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Approximating fixed points for continuous functions on an arbitrary interval

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Abstract

In this research article, we introduce a new iterative method for solving a fixed point problem of continuous functions on an arbitrary interval. We then prove the convergence theorem of the proposed algorithm. We finally give numerical examples to compare the result with Mann, Ishikawa and Noor iterations. Our main results extend the corresponding results in the literature.

MSC: 47H09; 47H10

Keywords: continuous function; convergence theorem; fixed point; iteration

1 Introduction

Let C be a closed interval on the real line and let $f : C \to C$ be a continuous function. A point $p \in C$ is called a *fixed point* of f if $f(p) = p$.

One classical way to approximate a fixed point of a nonlinear mapping was introduced in 1953 by Mann [1] as follows: a sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f(x_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$. Such an iteration process is known as *Mann iteration*. In 1991, Borwein and Borwein [2] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.1).

Another classical iteration process was introduced by Ishikawa [3] as follows: a sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n f(x_n), \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n f(y_n) \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Such an iterative method is known as *Ishikawa iteration*. In 2006, Qing and Qihou [4] proved the convergence theorem of the sequence generated by iteration (1.2) for a continuous function on the closed interval in the real line (see also [5]).

In 2000, Noor [6] defined the following iterative scheme by $x_1 \in C$ and

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)x_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n f(y_n) \end{aligned} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in $[0,1]$. Such an iterative method is known as *Noor iteration*. Phuengrattana and Suantai [7] considered the convergence of Noor iteration for continuous functions on an arbitrary interval in the real line.

In this paper, motivated by the previous ones, we introduce a new modified iteration process for solving a fixed point problem for continuous functions on an arbitrary interval in the real line. Numerical examples are also presented to compare the result with Mann, Ishikawa and Noor iterations.

2 Main results

We begin this section by proving the following crucial lemmas.

Lemma 2.1 *Let C be a closed interval on the real line (can be unbounded) and let $f: C \to C$ be a continuous function. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ with $0 \leq \tau_n + \beta_n \leq 1$ and $0 \leq \gamma_n + \alpha_n \leq 1$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned} \tag{2.1}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$.
If $x_n \to a$, then a is a fixed point of f.

Proof Let $x_n \to a$ and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}$, $\{z_n\}$, $\{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $z_n \to a$ and $\beta_n \to 0$.

From (2.1) we obtain

$$\begin{aligned} x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n) \\ &= z_n + \gamma_n(y_n - z_n) + \alpha_n\big(f(y_n) - z_n\big) \\ &= (1-\mu_n)x_n + \mu_n f(x_n) + \gamma_n(y_n - z_n) + \alpha_n\big(f(y_n) - z_n\big) \\ &= x_n + \mu_n\big(f(x_n) - x_n\big) + \gamma_n(y_n - z_n) + \alpha_n\big(f(y_n) - z_n\big) \\ &= x_n + \mu_n\big(f(x_n) - x_n\big) + \gamma_n\big((1-\tau_n-\beta_n)(x_n - z_n) + \beta_n\big(f(z_n) - z_n\big)\big) \\ &\quad + \alpha_n\big(f(y_n) - z_n\big) \\ &= x_n + \mu_n\big(f(x_n) - x_n\big) + \gamma_n\big((1-\tau_n-\beta_n)\mu_n\big(x_n - f(x_n)\big) + \beta_n\big(f(z_n) - z_n\big)\big) \\ &\quad + \alpha_n\big(f(y_n) - z_n\big) \end{aligned}$$

$$= x_n + \mu_n\big(1 - \gamma_n(1 - \tau_n - \beta_n)\big)\big(f(x_n) - x_n\big) + \gamma_n\beta_n\big(f(z_n) - z_n\big) + \alpha_n\big(f(y_n) - z_n\big). \tag{2.2}$$

Let $p_k = f(x_k) - x_k$, $q_k = f(z_k) - z_k$ and $r_k = f(y_k) - z_k$. Then we observe

$$\lim_{k\to\infty} p_k = \lim_{n\to\infty}\big(f(x_k) - x_k\big) = f(a) - a \neq 0,$$
$$\lim_{k\to\infty} q_k = \lim_{n\to\infty}\big(f(z_k) - z_k\big) = f(a) - a \neq 0,$$
$$\lim_{k\to\infty} r_k = \lim_{n\to\infty}\big(f(y_k) - z_k\big) = f(a) - a \neq 0.$$

So, from (2.2) we obtain

$$\begin{aligned} x_n &= x_1 + \sum_{k=1}^{n}\mu_k\big(1 - \gamma_k(1 - \tau_k - \beta_k)\big)\big(f(x_k) - x_k\big) + \sum_{k=1}^{n}\gamma_k\beta_k\big(f(z_k) - z_k\big) \\ &\quad + \sum_{k=1}^{n}\alpha_k\big(f(y_k) - z_k\big) \\ &= x_1 + \sum_{k=1}^{n}\mu_k\big(1 - \gamma_k(1 - \tau_k - \beta_k)\big)p_k + \sum_{k=1}^{n}\gamma_k\beta_k q_k + \sum_{k=1}^{n}\alpha_k r_k. \end{aligned}$$

It is easy to see that $\sum_{k=1}^{\infty}\mu_k(1 - \gamma_k(1 - \tau_k - \beta_k))p_k < \infty$ since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty}\mu_k < \infty$. Similarly, we have $\sum_{k=1}^{\infty}\gamma_k\beta_k q_k < \infty$ since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty}\beta_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} r_k \neq 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$. This contradicts the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is a fixed point of f. □

Lemma 2.2 *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ with $0 \le \tau_n + \beta_n \le 1$ and $0 \le \gamma_n + \alpha_n \le 1$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$z_n = (1 - \mu_n)x_n + \mu_n f(x_n),$$
$$y_n = (1 - \tau_n - \beta_n)x_n + \tau_n z_n + \beta_n f(z_n),$$
$$x_{n+1} = (1 - \gamma_n - \alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n), \quad n \ge 1,$$

where $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$ and $\sum_{n=1}^{\infty}\mu_n < \infty$.

If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.

Proof Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Since f is continuous, there exists δ with $0 < \delta < b - a$ such that for $|x - m| \le \delta$,

$$f(x) - x > 0.$$

Since $\{x_n\}$ is bounded and f is continuous, $\{f(x_n)\}$ is bounded. Hence $\{z_n\}$, $\{y_n\}$, $\{f(z_n)\}$ and $\{f(y_n)\}$ are all bounded. Using

$$\begin{aligned}
x_{n+1} - x_n &= (1-\gamma_n-\alpha_n)(z_n - x_n) + \gamma_n(y_n - x_n) + \alpha_n\bigl(f(y_n) - x_n\bigr),\\
y_n - x_n &= \tau_n(z_n - x_n) + \beta_n\bigl(f(z_n) - x_n\bigr),\\
z_n - x_n &= \mu_n\bigl(f(x_n) - x_n\bigr),
\end{aligned}$$

we can easily show that $|z_n - x_n| \to 0$, $|y_n - x_n| \to 0$ and $|x_{n+1} - x_n| \to 0$. Thus there exists a positive integer N such that for all $n > N$,

$$|x_{n+1} - x_n| < \frac{\delta}{2}, \qquad |y_n - x_n| < \frac{\delta}{2}, \qquad |z_n - x_n| < \frac{\delta}{2}. \tag{2.3}$$

Since $b = \limsup_n x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $n_{k_1} = k$, then $x_k > m$. For x_k, there exist two cases as follows.

(i) $x_k > m + \frac{\delta}{2}$, then $x_{k+1} > x_k - \frac{\delta}{2} \geq m$ using (2.3). So, we have $x_{k+1} > m$.

(ii) $m < x_k < m + \frac{\delta}{2}$, then $m - \frac{\delta}{2} < y_k < m + \delta$ and $m - \frac{\delta}{2} < z_k < m + \delta$ by (2.3). So, we obtain $|x_k - m| < \frac{\delta}{2} < \delta$, $|y_k - m| < \delta$, $|z_k - m| < \delta$. Hence

$$f(x_k) - x_k > 0, \qquad f(y_k) - y_k > 0, \qquad f(z_k) - z_k > 0. \tag{2.4}$$

We observe that

$$\begin{aligned}
y_k - z_k &= (1-\tau_k-\beta_k)(x_k - z_k) + \beta_k\bigl(f(z_k) - z_k\bigr)\\
&= \mu_k(1-\tau_k-\beta_k)\bigl(x_k - f(x_k)\bigr) + \beta_k\bigl(f(z_k) - z_k\bigr).
\end{aligned} \tag{2.5}$$

From (2.2), (2.4) and (2.5), we have

$$\begin{aligned}
x_{k+1} &= x_k + \mu_k\bigl(1-\gamma_k(1-\tau_k-\beta_k)\bigr)\bigl(f(x_k) - x_k\bigr) + \gamma_k\beta_k\bigl(f(z_k) - z_k\bigr)\\
&\quad + \alpha_k\bigl(f(y_k) - z_k\bigr)\\
&= x_k + \mu_k\bigl(1-\gamma_k(1-\tau_k-\beta_k)\bigr)\bigl(f(x_k) - x_k\bigr) + \gamma_k\beta_k\bigl(f(z_k) - z_k\bigr)\\
&\quad + \alpha_k\bigl(f(y_k) - y_k\bigr) + \alpha_k(y_k - z_k)\\
&= x_k + \mu_k\bigl(1-\gamma_k(1-\tau_k-\beta_k)\bigr)\bigl(f(x_k) - x_k\bigr) + \gamma_k\beta_k\bigl(f(z_k) - z_k\bigr)\\
&\quad + \alpha_k\bigl(f(y_k) - y_k\bigr) + \alpha_k\bigl(\mu_k(1-\tau_k-\beta_k)\bigl(x_k - f(x_k)\bigr) + \beta_k\bigl(f(z_k) - z_k\bigr)\bigr)\\
&= x_k + \mu_k\bigl(1-(\gamma_k+\alpha_k)(1-\tau_k-\beta_k)\bigr)\bigl(f(x_k) - x_k\bigr) + \alpha_k\bigl(f(y_k) - y_k\bigr)\\
&\quad + \beta_k(\gamma_k+\alpha_k)\bigl(f(z_k) - z_k\bigr)\\
&> x_k.
\end{aligned}$$

Thus $x_{k+1} > x_k > m$. From (i) and (ii), we have $x_{k+1} > m$. Similarly, we get that $x_{k+2} > m$, $x_{k+3} > m, \ldots$. Thus we have $x_n > m$ for all $n > k = n_{k_1}$. So, $a = \lim_{k\to\infty} x_{n_k} \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

We next consider the following two cases.

(i) There exists x_M such that $a < x_M < b$. Then $f(x_M) = x_M$. It follows that

$$z_M = (1 - \mu_M)x_M + \mu_M f(x_M) = x_M$$

and

$$\begin{aligned} y_M &= (1 - \tau_M - \beta_M)x_M + \tau_M z_M + \beta_M f(z_M) \\ &= (1 - \tau_M - \beta_M)x_M + \tau_M x_M + \beta_M f(x_M) \\ &= x_M. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} x_{M+1} &= (1 - \gamma_M - \alpha_M)z_M + \gamma_M y_M + \alpha_M f(y_M) \\ &= (1 - \gamma_M - \alpha_M)x_M + \gamma_M x_M + \alpha_M f(x_M) \\ &= x_M. \end{aligned}$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \cdots$. So, we conclude that $x_n \to x_M$. Since there exists $x_{n_k} \to a$, $x_M = a$. This shows that $x_n \to a$, which is a contradiction.

(ii) For all $n, x_n \leq a$ or $x_n \geq b$. Since $b - a > 0$ and $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, there exists $\bar{N}$ such that $|x_{n+1} - x_n| < \frac{(b-a)}{2}$ for $n > \bar{N}$. So, it is always that $x_n \leq a$ for $n > \bar{N}$; or it is always that $x_n \geq b$ for $n > \bar{N}$. If $x_n \leq a$ for $n > \bar{N}$, then $b = \lim_{l\to\infty} x_{n_l} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \bar{N}$, then $a = \lim_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. This completes the proof. □

We are now ready to prove the main results of this paper.

Theorem 2.3 *Let C be a closed interval on the real line* (*can be unbounded*) *and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in* $[0,1]$ *with $0 \leq \tau_n + \beta_n \leq 1$ and $0 \leq \gamma_n + \alpha_n \leq 1$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1 - \mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1 - \tau_n - \beta_n)x_n + \tau_n z_n + \beta_n f(z_n), \\ x_{n+1} &= (1 - \gamma_n - \alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$.

If $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point of f.

Proof Let $\{x_n\}$ be a bounded sequence. Then, by Lemma (2.2), $\{x_n\}$ is a convergent sequence. Hence, by Lemma (2.1), it converges to a fixed point of f. □

As a direct consequence of Theorem 2.3, we obtain the following result.

Theorem 2.4 *Let C be a closed interval on the real line* (*can be unbounded*) *and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in* $[0,1]$

with $0 \leq \tau_n + \beta_n \leq 1$ *and* $0 \leq \gamma_n + \alpha_n \leq 1$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in C$ *and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$ *and* $\sum_{n=1}^{\infty}\mu_n < \infty$.
Then $\{x_n\}$ *converges to a fixed point of* f *if and only if* $\{x_n\}$ *is bounded.*

Corollary 2.5 *Let* $f : [a,b] \to [a,b]$ *be a continuous function. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0,1]$ *with* $0 \leq \tau_n + \beta_n \leq 1$ *and* $0 \leq \gamma_n + \alpha_n \leq 1$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in [a,b]$ *and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$ *and* $\sum_{n=1}^{\infty}\mu_n < \infty$.
Then $\{x_n\}$ *converges to a fixed point of* f.

If we take $\tau_n = \gamma_n = 0$, then we obtain the following result.

Corollary 2.6 *Let* C *be a closed interval on the real line* (*can be unbounded*) *and let* $f : C \to C$ *be a continuous function. Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\mu_n\}$ *be sequences in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in C$ *and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)x_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)z_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$ *and* $\sum_{n=1}^{\infty}\mu_n < \infty$.
Then $\{x_n\}$ *converges to a fixed point of* f *if and only if* $\{x_n\}$ *is bounded.*

If we take $\tau_n + \beta_n = 1$ and $\gamma_n + \alpha_n = 1$, then we obtain the following result.

Corollary 2.7 *Let* C *be a closed interval on the real line* (*can be unbounded*) *and let* $f : C \to C$ *be a continuous function. Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\mu_n\}$ *be sequences in* $[0,1]$. *Let* $\{x_n\}$ *be a sequence generated iteratively by* $x_1 \in C$ *and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n f(y_n), \quad n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$.

Then $\{x_n\}$ *converges to a fixed point of* f *if and only if* $\{x_n\}$ *is bounded.*

Remark 2.8 Corollary 2.7 extends the main result obtained in [8] from the modified Ishikawa iteration to the modified Noor iteration.

3 Numerical examples

In this section, we give numerical examples to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (2.1) the CP-iteration.

Example 3.1 Let $f : [1,\infty) \longrightarrow [1,\infty)$ be defined by $f(x) = \sqrt{0.9 \ln x + 1}$. Then f is a continuous function. Use the initial point $x_1 = 3$ and the control conditions $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{2.5}}$, $\mu_n = \frac{1}{(n+1)^{1.5}}$, $\tau_n = \frac{1}{n+1}$ and $\gamma_n = \frac{1}{7}$.

Table 1 Comparison of the convergence rate of Mann, Ishikawa, Noor and CP iterations for the function given in Example 3.1

n	Mann	Ishikawa	Noor	CP-iteration	
	x_n	x_n	x_n	x_n	$\lvert f(x_n) - x_n\rvert$
1	3.000000	3.000000	3.000000	3.000000	1.589769
10	1.074110	1.071538	1.071437	1.043308	0.024408
20	1.015478	1.014946	1.014925	1.008428	0.004658
30	1.004821	1.004656	1.004650	1.002531	0.001394
40	1.001822	1.001760	1.001757	1.000934	0.000514
50	1.000777	1.000750	1.000749	1.000392	0.000216
60	1.000360	1.000348	1.000348	1.000180	0.000099
70	1.000178	1.000172	1.000172	1.000088	0.000048
80	1.000093	1.000089	1.000089	1.000045	0.000025
90	1.000050	1.000048	1.000048	1.000024	0.000013
100	1.000028	1.000027	1.000027	1.000014	0.000007

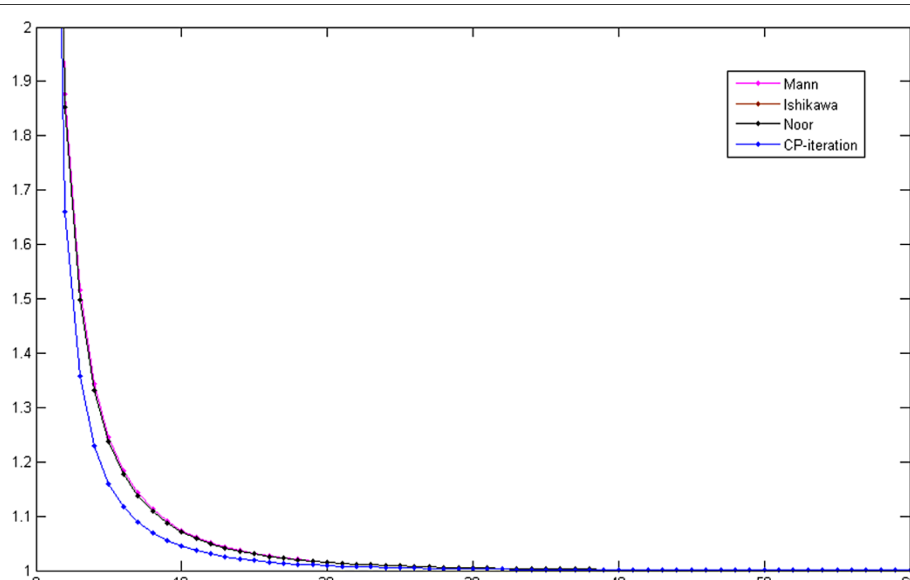

Figure 1 Convergence behavior of Mann, Ishikawa, Noor and CP iterations for the function given in Example 3.1.

Table 2 Comparison of the convergence rate of Mann, Ishikawa, Noor and CP iterations for the function given in Example 3.2

n	Mann	Ishikawa	Noor	CP-iteration	
	x_n	x_n	x_n	x_n	$\lvert f(x_n) - x_n \rvert$
1	5.000000	5.000000	5.000000	5.000000	2.363932
10	1.529828	1.504824	1.501931	1.359649	0.073668
20	1.293319	1.284893	1.283943	1.224654	0.023218
30	1.223891	1.219794	1.219335	1.187049	0.011033
40	1.193599	1.191253	1.190991	1.171120	0.006204
50	1.177752	1.176279	1.176115	1.162994	0.003827
60	1.168542	1.167561	1.167452	1.158380	0.002505
70	1.162807	1.162126	1.162050	1.155568	0.001710
80	1.159055	1.158568	1.158513	1.153766	0.001205
90	1.156509	1.156152	1.156112	1.152565	0.000870
100	1.154730	1.154463	1.154433	1.151740	0.000641
110	1.153458	1.153255	1.153233	1.151160	0.000480
120	1.152530	1.152374	1.152357	1.150743	0.000365
130	1.151842	1.151721	1.151707	1.150438	0.000281
140	1.151325	1.151230	1.151219	1.150212	0.000218
150	1.150931	1.150856	1.150847	1.150042	0.000171
160	1.150628	1.150568	1.150561	1.149912	0.000136
170	1.150393	1.150345	1.150339	1.149813	0.000108
180	1.150209	1.150169	1.150165	1.149735	0.000087
190	1.150063	1.150031	1.150027	1.149675	0.000070
200	1.149947	1.149920	1.149918	1.149627	0.000057

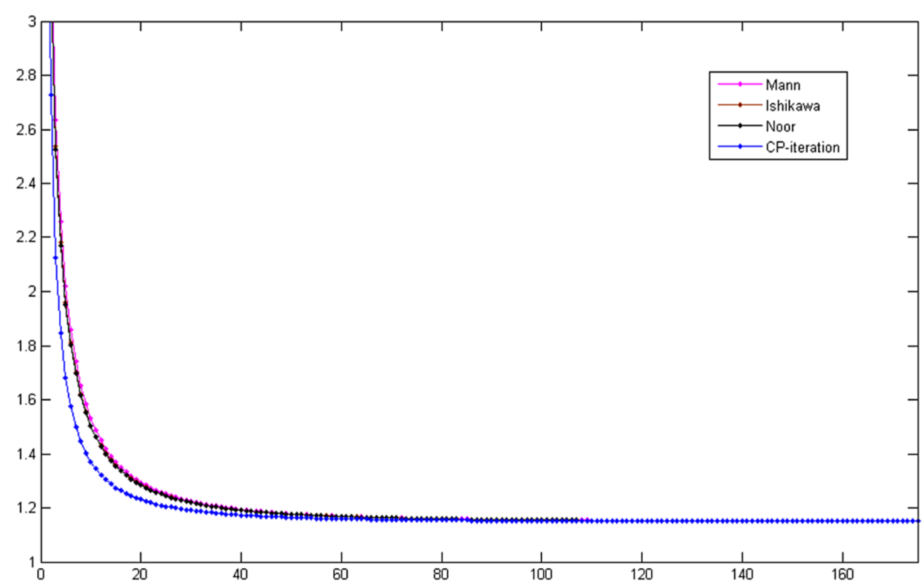

Figure 2 Convergence behavior of Mann, Ishikawa, Noor and CP iterations for the function given in Example 3.2.

Example 3.2 Let $f : [1, \infty) \longrightarrow [1, \infty)$ be defined by $f(x) = 0.2\sqrt{x-1} + \sqrt{x}$. Then f is a continuous function. Use the initial point $x_1 = 5$ and the control conditions $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^2}$, $\mu_n = \frac{1}{(n+1)^{1.1}}$, $\tau_n = \frac{1}{n+1}$ and $\gamma_n = \frac{1}{5}$.

Remark 3.3 From Table 1, Figure 1, Table 2 and Figure 2, we observe that the sequence generated by the CP-iteration converges to a fixed point faster than that of Mann, Ishikawa and Noor iterations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

PC and NP contributed equally. All authors read and approved the final manuscript.

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