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Some identities of higher-order Bernoulli, Euler, and Hermite polynomials arising from umbral calculus

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Abstract
In this paper, we study umbral calculus to have alternative ways of obtaining our results. That is, we derive some interesting identities of the higher-order Bernoulli, Euler, and Hermite polynomials arising from umbral calculus to have alternative ways.
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1 Introduction

As is well known, the Hermite polynomials are defined by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{1.1}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$ (see [1, 2]). In the special case, $x = 0$, $H_n(0) = H_n$ are called the *nth Hermite numbers*. The Bernoulli polynomials of order r are given by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.2}$$

From (1.2), the nth Bernoulli numbers of order r are defined by $B_n^{(r)}(0) = B_n^{(r)}$ (see [1–16]). The higher-order Euler polynomials are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}), \tag{1.3}$$

and $E_n^{(r)}(0) = E_n^{(r)}$ are called the *nth Euler numbers* of order r (see [1–16]).

The first Stirling number is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,k)x^l \quad \text{(see [8, 13])}, \tag{1.4}$$

and the second Stirling number is defined by the generating function to be

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad \text{(see [8, 10, 13])}. \tag{1.5}$$

For $\lambda\ (\neq 1) \in \mathbb{C}$, the *Frobenius-Euler polynomials* are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!} \quad (r\in\mathbb{R}) \text{ (see [3, 7])}. \tag{1.6}$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the *nth Frobenius-Euler numbers* of order r.

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and that $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$, and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\begin{aligned} &\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \\ &\langle cL|p(x)\rangle = c\langle L|p(x)\rangle, \end{aligned}$$

where c is a complex constant (see [8, 10, 13]).

The formal power series $f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \geq 0 \text{ (see [8, 10, 13])}. \tag{1.7}$$

Then, by (1.7), we get

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k\geq 0), \tag{1.8}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [8, 10, 13]).

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}t^k$ (see [13]). For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!}t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We will call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [8, 10, 13]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ having $o(f(t)) = 1$ is called a *delta series*, and a series $f(t)$ having $o(f(t)) = 0$ is called an *invertible series*. Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$, where $n,k\geq 0$. The sequence $S_n(x)$ is called a *Sheffer* *sequence* for $(g(t),f(t))$, which is denoted by $S_n(x) \sim (g(t),f(t))$. By (1.7) and (1.8), we see that $\langle e^{yt}|p(x)\rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k, \tag{1.9}$$

and, by (1.9), we get

$$p^{(k)}(0) = \langle t^k|p(x)\rangle, \qquad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.10}$$

Thus, from (1.10), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}. \tag{1.11}$$

In [8, 10, 13], we note that $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$.

For $S_n(x) \sim (g(t),f(t))$, we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k \quad \text{for all } y \in \mathbb{C}, \tag{1.12}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$. For $S_n(x) \sim (g(t),f(t))$ and $r_n(x) = (h(t), l(t))$, let us assume that

$$S_n(x) = \sum_{k=0}^{n} C_{n,k} r_k(x) \quad (\text{see [8, 10, 13]}). \tag{1.13}$$

Then we have

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l\big(\bar{f}(t)\big)^k \middle| x^n \right\rangle \quad (\text{see [13]}). \tag{1.14}$$

Equations (1.13) and (1.14) are called the alternative ways of Sheffer sequences.

In this paper, we study umbral calculus to have alternative ways of obtaining our results. That is, we derive some interesting identities of the higher-order Bernoulli, Euler, and Hermite polynomials arising from umbral calculus to have alternative ways.

2 Some identities of higher-order Bernoulli, Euler, and Hermite polynomials

In this section, we use umbral calculus to have alternative ways of obtaining our results. Let us consider the following Sheffer sequences:

$$E_n^{(r)}(x) \sim \left(\left(\frac{e^t + 1}{2} \right)^r, t \right), \qquad H_n(x) \sim \left(e^{\frac{1}{4}t^2}, \frac{t}{2} \right). \tag{2.1}$$

Then, by (1.13), we assume that

$$E_n^{(r)}(x) = \sum_{k=0}^{n} C_{n,k} H_k(x). \tag{2.2}$$

From (1.14) and (2.2), we have

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{e^{\frac{1}{4}t^2}}{(\frac{e^t+1}{2})^r}\left(\frac{t}{2}\right)^k \Big| x^n \right\rangle \\
&= \frac{1}{k!2^k}\left\langle \left(\frac{2}{e^t+1}\right)^r e^{\frac{1}{4}t^2} \Big| t^k x^n \right\rangle \\
&= 2^{-k}\binom{n}{k}\left\langle \left(\frac{2}{e^t+1}\right)^r \Big| e^{\frac{1}{4}t^2} x^{n-k} \right\rangle \\
&= 2^{-k}\binom{n}{k}\left\langle \left(\frac{2}{e^t+1}\right)^r \Big| \sum_{l=0}^{[\frac{n-k}{2}]} \frac{1}{4^l l!} t^{2l} x^{n-k} \right\rangle \\
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]} \frac{1}{2^{2l}l!}(n-k)_{2l}\left\langle 1 \Big| \left(\frac{2}{e^t+1}\right)^r x^{n-k-2l} \right\rangle \\
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]} \frac{1}{2^{2l}l!}(n-k)_{2l}E_{n-k-2l}^{(r)} \\
&= n! \sum_{0\le l\le n-k, l:\text{even}} \frac{E_{n-k-l}^{(r)}}{(\frac{l}{2})!2^{k+l}k!(n-k-l)!}.
\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1 *For $n \ge 0$, we have*

$$E_n^{(r)}(x) = n! \sum_{k=0}^{n} \left\{ \sum_{0\le l\le n-k, l:\text{even}} \frac{E_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!} \right\} H_k(x).$$

Let us consider the following two Sheffer sequences:

$$B_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right), \qquad H_n(x) \sim \left(e^{\frac{1}{4}t^2}, \frac{t}{2}\right). \tag{2.4}$$

Let us assume that

$$B_n^{(r)}(x) = \sum_{k=0}^{n} C_{n,k} H_k(x). \tag{2.5}$$

By (1.14) and (2.4), we get

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{e^{\frac{1}{4}t^2}}{(\frac{e^t-1}{t})^r}\left(\frac{t}{2}\right)^k \Big| x^n \right\rangle \\
&= 2^{-k}\binom{n}{k}\left\langle \left(\frac{t}{e^t-1}\right)^r \Big| \sum_{l=0}^{\infty}\left(\frac{1}{4}\right)^l \frac{1}{l!} t^{2l} x^{n-k} \right\rangle \\
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]} \frac{1}{l!4^l}(n-k)_{2l}\left\langle \left(\frac{t}{e^t-1}\right)^r \Big| x^{n-k-2l} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n-k)!}{l!2^{2l}(n-k-2l)!}\left\langle 1\middle|\left(\frac{t}{e^t-1}\right)^r x^{n-k-2l}\right\rangle \\
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n-k)!}{l!2^{2l}(n-k-2l)!}B_{n-k-2l}^{(r)} \\
&= n!\sum_{0\le l\le n-k, l:\text{even}}\frac{B_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}.
\end{aligned} \tag{2.6}
$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.2 *For $n \ge 0$, we have*

$$
B_n^{(r)}(x) = n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k, l:\text{even}}\frac{B_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}\right\}H_k(x).
$$

Consider

$$
H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r, t\right), \qquad H_n(x) \sim \left(e^{\frac{1}{4}t^2}, \frac{t}{2}\right). \tag{2.7}
$$

Let us assume that

$$
H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k}H_k(x). \tag{2.8}
$$

By (1.14), we get

$$
\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{e^{\frac{1}{4}t^2}}{(\frac{e^t-\lambda}{1-\lambda})^r}\left(\frac{t}{2}\right)^k\middle| x^n\right\rangle \\
&= \frac{1}{k!2^k}\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{\frac{1}{4}t^2}\middle| t^k x^n\right\rangle \\
&= 2^{-k}\binom{n}{k}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n-k)_{2l}}{l!4^l}\left\langle 1\middle|\left(\frac{1-\lambda}{e^t-\lambda}\right)^r x^{n-k-2l}\right\rangle \\
&= n!\sum_{l=0}^{[\frac{n-k}{2}]}\frac{H_{n-k-2l}^{(r)}(\lambda)}{l!2^{2l+k}(n-k-2l)!k!} \\
&= n!\sum_{0\le l\le n-k, l:\text{even}}\frac{H_{n-k-l}^{(r)}(\lambda)}{(\frac{l}{2})!2^{k+l}(n-k-l)!k!}.
\end{aligned} \tag{2.9}
$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.3 *For $n \ge 0$, we have*

$$
H_n^{(r)}(x|\lambda) = n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k, l:\text{even}}\frac{H_{n-k-l}^{(r)}(\lambda)}{k!(n-k-l)!(\frac{l}{2})!2^{k+l}}\right\}H_k(x).
$$

Let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k} E_k^{(r)}(x). \tag{2.10}$$

From (1.14), (2.1), and (2.10), we have

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{(\frac{e^{2t}+1}{2})^r}{e^{\frac{1}{4}(2t)^2}}(2t)^k \middle| x^n \right\rangle \\ &= \frac{1}{k!}\left\langle \frac{(\frac{e^{t}+1}{2})^r}{e^{\frac{1}{4}t^2}} t^k \middle| (2x)^n \right\rangle \\ &= \frac{1}{k!} 2^n \left\langle \left(\frac{e^t+1}{2}\right)^r e^{-\frac{1}{4}t^2} \middle| t^k x^n \right\rangle \\ &= 2^n \binom{n}{k} \left\langle \left(\frac{e^t+1}{2}\right)^r \middle| \sum_{l=0}^{\infty} \frac{(-1)^l}{l!4^l} t^{2l} x^{n-k} \right\rangle \\ &= 2^{n-r}\binom{n}{k} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l}{l!2^{2l}} (n-k)_{2l} \left\langle \left(e^t+1\right)^r | x^{n-k-2l} \right\rangle \\ &= \frac{1}{2^r} \sum_{j=0}^{r} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{\binom{n}{k}\binom{r}{j} 2^k (-1)^l (n-k)!}{l!(n-k-2l)!} (2j)^{n-k-2l}. \end{aligned} \tag{2.11}$$

Therefore, (2.10) and (2.11), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$H_n(x) = \frac{1}{2^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{\binom{n}{k}\binom{r}{j} 2^k (-1)^l (n-k)! (2j)^{n-k-2l}}{l!(n-k-2l)!} \right\} E_k^{(r)}(x).$$

Note that $H_n(x) \sim (e^{\frac{1}{4}t^2}, \frac{t}{2})$. Thus, we have

$$e^{\frac{1}{4}t^2} H_n(x) \sim \left(1, \frac{t}{2}\right), \quad \text{and} \quad (2x)^n \sim \left(1, \frac{t}{2}\right). \tag{2.12}$$

From (2.12), we have

$$e^{\frac{1}{4}t^2} H_n(x) = (2x)^n \quad \Leftrightarrow \quad H_n(x) = e^{-\frac{1}{4}t^2}(2x)^n. \tag{2.13}$$

By (2.11) and (2.13), we also see that

$$\begin{aligned} C_{n,k} &= \frac{1}{k!}\left\langle \frac{(\frac{e^{2t}+1}{2})^r}{e^{\frac{1}{4}(2t)^2}}(2t)^k \middle| x^n \right\rangle \\ &= \frac{1}{k!}\left\langle \left(\frac{e^t+1}{2}\right)^r t^k \middle| e^{-\frac{1}{4}t^2}(2x)^n \right\rangle \end{aligned}$$

$$= \frac{1}{k!2^r}\langle (e^t+1)^r | t^k H_n(x)\rangle$$
$$= \frac{1}{2^r}\binom{n}{k}2^k \sum_{j=0}^{r}\binom{r}{j}H_{n-k}(j). \tag{2.14}$$

Therefore, by (2.10) and (2.14), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$H_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\binom{n}{k}2^k\left[\sum_{j=0}^{r}\binom{r}{j}H_{n-k}(j)\right]E_k^{(r)}(x).$$

Let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k}B_k^{(r)}(x). \tag{2.15}$$

From (1.14), (2.4), and (2.15), we have

$$C_{n,k} = \frac{1}{k!}\left\langle \frac{(\frac{e^{2t}-1}{2t})^r}{e^{\frac{1}{4}(2t)^2}}(2t)^k \middle| x^n \right\rangle$$
$$= \frac{1}{k!}\left\langle \frac{(\frac{e^{t}-1}{t})^r}{e^{\frac{1}{4}t^2}}t^k \middle| (2x)^n \right\rangle$$
$$= \frac{1}{k!}\left\langle \left(\frac{e^t-1}{t}\right)^r t^k \middle| e^{-\frac{1}{4}t^2}(2x)^n \right\rangle. \tag{2.16}$$

From (2.13) and (2.16), we have

$$C_{n,k} = \frac{1}{k!}\left\langle \left(\frac{e^t-1}{t}\right)^r t^k \middle| H_n(x) \right\rangle. \tag{2.17}$$

For $r > n$, by (1.5) and (2.17), we get

$$C_{n,k} = \frac{1}{k!}\left\langle (e^t-1)^k \middle| \left(\frac{e^t-1}{t}\right)^{r-k} H_n(x) \right\rangle$$
$$= \frac{1}{k!}\left\langle (e^t-1)^k \middle| \sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_2(l+r-k, r-k)t^l H_n(x) \right\rangle$$
$$= \frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_2(l+r-k, r-k)2^l(n)_l\langle (e^t-1)^k | H_{n-l}(x)\rangle$$
$$= \frac{1}{k!}\sum_{l=0}^{n}\frac{(r-k)!}{(l+r-k)!}S_2(l+r-k, r-k)2^l\frac{n!}{(n-l)!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}H_{n-l}(j)$$
$$= n!\sum_{j=0}^{k}\sum_{l=0}^{n}\frac{(r-k)!S_2(l+r-k, r-k)(-1)^{k-j}\binom{k}{j}2^l H_{n-l}(j)}{(l+r-k)!k!(n-l)!}. \tag{2.18}$$

Therefore, by (2.15) and (2.18), we obtain the following theorem.

Theorem 2.6 *For $r > n \geq 0$, we have*

$$H_n(x) = n! \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)(-1)^{k-j}\binom{k}{j} 2^l H_{n-l}(j)}{(l+r-k)!k!(n-l)!} \right\} B_k^{(r)}(x).$$

Let us assume that $r \leq n$. For $0 \leq k < r$, by (2.18), we get

$$C_{n,k} = n! \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)(-1)^{k-j}\binom{k}{j} 2^l H_{n-l}(j)}{(l+r-k)!k!(n-l)!}. \tag{2.19}$$

For $r \leq k \leq n$, by (2.17), we get

$$\begin{aligned} C_{n,k} &= \frac{1}{k!} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \langle e^{jt} | D^{k-r} H_n(x) \rangle \\ &= \frac{2^{k-r} n!}{k!(n-k+r)!} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} H_{n-k+r}(j). \end{aligned} \tag{2.20}$$

Therefore, by (2.15), (2.19), and (2.20), we obtain the following theorem.

Theorem 2.7 *For $n \geq r$, we have*

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)(-1)^{k-j}\binom{k}{j} 2^l H_{n-l}(j)}{(l+r-k)!k!(n-l)!} \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} \frac{(-1)^{r-j}\binom{r}{j} 2^{k-r} H_{n-k+r}(j)}{k!(n-k+r)!} \right\} B_k^{(r)}(x). \end{aligned}$$

Let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k} H_k^{(r)}(x|\lambda). \tag{2.21}$$

Then, by (1.14), (2.7), and (2.21), we get

$$\begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \frac{(\frac{e^{2t}-\lambda}{1-\lambda})^r}{e^{\frac{1}{4}(2t)^2}} (2t)^k \middle| x^n \right\rangle \\ &= \frac{1}{k!} \left\langle \frac{(\frac{e^{t}-\lambda}{1-\lambda})^r}{e^{\frac{1}{4}t^2}} t^k \middle| (2x)^n \right\rangle \\ &= \frac{1}{k!} \left\langle \left(\frac{e^t-\lambda}{1-\lambda} \right)^r t^k \middle| e^{-\frac{1}{4}t^2} (2x)^n \right\rangle. \end{aligned} \tag{2.22}$$

By (2.13) and (2.22), we get

$$\begin{aligned} C_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{e^t-\lambda}{1-\lambda} \right)^r t^k \middle| H_n(x) \right\rangle \\ &= \frac{1}{k!(1-\lambda)^r} \langle (e^t-\lambda)^r | t^k H_n(x) \rangle \end{aligned}$$

$$= \frac{\binom{n}{k}2^k}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\langle e^{jt}|H_{n-k}(x)\rangle$$

$$= \frac{\binom{n}{k}2^k}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}H_{n-k}(j). \tag{2.23}$$

Therefore, by (2.21) and (2.23), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$, we have*

$$H_n(x) = \frac{1}{(1-\lambda)^r}\sum_{k=0}^{n}\binom{n}{k}2^k\left[\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}H_{n-k}(j)\right]H_k^{(r)}(x|\lambda).$$

Remark From (2.22), we have

$$C_{n,k} = \frac{1}{k!}\left\langle \frac{(\frac{e^t-\lambda}{1-\lambda})^r}{e^{\frac{1}{4}t^2}}t^k\middle|(2x)^n\right\rangle = \frac{2^n}{k!}\left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^r e^{-\frac{1}{4}t^2}\middle|t^kx^n\right\rangle$$

$$= \frac{(n)_k}{k!}2^n\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l}{l!4^l}\left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^r\middle|t^{2l}x^{n-k}\right\rangle$$

$$= \frac{\binom{n}{k}2^n}{(1-\lambda)^r}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l}{l!2^{2l}}(n-k)_{2l}\langle(e^t-\lambda)^r|x^{n-k-2l}\rangle$$

$$= \frac{1}{(1-\lambda)^r}\sum_{j=0}^{r}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{\binom{n}{k}\binom{r}{j}2^k(-1)^l(-\lambda)^{r-j}(n-k)!(2j)^{n-k-2l}}{l!(n-k-2l)!}. \tag{2.24}$$

Thus, by (2.21) and (2.24), we get

$$H_n(x) = \frac{1}{(1-\lambda)^r}\sum_{k=0}^{n}\left\{\sum_{j=0}^{r}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{\binom{n}{k}\binom{r}{j}2^k(-1)^l(-\lambda)^{r-j}(n-k)!(2j)^{n-k-2l}}{l!(n-k-2l)!}\right\}H_k^{(r)}(x|\lambda).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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