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Strong convergence of a general iterative algorithm in Hilbert spaces

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Abstract

In this paper, the problem of approximating a common element in the common fixed point set of an infinite family of nonexpansive mappings, in the solution set of a variational inequality involving an inverse-strongly monotone mapping and in the solution set of an equilibrium problem is investigated based on a general iterative algorithm. Strong convergence of the iterative algorithm is obtained in the framework of Hilbert spaces. The results obtained in this paper improve the corresponding results announced by many authors.

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1 Introduction and preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let C be a nonempty, closed and convex subset of H and $T: C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be a κ-contraction iff there exists a constant $\kappa \in (0,1)$ such that

$$\|Tx - Ty\| \le \kappa\|x - y\|, \quad \forall x, y \in C.$$

T is said to be a nonexpansive mapping iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Let $B: C \to H$ be a mapping. Recall that B is said to be an α-inverse-strongly monotone iff there exits a positive constant α such that

$$\langle Bx - By, x - y\rangle \ge \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

The classical variational inequality is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \ge 0, \quad \forall v \in C. \tag{1.1}$$

In this paper, we use $VI(C, B)$ to denote the solution set of the variational inequality.

Let P_C be the metric projection from H onto C. It is also known that P_C satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2, \quad \forall x,y\in H.$$

Moreover, P_Cx is characterized by the properties $P_Cx\in C$ and $\langle x-P_Cx, P_Cx-y\rangle \geq 0$ for all $y\in C$. One can see that the variational inequality is equivalent to a fixed point problem. The element $u\in C$ is a solution of the variational inequality if and only if u is a fixed point of the mapping $P_C(I-\lambda B)$, where $\lambda > 0$ is a constant and I is the identity mapping. This alternative equivalent formulation has played a significant role in the studies of the variational inequality and related optimization problems.

Recall that an operator A is strongly positive on H iff there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax,x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x\in H.$$

Recall that a set-valued mapping $S: H\to 2^H$ is said to be monotone if for all $x,y\in H$, $f\in Sx$ and $g\in Sy$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $S: H\to 2^H$ is maximal if the graph of Graph(S) of S is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping S is maximal iff for $(x,f)\in H\times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g)\in \operatorname{Graph}(S)$ implies $f\in Sx$. Let B be a monotone map of C into H and let N_Cv be the normal cone to C at $v\in C$, *i.e.*, $N_Cv = \{w\in H: \langle v-u, w\rangle \geq 0, \forall u\in C\}$ and define

$$Sv = \begin{cases} Bv + N_Cv, & v\in C, \\ \emptyset, & v\notin C. \end{cases}$$

Then S is maximal monotone and $0\in Sv$ iff $v\in VI(C,B)$; see [1] and the references therein.

Let F be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F: C\times C\to\mathbb{R}$ is to find $x\in C$ such that

$$F(x,y)\geq 0, \quad \forall y\in C. \tag{1.2}$$

The set of solutions of the problem (1.2) is denoted by $EP(F)$. Numerous problems in physics, optimization and economics reduce to finding a solution of (1.2). Recently, many iterative algorithms have been studied to solve the equilibrium problem (1.2); see, for instance, [2–19].

For solving the equilibrium problem (1.2), let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x\in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz+(1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous.

In 2007, Takahashi and Takahashi [17] proved the following result.

Theorem TT *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)-(A4) *and let T be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, & n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{y_n\}$ strongly converge to some point z, where $z = P_C F(T) \cap EP(T) f(z)$.

Recently, Plubtieng and Punpaeng [19] further improved the above results by involving a strongly positive self-adjoint operator. To be more precise, they proved the following results.

Theorem PP *Let H be a real Hilbert space, let F be a bifunction from $H \times H \to R$ satisfying* (A1)-(A4) *and let T be a nonexpansive mapping on H such that $F(T) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive bounded linear operator on H with the coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Ty_n, & n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{y_n\}$ strongly converge to some point z, where $z = P_{F(T)\cap EP(T)}(I - A + \gamma f)(z)$.

In 2008, Su, Shang and Qin [2] considered the variational inequality (1.1), and the equilibrium problem (1.2) based on a composite iterative algorithm and proved the following theorem.

Theorem SSQ *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)-(A4)*. Let A be α-inverse-strongly monotone and let T be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \cap VI(C,A) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)TP_C(I - \lambda_n A)y_n, & n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$, where $0 < a < b < 2\alpha$, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{y_n\}$ strongly converge to some point z, where $z = P_C F(T) \cap EP(T) f(z)$.

The above results only involve a single mapping, we will consider an infinite family of mappings in this paper. To be more precise, we study the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{1.3}$$

where $\{\gamma_1\}, \{\gamma_2\}, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$, $T_1, T_2, \ldots$ are an infinite family of mappings of C into itself.

Considering W_n, we have the following lemmas which are important in proving our main results.

Lemma 1.1 [20] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 1.1, one can define the mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. Throughout this paper, we will assume that $0 < \gamma_n \le b < 1$, where b is some constant.

Lemma 1.2 [20] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

In this paper, based on a general iterative algorithm, we study the problem of approximating a common element in the common fixed point set of an infinite family of nonexpansive mappings, in the solution set of a variational inequality involving an inverse-strongly monotone mapping and in the solution set of an equilibrium problem. Strong convergence of the iterative algorithm is obtained in the framework of Hilbert spaces.

In order to obtain the strong convergence, we need the following tools.

Lemma 1.3 *In Hilbert spaces, the following inequality holds*:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in H.$$

Lemma 1.4 [21] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.5 [22] *Assume* B *is a strong positive linear bounded operator on a Hilbert space* H *with the coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq \|B\|^{-1}$. *Then* $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.

Lemma 1.6 [22] *Let* H *be a Hilbert space. Let* B *be a strongly positive linear bounded self-adjoint operator with the constant* $\bar{\gamma} > 0$ *and* f *be a contraction with the constant* κ. *Assume that* $0 < \gamma < \bar{\gamma}/\kappa$. *Let* T *be a nonexpansive mapping with a fixed point* $x_t \in H$ *of the contraction* $x \mapsto t\gamma f(x) + (I - tB)Tx$. *Then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a fixed point* $\bar{x}$ *of* T, *which solves the variational inequality*

$$\langle (A-\gamma f)\bar{x}, z-\bar{x}\rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\bar{x} = \bar{x}$.

Lemma 1.7 [23, 24] *Let* C *be a nonempty closed convex subset of* H *and let* B *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $r > 0$ *and* $x \in H$. *Then there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : H \to C$ *as follows*:

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:

(1) T_r *is single-valued*;

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 1.8 [25] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* β_n *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n +$

$\beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}\bigl(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\bigr)\leq 0.$$

Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 1.9 [26, 27] *Let* K *be a nonempty closed convex subset of a Hilbert space* H, $\{T_i : C\to C\}$ *be a family of infinitely nonexpansive mappings with* $\bigcap_{i=1}^{\infty}F(T_i)$, $\{\gamma_n\}$ *be a real sequence such that* $0<\gamma_n\leq b<1$ *for each* $n\geq 1$. *If* C *is any bounded subset of* K, *then* $\lim_{n\to\infty}\sup_{x\in C}\|Wx-W_nx\|=0$.

2 Main results

Theorem 2.1 *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* F *be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4). *Let* $\{T_n\}_{n=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* C *into* C. *Let* $B : C\to H$ *be an* α*-inverse-strongly monotone mapping. Let* A *be a strongly positive linear bounded self-adjoint operator on* H *with the coefficient* $\bar{\gamma}>0$. *Assume that* $0<\gamma<\bar{\gamma}/\kappa$ *and* $F:=\bigcap_{i=1}^{\infty}F(T_i)\cap EP(F)\cap VI(C,B)\neq\emptyset$. *Let* $f : C\to H$ *be a* κ*-contraction. Let* $\{x_n\}$ *be a sequence generated in the following iterative process*:

$$\begin{cases} x_1\in H,\\ F(y_n,z)+\frac{1}{r_n}\langle z-y_n,y_n-x_n\rangle\geq 0, \quad \forall z\in C,\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)\beta_n\gamma f(y_n)+(1-\alpha_n)(I-\beta_n A)W_nP_C(I-s_nB)y_n, \quad n\geq 1,\end{cases}$$

where W_n *is generated in* (1.3), $\{\alpha_n\}$, $\{\beta_n\}$ *are real number sequences in* $(0,1)$, $\{r_n\}$ *and* $\{s_n\}$ *are positive real number sequences. Assume that the following restrictions are satisfied*:

(a) $0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1$;

(b) $\lim_{n\to\infty}\beta_n=0$, $\sum_{n=1}^{\infty}\beta_n=\infty$;

(c) $\lim_{n\to\infty}|r_{n+1}-r_n|=0$, $\lim_{n\to\infty}|s_{n+1}-s_n|=0$;

(d) $\liminf_{n\to\infty}r_n>0$, $\{s_n\}\subset[s,s']$ *for some* s, s' *with* $0<s<s'<2\alpha$.

Then $\{x_n\}$ *converges strongly to* $q\in F$, *where* $q=P_F(\gamma f+(I-A))(q)$, *which solves the following variational inequality*:

$$\bigl\langle\gamma f(q)-Aq,p-q\bigr\rangle\leq 0, \quad \forall p\in F.$$

Proof We divide the proof into five steps.

Step 1. Show that the sequence $\{x_n\}$ is bounded.

Notice that $I-s_nB$ is nonexpansive. Indeed, we see from the restriction (d) that

$$\begin{aligned}&\bigl\|(I-s_nB)x-(I-s_nB)y\bigr\|^2\\ &\quad=\bigl\|(x-y)-s_n(Bx-By)\bigr\|^2\\ &\quad=\|x-y\|^2-2s_n\langle x-y,Bx-By\rangle+s_n^2\|Bx-By\|^2\\ &\quad\leq\|x-y\|^2-s_n(2\alpha-s_n)\|Bx-By\|^2\\ &\quad\leq\|x-y\|^2,\end{aligned}$$

which implies the mapping $I - s_nB$ is nonexpansive. Fix $p \in F$. Since $z_n = T_{r_n}x_n$, we have

$$\|y_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|.$$

Put

$$\zeta_n = \beta_n\gamma f(y_n) + (I - \beta_nA)W_n\rho_n,$$

where

$$\rho_n = P_C(I - s_nB)y_n.$$

It follows that

$$\|\rho_n - p\| \leq \|y_n - p\| \leq \|x_n - p\|.$$

Since $\beta_n \to 0$ as $n \to \infty$, we may assume, with no loss of generality, that $\beta_n < \|A\|^{-1}$ for all n. It follows that

$$\begin{aligned}
\|\zeta_n - p\| &= \big\|\beta_n\big(\gamma f(y_n) - Ap\big) + (I - \beta_nA)(W_n\rho_n - p)\big\| \\
&\leq \beta_n\big\|\gamma f(y_n) - Ap\big\| + \|I - \beta_nA\|\|W_n\rho_n - p\| \\
&\leq \beta_n\big[\gamma\,\big\|f(y_n) - f(p)\big\| + \big\|\gamma f(p) - Ap\big\|\big] + (1 - \beta_n\bar{\gamma})\|\rho_n - p\| \\
&\leq \beta_n\big[\gamma\,\big\|f(y_n) - f(p)\big\| + \big\|\gamma f(p) - Ap\big\|\big] + (1 - \beta_n\bar{\gamma})\|x_n - p\| \\
&\leq \big[1 - (\bar{\gamma} - \gamma\kappa)\beta_n\big]\|x_n - p\| + \beta_n\big\|\gamma f(p) - Ap\big\|,
\end{aligned}$$

which yields

$$\begin{aligned}
\|x_{n+1} - p\| &= \big\|\alpha_n(x_n - p) + (1 - \alpha_n)(\zeta_n - p)\big\| \\
&\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|\zeta_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\big[1 - (\bar{\gamma} - \gamma\alpha)\beta_n\big]\|x_n - p\| \\
&\quad + (1 - \alpha_n)\beta_n\big\|\gamma f(p) - Ap\big\|.
\end{aligned}$$

This in turn implies that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\kappa}\right\}.$$

This completes the proof that the sequence $\{x_n\}$ is bounded. This completes the proof of Step 1.

Step 2. Show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

In view of $y_n = T_{r_n}x_n$ and $y_{n+1} = T_{r_{n+1}}x_{n+1}$, we see that

$$F(y_n, z) + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0, \quad \forall z \in C, \tag{2.1}$$

and

$$F(y_{n+1}, z) + \frac{1}{r_{n+1}} \langle z - y_{n+1}, y_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall z \in C. \tag{2.2}$$

Putting $z = y_{n+1}$ in (2.1) and $z = y_n$ in (2.2), we find that

$$F(y_n, y_{n+1}) + \frac{1}{r_n} \langle y_{n+1} - y_n, y_n - x_n \rangle \geq 0$$

and

$$F(y_{n+1}, y_n) + \frac{1}{r_{n+1}} \langle y_n - y_{n+1}, y_{n+1} - x_{n+1} \rangle \geq 0.$$

It follows from (A2) that

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0.$$

That is,

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}} (y_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number m such that $r_n > m > 0$ for all n. It follows that

$$\|y_{n+1} - y_n\|^2 \leq \|y_{n+1} - y_n\| \left(\|x_{n+1} - x_n\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|y_{n+1} - x_{n+1}\| \right).$$

It follows that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|y_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{M_1}{m} |r_{n+1} - r_n|, \end{aligned} \tag{2.3}$$

where M_1 is some real constant such that $M_1 \geq \sup_{n\geq 1}\{\|y_n - x_n\|\}$.

On the other hand, we have

$$\begin{aligned} \|\rho_{n+1} - \rho_n\| &= \big\| P_C(I - s_{n+1}B)y_{n+1} - P_C(I - s_nB)y_n \big\| \\ &\leq \big\| (I - s_{n+1}B)y_{n+1} - (I - s_nB)y_n \big\| \\ &= \big\| (I - s_{n+1}B)y_{n+1} - (I - s_{n+1}B)y_n + (s_n - s_{n+1})By_n \big\| \\ &\leq \|y_{n+1} - y_n\| + |s_n - s_{n+1}| M_2, \end{aligned} \tag{2.4}$$

where $M_2 \geq \sup_{n\geq 1}\{\|By_n\|\}$. Substituting (2.3) into (2.4) yields

$$\|\rho_{n+1} - \rho_n\| \leq \|x_{n+1} - x_n\| + M_3 \big(|r_{n+1} - r_n| + |s_n - s_{n+1}| \big), \tag{2.5}$$

where $M_3 = \max\{M_1, M_2\}$. Notice that

$$\begin{aligned}
\|\zeta_n - \zeta_{n+1}\| &= \big\|(I - \beta_{n+1}A)(W_{n+1}\rho_{n+1} - W_n\rho_n) - (\beta_{n+1} - \beta_n)AW_n\rho_n \\
&\quad + \gamma\big[\beta_{n+1}\big(f(y_{n+1}) - f(z_n)\big) + f(y_n)(\beta_{n+1} - \beta_n)\big]\big\| \\
&\le (1 - \beta_{n+1}\bar{\gamma})\big(\|\rho_{n+1} - \rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\|\big) \\
&\quad + |\beta_{n+1} - \beta_n|\|AW_n\rho_n\| + \gamma\big(\beta_{n+1}\kappa\|y_{n+1} - y_n\| + |\beta_{n+1} - \beta_n|\big\|f(y_n)\big\|\big). \qquad (2.6)
\end{aligned}$$

Since T_i and $U_{n,i}$ are nonexpansive, we see from (1.3) that

$$\begin{aligned}
\|W_{n+1}\rho_n - W_n\rho_n\| &= \|\gamma_1 T_1 U_{n+1,2}\rho_n - \gamma_1 T_1 U_{n,2}\rho_n\| \\
&\le \gamma_1\|U_{n+1,2}\rho_n - U_{n,2}\rho_n\| \\
&= \gamma_1\|\gamma_2 T_2 U_{u+1,3}\rho_n - \gamma_2 T_2 U_{n,3}\rho_n\| \\
&\le \gamma_1\gamma_2\|U_{u+1,3}\rho_n - U_{n,3}\rho_n\| \\
&\le \cdots \\
&\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\
&\le M_4\prod_{i=1}^{n}\gamma_i, \qquad (2.7)
\end{aligned}$$

where M_4 is a constant such that $M_4 \ge \sup_{n\ge 1}\{\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\|\}$. Substituting (2.3), (2.5) and (2.7) into (2.6) yields

$$\|\zeta_n - \zeta_{n+1}\| \le \|x_{n+1} - x_n\| + M_5\left(\prod_{i=1}^{n}\gamma_i + |r_{n+1} - r_n| + |s_n - s_{n+1}| + |\beta_n - \beta_{n+1}|\right),$$

where M_5 is a constant such that

$$M_5 = \max\left\{M_3 + \bar{\gamma}\frac{M_1}{m}, M_4, \sup_{n\ge 1}\big\{\|AW_n\rho_n\| + \gamma\big\|f(y_n)\big\|\big\}\right\}.$$

It follows from the restrictions (b) and (c) that

$$\limsup_{n\to\infty}\big(\|\zeta_n - \zeta_{n+1}\| - \|x_{n+1} - x_n\|\big) \le 0.$$

By virtue of Lemma 1.8, we obtain that

$$\lim_{n\to\infty}\|\zeta_n - x_n\| = 0. \qquad (2.8)$$

On the other hand, we have

$$\|x_{n+1} - x_n\| = (1 - \alpha_n)\|x_n - \zeta_n\|.$$

This implies from (2.8) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (2.9)$$

This completes the proof of Step 2.

Step 3. Show that $\lim_{n\to\infty}\|y_n - Wy_n\| = 0$.

Notice that $\zeta_n = \beta_n\gamma f(z_n) + (I - \beta_n A)W_n\rho_n$. It follows that

$$\|\zeta_n - W_n\rho_n\| = \beta_n\big\|\gamma f(y_n) - AW_n\rho_n\big\|.$$

This implies from the restriction (b) that

$$\lim_{n\to\infty}\|\zeta_n - W_n\rho_n\| = 0. \tag{2.10}$$

For any $p \in F$, we find that

$$\begin{aligned}
\|y_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\
&= \langle y_n - p, x_n - p\rangle \\
&= 1/2\big(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2\big).
\end{aligned}$$

That is,

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2.$$

This in turn implies that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad= \big\|\alpha_n(x_n - p) + (1-\alpha_n)(\zeta_n - p)\big\|^2 \\
&\quad\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\zeta_n - p\|^2 \\
&\quad= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big\|\beta_n\big(\gamma f(y_n) - Ap\big) + (I - \beta_n A)(W_n\rho_n - p)\big\|^2 \\
&\quad\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big(\beta_n\big\|\gamma f(y_n) - Ap\big\| + (1-\beta_n\bar{\gamma})\|\rho_n - p\|\big)^2 \\
&\quad\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|\rho_n - p\|^2 \\
&\qquad\quad + 2(1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\| \\
&\quad\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|y_n - p\|^2 \\
&\qquad\quad + 2(1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\| \\
&\quad\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\qquad\quad - (1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - y_n\|^2 + 2(1-\alpha_n)\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\|,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - y_n\|^2 \\
&\quad\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\big\|\gamma f(y_n) - Ap\big\|^2 \\
&\qquad\quad + 2\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\|
\end{aligned}$$

$$\begin{aligned}&\le \big(\|x_n-p\|-\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+\beta_n\big\|\gamma f(y_n)-Ap\big\|^2\\&\quad+2\beta_n\big\|\gamma f(y_n)-Ap\big\|\|\rho_n-p\|.\end{aligned}$$

It follows from the restriction (b) and (2.9) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{2.11}$$

Notice that

$$\begin{aligned}\|\rho_n-p\|^2 &= \big\|P_C(I-s_nB)y_n-P_C(I-s_nB)p\big\|^2\\&\le \big\|(y_n-p)-s_n(By_n-Bp)\big\|^2\\&= \|y_n-p\|^2-2s_n\langle y_n-p,By_n-Bp\rangle+s_n^2\|By_n-Bp\|^2\\&\le \|x_n-p\|^2-2s_n\alpha\|By_n-Bp\|^2+s_n^2\|By_n-Bp\|^2\\&\le \|x_n-p\|^2-s_n(2\alpha-s_n)\|By_n-Bp\|^2.\end{aligned} \tag{2.12}$$

On the other hand, we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad=\big\|\alpha_n(x_n-p)+(1-\alpha_n)(\zeta_n-p)\big\|^2\\&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|\zeta_n-p\|^2\\&\quad=\alpha_n\|x_n-p\|^2+(1-\alpha_n)\big\|\beta_n\big(\gamma f(y_n)-Ap\big)+(I-\beta_nA)(W_n\rho_n-p)\big\|^2\\&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\big(\beta_n\big\|\gamma f(y_n)-Ap\big\|+\|I-\beta_nA\|\|W_n\rho_n-p\|\big)^2\\&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\big(\beta_n\big\|\gamma f(y_n)-Ap\big\|+(1-\beta_n\bar{\gamma})\|\rho_n-p\|\big)^2\\&\quad\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\big\|\gamma f(y_n)-Ap\big\|^2+(1-\alpha_n)\|\rho_n-p\|^2\\&\qquad+2(1-\alpha_n)\beta_n\big\|\gamma f(y_n)-Ap\big\|\|\rho_n-p\|.\end{aligned} \tag{2.13}$$

Substituting (2.12) into (2.13), we find that

$$\begin{aligned}\|x_{n+1}-p\|^2\le{}& \|x_n-p\|^2+\beta_n\big\|\gamma f(y_n)-Ap\big\|^2+2\beta_n\big\|\gamma f(y_n)-Ap\big\|\|\rho_n-p\|\\&-(1-\alpha_n)s_n(2\alpha-s_n)\|By_n-Bp\|^2.\end{aligned}$$

This in turn implies that

$$\begin{aligned}&(1-\alpha_n)s_n(2\alpha-s_n)\|By_n-Bp\|^2\\&\quad\le \|x_n-p\|^2+\beta_n\big\|\gamma f(y_n)-Ap\big\|^2-\|x_{n+1}-p\|^2\\&\qquad+2\beta_n\big\|\gamma f(y_n)-Ap\big\|\|\rho_n-p\|\\&\quad\le \big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+\beta_n\big\|\gamma f(y_n)-Ap\big\|^2\\&\qquad+2\beta_n\big\|\gamma f(y_n)-Ap\big\|\|\rho_n-p\|.\end{aligned}$$

It follows from the restrictions (a), (b) and (d) that

$$\lim_{n\to\infty} \|By_n - Bp\| = 0. \tag{2.14}$$

On the other hand, we have

$$\begin{aligned}
\|\rho_n - p\|^2 &= \big\|P_C(I - s_nB)y_n - P_C(I - s_nB)p\big\|^2 \\
&\leq \big\langle (I - s_nB)y_n - (I - s_nB)p, \rho_n - p\big\rangle \\
&= \frac{1}{2}\big(\big\|(I - s_nB)y_n - (I - s_nB)p\big\|^2 + \|\rho_n - p\|^2 \\
&\quad - \big\|(I - s_nB)y_n - (I - s_nB)p - (\rho_n - p)\big\|^2\big) \\
&\leq \frac{1}{2}\big(\|y_n - p\|^2 + \|\rho_n - p\|^2 - \big\|(y_n - \rho_n) - s_n(By_n - Bp)\big\|^2\big) \\
&= \frac{1}{2}\big(\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|y_n - \rho_n\|^2 - s_n^2\|By_n - Bp\|^2 \\
&\quad + 2s_n\|y_n - \rho_n\|By_n - Bp\|\big),
\end{aligned}$$

which yields

$$\|\rho_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - \rho_n\|^2 + 2s_n\|y_n - \rho_n\|\|By_n - Bp\|. \tag{2.15}$$

Substituting (2.15) into (2.13) yields

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + \beta_n\big\|\gamma f(y_n) - Ap\big\|^2 - (1 - \alpha_n)\|y_n - \rho_n\|^2 \\
&\quad + 2s_n\|y_n - \rho_n\|\|By_n - Bp\| + 2\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
(1 - \alpha_n)\|y_n - \rho_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\big\|\gamma f(y_n) - Ap\big\|^2 \\
&\quad + 2s_n\|y_n - \rho_n\|\|By_n - Bp\| + 2\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\| \\
&\leq \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + \beta_n\big\|\gamma f(y_n) - Ap\big\|^2 \\
&\quad + 2s_n\|y_n - \rho_n\|\|By_n - Bp\| + 2\beta_n\big\|\gamma f(y_n) - Ap\big\|\|\rho_n - p\|.
\end{aligned}$$

In view of the restrictions (a), (b) and (d), we find from (2.9) that

$$\lim_{n\to\infty} \|y_n - \rho_n\| = 0. \tag{2.16}$$

Notice that

$$\begin{aligned}
\|y_n - W_ny_n\| &\leq \|W_ny_n - W_n\rho_n\| + \|W_n\rho_n - \zeta_n\| + \|\zeta_n - x_n\| + \|x_n - y_n\| \\
&\leq \|y_n - \rho_n\| + \|W_n\rho_n - \zeta_n\| + \|\zeta_n - x_n\| + \|x_n - y_n\|.
\end{aligned}$$

In the light of (2.8), (2.10), (2.11) and (2.16), we find that $\lim_{n\to\infty}\|y_n - W_n y_n\| = 0$. On the other hand, we have

$$\|Wy_n - y_n\| \le \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\|.$$

It follows from Lemma 1.9 that

$$\lim_{n\to\infty}\|y_n - Wy_n\| = 0. \tag{2.17}$$

This completes the proof of Step 3.

Step 4. Show that $\limsup_{n\to\infty}\langle\gamma f(q) - Aq, x_n - q\rangle \le 0$, where $q = P_F(\gamma f + (I - A))(q)$.

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\bigl\langle\gamma f(q) - Aq, x_n - q\bigr\rangle = \lim_{i\to\infty}\bigl\langle\gamma f(q) - Aq, x_{n_i} - q\bigr\rangle. \tag{2.18}$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$. Since $y_n = T_{r_n}x_n$, we have

$$F(y_n, z) + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \ge 0, \quad \forall z \in C.$$

It follows from (A2) that

$$\left\langle z - y_n, \frac{y_n - x_n}{r_n}\right\rangle \ge F(z, y_n).$$

It follows that

$$\left\langle z - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge F(z, y_{n_i}).$$

In view of the restriction (c), we obtain from (2.11) that

$$\lim_{n\to\infty}\frac{y_n - x_n}{r_n} = 0.$$

Since $y_{n_i} \rightharpoonup w$, we have from (A4) that $F(z, w) \le 0$ for all $z \in C$. For t with $0 < t \le 1$ and $z \in C$, let $z_t = tz + (1 - t)w$. Since $z \in C$ and $w \in C$, we have $z_t \in C$ and hence $F(z_t, w) \le 0$. So, from (A1) and (A4), we have

$$0 = F(z_t, z_t) \le tF(z_t, z) + (1 - t)F(z_t, w) \le tF(z_t, z).$$

That is, $F(z_t, z) \ge 0$. It follows from (A3) that $F(w, z) \ge 0$ for all $z \in C$ and hence $w \in EP(F)$. On the other hand, we see that $w \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. If $w \ne Ww$, then we have the following. Since Hilbert spaces are *Opial's* spaces, we find from (2.17) that

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Ww\| \\ &= \liminf_{i\to\infty}\|y_{n_i} - Wy_{n_i} + Wy_{n_i} - Ww\|\end{aligned}$$

$$\begin{aligned}&\le \liminf_{i\to\infty} \| Wy_{n_i} - Ww\| \\ &\le \liminf_{i\to\infty} \|y_{n_i} - w\|,\end{aligned}$$

which derives a contradiction. Thus, we have $w \in F(W)$. Next, let us first show that $w \in VI(C,B)$. Put

$$S\xi = \begin{cases} B\xi + N_C\xi, & \xi \in C, \\ \emptyset, & \xi \overline{\in} C. \end{cases}$$

Since B is monotone, we see that S is maximal monotone. Let $(\xi,\xi') \in \mathrm{Graph}(S)$. Since $\xi' - B\xi \in N_C\xi$ and $\rho_n \in C$, we have

$$\langle \xi - \rho_n, \xi' - B\xi\rangle \ge 0.$$

On the other hand, we have from $\rho_n = P_C(I - s_nB)y_n$ that

$$\langle \xi - \rho_n, \rho_n - (I - s_nB)y_n\rangle \ge 0.$$

That is,

$$\left\langle \xi - \rho_n, \frac{\rho_n - y_n}{s_n} + By_n\right\rangle \ge 0.$$

It follows from the above that

$$\begin{aligned}\langle \xi - \rho_{n_i}, \xi'\rangle &\ge \langle \xi - \rho_{n_i}, B\xi\rangle \\ &\ge \left\langle \xi - \rho_{n_i}, B\xi - \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}} - By_{n_i}\right\rangle \\ &= \langle \xi - \rho_{n_i}, B\xi - B\rho_{n_i}\rangle + \langle \xi - \rho_{n_i}, B\rho_{n_i} - By_{n_i}\rangle \\ &\quad - \left\langle \xi - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}}\right\rangle \\ &\ge \langle \xi - \rho_{n_i}, B\rho_{n_i} - By_{n_i}\rangle - \left\langle \xi - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{s_{n_i}}\right\rangle,\end{aligned}$$

which implies from (2.16) that $\langle \xi - w, \xi'\rangle \ge 0$. We have $w \in S^{-1}0$ and hence $w \in VI(C,B)$. This completes the proof $w \in F$. On the other hand, we find from (2.18) that

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle &= \lim_{n\to\infty}\langle \gamma f(q) - Aq, x_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, w - q\rangle \le 0. \end{aligned} \tag{2.19}$$

This completes the proof of Step 4.

Step 5. Show $\lim_{n\to\infty} \|x_n - q\| = 0$.

It follows from Lemma 1.3 that

$$\begin{aligned}
\|\zeta_n - q\|^2 &= \big\|(I-\beta_n A)(W_n\rho_n - q) + \beta_n\big(\gamma f(y_n) - Aq\big)\big\|^2 \\
&\le \big\|(I-\beta_n A)(W_n\rho_n - q)\big\|^2 + 2\beta_n\langle\gamma f(z_n) - Aq, \zeta_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|\rho_n - q\|^2 + 2\beta_n\langle\gamma f(y_n) - Aq, \zeta_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|y_n - q\|^2 + 2\beta_n\gamma\langle f(y_n) - f(q), \zeta_n - q\rangle \\
&\quad + 2\beta_n\langle\gamma f(q) - Aq, \zeta_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + 2\beta_n\gamma\kappa\|y_n - q\|\|\zeta_n - q\| \\
&\quad + 2\beta_n\langle\gamma f(q) - Aq, \zeta_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\kappa\big(\|y_n - q\|^2 + \|\zeta_n - q\|^2\big) \\
&\quad + 2\beta_n\langle\gamma f(q) - Aq, \zeta_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\kappa\big(\|x_n - q\|^2 + \|\zeta_n - q\|^2\big) \\
&\quad + 2\beta_n\langle\gamma f(q) - Aq, \zeta_n - q\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\zeta_n - q\|^2 &\le \frac{(1-\beta_n\bar{\gamma})^2 + \beta_n\gamma\kappa}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1-\beta_n\gamma\kappa}\langle\gamma f(q) - Aq, \zeta_n - q\rangle \\
&= \frac{(1-2\beta_n\bar{\gamma} + \beta_n\kappa\gamma)}{1-\beta_n\gamma\kappa}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\kappa}\|x_n - q\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n\gamma\kappa}\langle\gamma f(q) - Aq, \zeta_n - q\rangle \\
&\le \left(1 - \frac{2\beta_n(\bar{\gamma} - \kappa\gamma)}{1-\beta_n\gamma\kappa}\right)\|x_n - q\|^2 \\
&\quad + \frac{2\beta_n(\bar{\gamma} - \kappa\gamma)}{1-\beta_n\gamma\kappa}\left(\frac{1}{\bar{\gamma} - \kappa\gamma}\langle\gamma f(q) - Aq, \zeta_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \kappa\gamma)}M_6\right),
\end{aligned} \tag{2.20}$$

where M_6 is a constant such that $M_6 \ge \sup_{n\ge1}\{\|x_n - q\|^2\}$. On the other hand, we have

$$\|x_{n+1} - p\|^2 \le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\zeta_n - p\|^2. \tag{2.21}$$

Substituting (2.20) into (2.21) yields

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \left[1 - (1-\alpha_n)\frac{2\beta_n(\bar{\gamma} - \kappa\gamma)}{1-\beta_n\gamma\kappa}\right]\|x_n - q\|^2 \\
&\quad + (1-\alpha_n)\frac{2\beta_n(\bar{\gamma} - \kappa\gamma)}{1-\beta_n\gamma\alpha} \\
&\quad \times\left(\frac{1}{\bar{\gamma} - \kappa\gamma}\langle\gamma f(q) - Aq, \zeta_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \kappa\gamma)}M_6\right).
\end{aligned} \tag{2.22}$$

Let $\lambda_n = (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\kappa\gamma)}{1-\beta_n\kappa\gamma}$ and

$$\theta_n = \frac{1}{\bar{\gamma}-\kappa\gamma}\langle \gamma f(q) - Aq, \zeta_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\kappa\gamma)}M_6.$$

This implies that

$$\|x_{n+1} - q\|^2 \le (1-\lambda_n)\|x_n - q\|^2 + \lambda_n t_n. \tag{2.23}$$

In view of the restriction (b), we find from (2.8) and (2.11) that

$$\lim_{n\to\infty} \lambda_n = 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \theta_n \le 0.$$

We can easily draw the desired conclusion with the aid of Lemma 1.4. This completes the proof of Step 5. The proof is completed. □

From Theorem 2.1, we have the following results.

Corollary 2.2 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C. Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Let A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\kappa$ and $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,B) \neq \emptyset$. Let $f : C \to H$ be a κ-contraction. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\beta_n \gamma f(y_n) + (1-\alpha_n)(I - \beta_n A)W_n P_C(I - s_n B)P_C x_n, \quad n \ge 1, \end{cases}$$

where W_n is generated in (1.3), $\{\alpha_n\}$, $\{\beta_n\}$ *are real number sequences in* $(0,1)$, $\{r_n\}$ *and* $\{s_n\}$ *are positive real number sequences. Assume that the following restrictions are satisfied*:
(a) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(b) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(c) $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(d) $\{s_n\} \subset [s, s']$ *for some* s, s' *with* $0 < s < s' < 2\alpha$.
Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q\rangle \le 0, \quad \forall p \in F.$$

Proof Putting $F(x,y) = 0$ and $r_n = 1$, we can immediately draw the desired conclusion from Theorem 2.1. □

Corollary 2.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$, which satisfies* (A1)-(A4). *Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Let A be a strongly positive linear bounded self-adjoint operator on H with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\kappa$ and $F := EP(F) \cap VI(C,B) \neq \emptyset$. Let*

$f : C \to H$ be a κ-contraction. Let $\{x_n\}$ be a sequence generated in the following iterative process:

$$\begin{cases} x_1 \in H, \\ F(y_n, z) + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\beta_n \gamma u + (1-\alpha_n)(I - \beta_n A)P_C(I - s_n B)y_n, \quad n \geq 1, \end{cases}$$

where u is a fixed element in C, $\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ and $\{s_n\}$ are positive real number sequences. Assume that the following restrictions are satisfied:

(a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(b) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(c) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(d) $\liminf_{n\to\infty} r_n > 0$, $\{s_n\} \subset [s, s']$ *for some s, s' with* $0 < s < s' < 2\alpha$.

Then, $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma u + (q - Aq))$, which solves the following variational inequality:

$$\langle \gamma u - Aq, p - q\rangle \leq 0, \quad \forall p \in F.$$

Proof Putting $T_i = I$, where I is the identity mapping and $f(x) = u$, for all $x \in C$, we can immediately draw the desired conclusion from Theorem 2.1. □

Corollary 2.4 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C. Let $B : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(C,B) \neq \emptyset$. Let $f : C \to H$ be a κ-contraction. Let $\{x_n\}$ be a sequence generated in the following iterative process*:

$$\begin{cases} x_1 \in H, \\ F(y_n, z) + \frac{1}{r_n}\langle z - y_n, y_n - x_n\rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\beta_n f(y_n) + (1-\alpha_n)(1-\beta_n)W_n P_C(I - s_n B)y_n, \quad n \geq 1, \end{cases}$$

where W_n is generated in (1.3), *$\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ and $\{s_n\}$ are positive real number sequences. Assume that the following restrictions are satisfied*:

(a) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(b) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(c) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$;
(d) $\liminf_{n\to\infty} r_n > 0$, $\{s_n\} \subset [s, s']$ *for some s, s' with* $0 < s < s' < 2\alpha$.

Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F f(q)$, which solves the following variational inequality:

$$\langle f(q) - q, p - q\rangle \leq 0, \quad \forall p \in F.$$

Proof Putting $A = I$, where I is the identity mapping and $\gamma = 1$, we can immediately draw the desired conclusion from Theorem 2.1. □

Competing interests
The author declares that they have no competing interests.

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