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Some notes on T-partial order

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Abstract

In this study, by means of the T-partial order defined in Karaçal and Kesicioğlu (Kybernetika 47:300-314, 2011), an equivalence relation on the class of t-norms on $([0,1],\leq,0,1)$ is defined. Then, it is showed that the equivalence class of the weakest t-norm T_D on $[0,1]$ contains a t-norm which is different from T_D. Finally, defining the sets of some incomparable elements with any $x\in(0,1)$ according to $\preceq_T$, these sets are studied.

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1 Introduction

Triangular norms were originally studied in the framework of probabilistic metric spaces [1–4] aiming at an extension of the triangle inequality and following some ideas of Menger [5]. Later on, they turned out to be interpretations of the conjunction in many-valued logics [6–8], in particular in fuzzy logics, where the unit interval serves as a set of truth values.

In [9], a natural order for semigroups was defined. Similarly, in [10], a partial order defined by means of t-norms on a bounded lattice was introduced. For any elements x, y of a bounded lattice

$$x\preceq_T y:\quad\Leftrightarrow\quad T(\ell,y)=x\quad\text{for some }\ell,$$

where T is a t-norm. This order $\preceq_T$ is called a t-partial order of T. Moreover, some connections between the natural order and the t-partial order $\preceq_T$ were studied.

In [10], it was investigated that $\preceq_T$ implies the natural order but its converse needs not be true. It was showed that a partially ordered set is not a lattice with respect to $\preceq_T$. Some sets which are lattices with respect to $\preceq_T$ under some special conditions were determined. For more details on t-norms on bounded lattices, we refer to [11–17].

In the present paper, we introduce an equivalence on the class of t-norms on $([0,1],\leq,0,1)$ based on the equality of the sets of all incomparable elements with respect to $\preceq_T$. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we define an equivalence on the class of t-norms on $([0,1],\leq,0,1)$, and we determine the equivalence class of the weakest t-norm T_D on $[0,1]$. Thus, we obtain that the equivalence class of the weakest t-norm T_D contains a t-norm which is different from a t-norm T_D. We obtain that for arbitrary $m\in K_T$, there exists an element $y_m\in K_T$ such that $T(m,\cdot):[0,1]\to[0,m]$ or $T(y_m,\cdot):[0,1]\to[0,y_m]$ is not continuous.

2 Basic definitions and properties

Definition 2.1 [18] A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, *i.e.*, a function $T : [0,1] \times [0,1] \to [0,1]$, such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$ (commutativity);
(T2) $T(x, T(y,z)) = T(T(x,y),z)$ (associativity);
(T3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$ (monotonicity);
(T4) $T(x,1) = x$ (boundary condition).

Example 2.1 [18] The following are the four basic t-norms T_M, T_P, T_L, T_D given by, respectively,

$$T_M(x,y) = \min(x,y),$$
$$T_P(x,y) = x \cdot y,$$
$$T_L(x,y) = \max(x+y-1, 0),$$
$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Also, t-norms on a bounded lattice $(L, \leq, 0, 1)$ are defined in a similar way, and then extremal t-norms T_W as well as $T_\wedge$ on L are defined as T_D and T_M on $[0,1]$.

Remark 2.1 [18]

(i) Directly from Definition 2.1, we can deduce that, for all $x \in [0,1]$, each t-norm T satisfies the following additional boundary conditions:

$$T(0,x) = 0,$$
$$T(1,x) = x.$$

Therefore, all t-norms coincide on the boundary of the unit square $[0,1] \times [0,1]$.

(ii) The monotonicity of a t-norm T in its second component described by (T3) is, together with the commutativity (T1), equivalent to the monotonicity in both components, *i.e.*, to

$$T(x_1,y_1) \leq T(x_2,y_2) \quad \text{whenever} \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2. \tag{2.1}$$

Definition 2.2 [18] A function $F : [0,1] \times [0,1] \to [0,1]$ is called continuous if for all convergent sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$,

$$F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = \lim_{n\to\infty} F(x_n, y_n).$$

Proposition 2.1 [18] *A function* $F : [0,1] \times [0,1] \to [0,1]$ *which is non-decreasing, i.e., which satisfies* (2.1), *is continuous if and only if it is continuous in each component, i.e., if for all* $x_0, y_0 \in [0,1]$, *both the vertical section* $F(x_0, \cdot) : [0,1] \to [0,1]$ *and the horizontal section* $F(\cdot, y_0) : [0,1] \to [0,1]$ *are continuous functions in one variable.*

Proposition 2.2 [18] *A non-decreasing function $F : [0,1] \times [0,1] \to [0,1]$ is lower semi-continuous if and only if it is left-continuous in each component, i.e., if for all $x_0, y_0 \in [0,1]$ and for all sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, we have*

$$\sup\{F(x_n, y_0) \mid n \in \mathbb{N}\} = F(\sup\{x_n \mid n \in \mathbb{N}\}, y_0),$$
$$\sup\{F(x_0, y_n) \mid n \in \mathbb{N}\} = F(x_0, \sup\{y_n \mid n \in \mathbb{N}\}).$$

By the same token, the upper semicontinuity of a non-decreasing function $F : [0,1] \times [0,1] \to [0,1]$ is equivalent to its right-continuity in each component.

Proposition 2.3 [7] *For a non-decreasing function $F : \mathbb{I}^n \to \mathbb{R}$ ($\mathbb{I}$ an interval), the following conditions are equivalent*:

(i) *F is continuous*;

(ii) *F is continuous in each variable, i.e., for any $x \in \mathbb{I}^n$ and any $i \in \{1, 2, \dots, n\}$, the unary function*

$$u \to F(x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n)$$

is continuous;

(iii) *F has the intermediate value property*: *For any $x, y \in \mathbb{I}^n$, with $x \le y$, and any $c \in [F(x), F(y)]$, there exists $z \in \mathbb{I}^n$, with $x \le z \le y$, such that $F(z) = c$.*

Definition 2.3 [10] Let L be a bounded lattice, let T be a t-norm on L. The order defined as follows is called a t-order (triangular order) for a t-norm T.

$$x \preccurlyeq_T y : \quad \Leftrightarrow \quad T(\ell, y) = x \quad \text{for some } \ell \in L.$$

Example 2.2 [10] Let $L = \{0, a, b, c, 1\}$ and consider the order $\le$ on L as in Figure 1.

We choose the t-norm T_W. Then $a \le b$, but $a \not\preceq_{T_W} b$. We suppose that $a \preceq_{T_W} b$. Then there exists an element $\ell \in L$ such that $T_W(\ell, b) = a$. If $\ell = 0$, then it is obtained that $a = 0$, a contradiction. If $\ell = a, b$ or c, then we have that $T_W(\ell, b) = 0 = a$, a contradiction. If $\ell = 1$, then it is obtained that $T_W(1, b) = b = a$, which is not possible. So, there does not exist any element $\ell \in L$ satisfying $T_W(\ell, b) = a$. Thus, $a \not\preceq_{T_W} b$. Then the order $\preceq_{T_W}$ on L is as in Figure 2.

Proposition 2.4 [10] *Let L be a bounded lattice, let T be a t-norm on L. Then the binary relation $\preccurlyeq_T$ is a partial order on L.*

Definition 2.4 [10] This partial order $\preccurlyeq_T$ is called a T-partial order on L.

Figure 1 The order $\le$ on *L*.

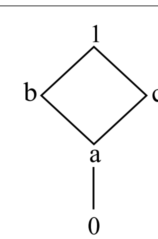
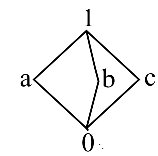


Figure 2 The order $\preceq_{T_W}$ on *L*.



Definition 2.5 Let T be a t-norm on $[0,1]$ and let K_T be defined by

$$K_T = \{x \in [0,1] \mid \text{ for some } y \in [0,1], [x \leq y \text{ implies } x \npreceq_T y] \text{ or } [y \leq x \text{ implies } y \npreceq_T x]\}.$$

We will use the notation K_T to denote the set of all incomparable elements with respect to $\preceq_T$.

3 The equivalence of any two *t*-norms

Let L be a lattice and let T be any t-norm on L. In [10], a partial order for a t-norm T on L was defined. In this section, we define an equivalence relation with the help of the sets of all incomparable elements with respect to $\preceq_T$. The above introduced T-partial order allows us to introduce the next equivalence relation on the class of all t-norms on $([0,1], \leq, 0, 1)$.

Definition 3.1 Let $([0,1], \leq, 0, 1)$ be the unit interval. Define a relation $\sim$ on the class of all t-norms on $([0,1], \leq, 0, 1)$ by $T_1 \sim T_2$ if and only if the set of all incomparable elements with respect to the T_1-partial order is equal to the set of all incomparable elements with respect to the T_2-partial order, that is,

$$T_1 \sim T_2 : \quad \Leftrightarrow \quad K_{T_1} = K_{T_2}.$$

Proposition 3.1 *The relation $\sim$ given in Definition* 3.1 *is an equivalence relation.*

Proof Let T_1, T_2 and T_3 be t-norms on $([0,1], \leq, 0, 1)$. Since $K_{T_1} = K_{T_1}$, it is obtained that $T_1 \sim T_1$. Thus, the reflexivity is satisfied.

Let $T_1 \sim T_2$. Then we have that $K_{T_1} = K_{T_2}$, and since $K_{T_2} = K_{T_1}$, it is obtained that $T_2 \sim T_1$. Thus, the symmetry is satisfied. Let $T_1 \sim T_2$ and $T_2 \sim T_3$. Then we have $K_{T_1} = K_{T_2}$ and $K_{T_2} = K_{T_3}$. Since $K_{T_1} = K_{T_3}$, it is obtained that $T_1 \sim T_3$. This means that the relation $\sim$ satisfies the transitivity. So, we have that $\sim$ is an equivalence relation. □

Definition 3.2 For a given t-norm T on $([0,1], \leq, 0, 1)$, we denote by $\overline{T}$ the $\sim$ equivalence class linked to T, *i.e.*,

$$\overline{T} = \{T' \mid T' \text{ is a } t\text{-norm on } [0,1] \text{ and } T \sim T'\}.$$

Proposition 3.2 shows that the equivalence class of the t-norm T_D contains a t-norm which is different from T_D.

Proposition 3.2 *Let the t-norm T_D be on* $[0,1]$. *Then* $\overline{T_D} \neq \{T_D\}$.

We give a contrary example as follows for the proof of Proposition 3.2.

Example 3.1 Consider the t-norm $T : [0,1] \times [0,1] \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \frac{xy}{2} & \text{if } (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Then $K_T = K_{T_D}$. Firstly, let us show that $K_T = (0,1)$. Let $x \in (0,1)$ and $y = \frac{2x}{3}$. Then $y < x$, but $y \not\preceq_T x$. Suppose that $y \preccurlyeq_T x$. Then, for some ℓ, $T(\ell,x) = \frac{2x}{3}$. Since $x \neq y$, it is not possible $\ell = 1$. Then $\frac{2x}{3} = T(\ell,x) = \frac{\ell x}{2}$, whence it is obtained that $\ell = \frac{4}{3}$, a contradiction. Since for any $x \in (0,1)$ there exists an element $y = \frac{2x}{3}$ such that $\frac{2x}{3} < x$ but $\frac{2x}{3} \not\preceq_T x$, $x \in K_T$. Conversely, for any t-norm T, it is clear that $K_T \subseteq (0,1)$. So, it is obtained that $K_T = (0,1)$.

Now, we will show that $K_{T_D} = (0,1)$. Let $x \in (0,1)$. For any $y \in (0,1)$ with $x < y$, it is obvious that $x \not\preceq_{T_D} y$. Otherwise, it would be $T_D(\ell,y) = x$ for some ℓ. Since $x \neq y$, $\ell \neq 1$. Thus, it must be $x = 0$ for $\ell, y \in (0,1)$, a contradiction. Since there is an element y with $x < y$ such that $x \not\preceq_{T_D} y$, $x \in K_{T_D}$. This shows that $K_{T_D} = (0,1)$. So, it is obtained that $K_T = K_{T_D}$.

Definition 3.3 Let T be a t-norm on $[0,1]$ and let $K_{x\downarrow}$, $K_{x\uparrow}$ be defined by

$$K_{x\downarrow} = \{y \in [0,1] \mid y \leq x \text{ and } y \not\preceq_T x\},$$
$$K_{x\uparrow} = \{y' \in [0,1] \mid x \leq y' \text{ and } x \not\preceq_T y'\} \quad \text{for any } x \in (0,1).$$

Lemma 3.1 *Let T be a t-norm on $[0,1]$ and $x \in K_T$ be arbitrarily chosen. If T is continuous at (x,y) for all $y \in [0,1]$, then $K_{x\downarrow} = \emptyset$.*

Proof Let T be a t-norm on $[0,1]$ and let $x \in K_T$ be arbitrarily chosen. Suppose that $K_{x\downarrow} \neq \emptyset$. Then there exists an element $y_0 \in [0,1]$ such that $y_0 \leq x$, but $y_0 \not\preceq_T x$. Since the t-norm $T(x,\cdot) : [0,1] \to [0,x]$ is continuous, there exists an element $z \in [0,1]$ such that $T(x,z) = y_0$ for $y_0 \in [0,x]$ by Proposition 2.3. So, it is obtained that $y_0 \preceq_T x$, a contradiction. Therefore we have that $K_{x\downarrow} = \emptyset$. □

Lemma 3.2 *Let T be a t-norm on $[0,1]$ and the function $T(x_0,\cdot) : [0,1] \to [0,x_0]$ be continuous. Then, for all $y \in [0,1]$ with $y \leq x_0$, we have that $y \preceq_T x_0$.*

Proof Let T be a t-norm on $[0,1]$ and the function $T(x_0,\cdot) : [0,1] \to [0,x_0]$ be continuous. Suppose that there exists an element $y_0 \in [0,1]$ such that $y_0 \leq x_0$ and $y_0 \not\preceq_T x_0$. Since $T(x_0,\cdot) : [0,1] \to [0,x_0]$ is continuous, there exists an element $t \in [0,1]$ such that $T(x_0,t) = y_0$ for $y_0 \in [0,x_0]$ by Proposition 2.3. Thus, it is obtained that $y_0 \preceq_T x_0$, a contradiction. Therefore, for all $y \in [0,1]$ with $y \leq x_0$, we have that $y \preceq_T x_0$. □

Theorem 3.1 *Let T be a t-norm on $[0,1]$ and $K_T \neq \emptyset$. Then, for arbitrary $m \in K_T$, there exists an element $y_m \in K_T$ such that $T(m,\cdot) : [0,1] \to [0,m]$ or $T(y_m,\cdot) : [0,1] \to [0,y_m]$ is not continuous.*

Proof Let T be a t-norm on $[0,1]$ and $K_T \neq \emptyset$. Suppose that $T(x,\cdot) : [0,1] \to [0,x]$ is continuous for $x \in K_T$. Choose $m \in K_T$ arbitrarily. Then there exists an element $y_m \in K_T$ such that $m < y_m$ but $m \not\preceq_T y_m$, or $y_m < m$ but $y_m \not\preceq_T m$. Let $m < y_m$ but $m \not\preceq_T y_m$. Since $T(y_m,\cdot) :$

$[0,1] \to [0, y_m]$ is continuous, then it is obtained that $m \preceq_T y_m$ by Lemma 3.2, a contradiction. Let $y_m < m$ but $y_m \npreceq_T m$. Since $T(m, \cdot) : [0,1] \to [0, m]$ is continuous, then it is obtained that $y_m \preceq_T m$ by Lemma 3.2, a contradiction. Therefore, for arbitrary $m \in K_T$, there exists an element $y_m \in K_T$ such that $T(m, \cdot) : [0,1] \to [0, m]$ or $T(y_m, \cdot) : [0,1] \to [0, y_m]$ is not continuous. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The present study was proposed by FK. All authors read and approved the final manuscript.

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