma\alpha}\right)\left(\frac{k_n - 1}{\beta_n}\right)M'' + \frac{1}{(\varphi(1)\bar{\gamma} - \gamma\alpha)}\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle.$$

Then (3.23) is reduced to

$$\Phi\big(\|x_{n+1} - \tilde{x}\|\big) \le (1 - s_n)\Phi\big(\|x_n - \tilde{x}\|\big) + s_n\sigma_n. \tag{3.24}$$

Applying Lemma 2.6 to (3.24), we conclude that $\Phi(\|x_{n+1} - \tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Using Theorem 3.4, we obtain the following two strong convergence theorems of new iterative approximation methods for an asymptotically nonexpansive semigroup $\mathcal{S}$.

Theorem 3.5 *Let E be a real Banach space with uniform normal structure, which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E such that $K \pm K \subset K$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator on K with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1$, $t_n \geq 0$,*

(C1) $\lim_{n\to\infty} \beta_n = 0$;
(C2) $\lim_{n\to\infty} \frac{k_n - 1}{\beta_n} = 0$;
(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $w_0 \in K$, let the sequence $\{w_n\}$ be defined by

$$w_{n+1} = \beta_n \gamma f\bigl(T^n(t_n)w_n\bigr) + (I - \beta_n A)T^n(t_n)w_n, \quad n \geq 0. \tag{3.25}$$

Then $\{w_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (3.2).

Proof Let $\{x_n\}$ be the sequence given by $x_0 = w_0$ and

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n A)T^n(t_n)x_n, \quad \forall n \geq 0.$$

By Theorem 3.4, $x_n \to \tilde{x}$. We claim that $w_n \to \tilde{x}$. We calculate the following:

$$\begin{aligned}
\|x_{n+1} - w_{n+1}\| &= \beta_n \gamma \bigl\|f(x_n) - f\bigl(T^n(t_n)w_n\bigr)\bigr\| + \|I - \beta_n A\| \bigl\|T^n(t_n)x_n - T^n(t_n)w_n\bigr\| \\
&\leq \beta_n \gamma \alpha \bigl\|x_n - T^n(t_n)w_n\bigr\| + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \bigl\|x_n - T^n(t_n)\tilde{x}\bigr\| + \beta_n \gamma \alpha \bigl\|T^n(t_n)\tilde{x} - T^n(t_n)w_n\bigr\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - \tilde{x}\| + \beta_n \gamma \alpha k_n \|\tilde{x} - w_n\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - \tilde{x}\| + \beta_n \gamma \alpha \|w_n - x_n\| + \beta_n \gamma \alpha k_n \|\tilde{x} - x_n\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&= \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| + \beta_n \gamma \alpha \|w_n - x_n\| \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\| \\
&= \bigl(\varphi(1)(1 - \beta_n \bar{\gamma})(k_n) + \beta_n \gamma \alpha\bigr)\|x_n - w_n\| \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\| \\
&\leq \bigl[\bigl(1 - \varphi(1)\beta_n \bar{\gamma}\bigr)(k_n - 1) + 1 - \beta_n\bigl(\varphi(1)\bar{\gamma} - \gamma \alpha\bigr)\bigr]\|x_n - w_n\| \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\| \\
&\leq \bigl(1 - \beta_n\bigl(\varphi(1)\bar{\gamma} - \gamma \alpha\bigr)\bigr)\|x_n - w_n\| + \bigl(1 - \varphi(1)\beta_n \bar{\gamma}\bigr)(k_n - 1)M \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\|,
\end{aligned}$$

where $M > 0$ such that $\sup_{n\geq 1}\|x_n - w_n\| \leq M$. Put

$$s_n = \beta_n\bigl(\varphi(1)\bar{\gamma} - \gamma\alpha\bigr)$$

and

$$\sigma_n = \left(\frac{1-\varphi(1)\beta_n\bar{\gamma}}{\varphi(1)\bar{\gamma}-\gamma\alpha}\right)\left(\frac{k_n-1}{\beta_n}\right)M + \frac{(\gamma\alpha + k_n)}{(\varphi(1)\bar{\gamma}-\gamma\alpha)}\|x_n - \tilde{x}\|.$$

Then we have that

$$\|x_{n+1} - w_{n+1}\| \leq (1-s_n)\|x_n - w_n\| + s_n\sigma_n. \tag{3.26}$$

It follows from (C3), $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$ and Lemma 2.6 that $\|x_n - w_n\| \to 0$ as $n \to \infty$; that is, $w_n \to \tilde{x}$ as $n \to \infty$. □

Theorem 3.6 *Let E be a real Banach space with uniform normal structure which has a uniformly Gâteaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E such that $K \pm K \subset K$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator on K with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1$, $t_n \geq 0$,*

(C1) $\lim_{n\to\infty}\beta_n = 0$;
(C2) $\lim_{n\to\infty}\frac{k_n-1}{\beta_n} = 0$;
(C3) $\sum_{n=0}^{\infty}\beta_n = \infty$.

For any $z_0 \in K$, let the sequence $\{z_n\}$ be defined by

$$z_{n+1} = T^n(t_n)\bigl(\beta_n\gamma f(z_n) + (I - \beta_n A)z_n\bigr), \quad n \geq 0. \tag{3.27}$$

Then $\{z_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (3.2).

Proof Define the sequences $\{w_n\}$ and $\{\sigma_n\}$ by

$$w_n = \beta_n\gamma f(z_n) + (I - \beta_n A)z_n \quad \text{and} \quad \sigma_n = \beta_{n+1}, \quad n \geq 0.$$

We have that

$$w_{n+1} = \beta_{n+1}\gamma f(z_{n+1}) + (I - \beta_{n+1}A)z_{n+1} = \sigma_n\gamma f\bigl(T^n(t_n)w_n\bigr) + (I - \sigma_n A)T^n(t_n)w_n.$$

It follows from Theorem 3.5 that $\{w_n\}$ converges strongly to $\tilde{x}$. Thus, we have

$$\begin{aligned}\|z_n - \tilde{x}\| &\leq \|z_n - w_n\| + \|w_n - \tilde{x}\| = \beta_n\bigl\|\gamma f(z_n) - Az_n\bigr\| + \|w_n - \tilde{x}\| \\ &\leq \beta_n M + \|w_n - \tilde{x}\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

where $M > 0$ such that $M \geq \sup_{n\geq 1}\|\gamma f(z_n) - Az_n\|$. Hence, $\{z_n\}$ converges strongly to $\tilde{x}$. □

If $A \equiv I$, the identity mapping on E, and $\gamma = 1$, then Theorem 3.4 reduces to the following corollary.

Corollary 3.7 *Let E be a real Banach space with uniform normal structure which has a uniformly Gâteaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1$, $t_n \geq 0$,*

(C1) $\lim_{n\to\infty} \beta_n = 0$;
(C2) $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$;
(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $x_0 \in K$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) T^n(t_n) x_n \, ds, \quad n \geq 0. \tag{3.28}$$

Suppose, in addition, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ uniformly in $t \in [0,\infty)$. Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{x}$ in $F(\mathcal{S})$, which solves the variational inequality (3.11).

If $f \equiv u$, then Corollary 3.7 reduces to the following corollary.

Corollary 3.8 *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, K be a nonempty, bounded, closed and convex subset of E such that $K \pm K \subset K$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on K with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < N(E)^{1/2}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_K$ with coefficient $\alpha \in (0,1)$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1$, $t_n \geq 0$,*

(C1) $\lim_{n\to\infty} \beta_n = 0$;
(C2) $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$;
(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $x_0 \in K$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \beta_n u + (1 - \beta_n) T^n(t_n) x_n \, ds, \quad n \geq 0. \tag{3.29}$$

Suppose, in addition, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ uniformly in $t \in [0,\infty)$. Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (3.13).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors read and approved the final manuscript.

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