Adiyasuren and Batbold *Journal of Inequalities and Applications* 2013, **2013**:189
http://www.journalofinequalitiesandapplications.com/content/2013/1/189

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On a generalization of a Hilbert-type integral inequality in the whole plane with a hypergeometric function

Vandanjav Adiyasuren[1] and Tserendorj Batbold[2*]

*Correspondence: tsbatbold@hotmail.com
[2]Institute of Mathematics, National University of Mongolia, P.O. Box 46A/104, Ulaanbaatar, 14201, Mongolia
Full list of author information is available at the end of the article

Abstract

By introducing some parameters, we establish a generalization of the Hilbert-type integral inequality in the whole plane with the homogeneous kernel of degree -2λ and the best constant factor which involves the hypergeometric function.

MSC: 26D15

Keywords: Hilbert's inequality; Hölder's inequality; homogeneous kernel; weight function; equivalent form

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x) \geq 0$ satisfy

$$0 < \int_0^\infty f^p(x)\,dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)\,dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)} \left\{\int_0^\infty f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\pi/(\sin\pi/p)$ is the best possible. Inequality (1) is called Hardy-Hilbert's inequality [1] and is important in analysis and applications [2].

In 2001, Yang gave an extension of (1) involving beta function as (see [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy$$
$$< B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left\{\int_0^\infty x^{1-\lambda} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{1-\lambda} g^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ $(\lambda > 2 - \min\{p,q\})$ is the best possible.

Recently, some new Hilbert-type inequalities in the whole plane have been obtained [4, 5]. Xin and Yang in [5] established the following:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $|\beta| < 1$, $0 < \alpha_1 < \alpha_2 < \pi$, $f, g \geq 0$, satisfy

$$0 < \int_{-\infty}^\infty |x|^{-p\beta-1} f^p(x)\,dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^\infty |y|^{q\beta-1} g^q(y)\,dy < \infty,$$

then we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)g(y)\,dx\,dy\\&\quad<k(\beta)\left(\int_{-\infty}^{\infty}|x|^{-p\beta-1}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy\right)^{\frac{1}{q}},\end{aligned}\tag{3}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty}|y|^{p(1-\beta)-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{x^2+2xy\cos\alpha_i+y^2}\right\}f(x)\,dx\right)^p dy\\&\quad<k^p(\beta)\int_{-\infty}^{\infty}|x|^{-p\beta-1}f^p(x)\,dx,\end{aligned}\tag{4}$$

where the constant factors $k(\beta)=\frac{\pi}{\sin\beta\pi}(\frac{\sin\beta\alpha_1}{\sin\alpha_1}+\frac{\sin\beta(\pi-\alpha_2)}{\sin\alpha_2})$ and $k^p(\beta)$ are the best possible. Inequalities (3) and (4) are equivalent.

By introducing some parameters, we establish generalizations of inequalities (3) and (4) with the homogeneous kernel of degree -2λ and the best constant factor which involves the hypergeometric function.

2 Preliminary lemmas

In order to prove our assertions, we need the following lemmas.

Recall that the hypergeometric function $F(\alpha,\beta;\gamma;x)$ is defined [6] by

$$F(\alpha,\beta;\gamma;x)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}\frac{x^r}{r!},\tag{5}$$

where $(\alpha)_r$ is the Pochhammer symbol defined by

$$(\alpha)_r=\alpha(\alpha+1)\cdots(\alpha+r-1)=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}.$$

It is known the series (5) converges for $|x|<1$ and diverges for $|x|>1$. The hypergeometric function satisfies the integral representation

$$F(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt,\quad \text{if } \gamma>\beta>0.$$

Lemma 2.1 (See [7]) *Suppose that* $a,c>0$, $b^2<ac$, $0<\alpha<2\lambda$. *Then we have*

$$\int_0^{\infty}\frac{x^{\alpha-1}}{(ax^2+2bx+c)^{\lambda}}\,dx=a^{-\frac{\alpha}{2}}c^{\frac{\alpha}{2}-\lambda}B(\alpha,2\lambda-\alpha)F\left(\frac{\alpha}{2},\lambda-\frac{\alpha}{2};\lambda+\frac{1}{2};1-\frac{b^2}{ac}\right).$$

Lemma 2.2 *Let* $a,c,\lambda>0$, $b\geq 0$, $1-2\lambda<\beta<1$ *and* $0<\alpha_1<\alpha_2<\pi$ *be real parameters such that* $b^2\max\{\cos^2\alpha_1,\cos^2(\pi-\alpha_2)\}<ac$. *Define the weight functions* $\omega(x)$ *and* $\varpi(y)$ $(x,y\in(-\infty,\infty))$ *as follows*:

$$\omega(x):=\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^{\lambda}}\right\}\frac{|x|^{\beta+2\lambda-1}}{|y|^{\beta}}\,dy,$$

$$\varpi(y) := \int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{(ax^2 + 2bxy\cos\alpha_i + cy^2)^\lambda} \right\} \frac{|y|^{1-\beta}}{|x|^{-\beta-2\lambda+2}}\, dx.$$

Then we have $\omega(x) = \varpi(y) = C_\lambda$ $(x, y \neq 0)$, *where*

$$\begin{aligned} C_\lambda &= a^{\frac{1-\beta}{2}-\lambda} c^{\frac{1-\beta}{2}} B(1-\beta, 2\lambda+\beta-1) \\ &\quad\times \left[F\left(\frac{1-\beta}{2}, \lambda - \frac{1-\beta}{2}; \lambda + \frac{1}{2}; 1 - \frac{b^2\cos^2\alpha_1}{ac} \right) \right. \\ &\quad \left. + F\left(\frac{1-\beta}{2}, \lambda - \frac{1-\beta}{2}; \lambda + \frac{1}{2}; 1 - \frac{b^2\cos^2(\pi-\alpha_2)}{ac} \right) \right]. \end{aligned}$$

Proof For $x \in (-\infty, 0)$, setting $u = y/x$, $u = -y/x$ in the following two integrals, respectively, and using Lemma 2.1, we get

$$\begin{aligned} \omega(x) &= \int_{-\infty}^{0} \frac{1}{(ax^2 + 2bxy\cos\alpha_1 + cy^2)^\lambda} \frac{(-x)^{\beta+2\lambda-1}}{(-y)^\beta}\, dy \\ &\quad + \int_{0}^{\infty} \frac{1}{(ax^2 + 2bxy\cos\alpha_2 + cy^2)^\lambda} \frac{(-x)^{\beta+2\lambda-1}}{y^\beta}\, dy \\ &= \int_0^\infty \frac{u^{-\beta}}{(cu^2 + 2bu\cos\alpha_1 + a)^\lambda}\, du + \int_0^\infty \frac{u^{-\beta}}{(cu^2 + 2bu\cos(\pi-\alpha_2) + a)^\lambda}\, du \\ &= C_\lambda. \end{aligned}$$

For $x \in (0, \infty)$, setting $u = -y/x$, $u = y/x$ in the following two integrals, respectively, and using Lemma 2.1, we get

$$\begin{aligned} \omega(x) &= \int_{-\infty}^{0} \frac{1}{(ax^2 + 2bxy\cos\alpha_2 + cy^2)^\lambda} \frac{x^{\beta+2\lambda-1}}{(-y)^\beta}\, dy \\ &\quad + \int_{0}^{\infty} \frac{1}{(ax^2 + 2bxy\cos\alpha_1 + cy^2)^\lambda} \frac{x^{\beta+2\lambda-1}}{y^\beta}\, dy \\ &= \int_0^\infty \frac{u^{-\beta}}{(cu^2 + 2bu\cos(\pi-\alpha_2) + a)^\lambda}\, du + \int_0^\infty \frac{u^{-\beta}}{(cu^2 + 2bu\cos\alpha_1 + a)^\lambda}\, du \\ &= C_\lambda. \end{aligned}$$

By the same way, we still can find that $\omega(x) = \varpi(y) = C_\lambda$ $(x, y \neq 0)$. The lemma is proved. □

Lemma 2.3 *Let p and q be conjugate parameters with* $p > 1$, *and let* $a, c, \lambda > 0$, $b \geq 0$, $1 - 2\lambda < \beta < 1$, $0 < \alpha_1 < \alpha_2 < \pi$ *and* $b^2 \max\{\cos^2\alpha_1, \cos^2(\pi-\alpha_2)\} < ac$, *and* $f(x)$ *be a nonnegative measurable function in* $(-\infty, \infty)$, *then we have*

$$\begin{aligned} J &:= \int_{-\infty}^{\infty} |y|^{p(1-\beta)-1} \left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}} \left\{ \frac{1}{(ax^2 + 2bxy\cos\alpha_i + cy^2)^\lambda} \right\} f(x)\, dx \right)^p dy \\ &\leq C_\lambda^p \int_{-\infty}^{\infty} |x|^{-p(\beta+2\lambda-2)-1} f^p(x)\, dx. \end{aligned} \tag{6}$$

Proof By Lemma 2.2 and Hölder's inequality [8], we have

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}f(x)\,dx\right)^p \\
&\quad= \left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\right. \\
&\qquad\left.\times\left(\frac{|x|^{(-\beta-2\lambda+2)/q}}{|y|^{\beta/p}}f(x)\right)\left(\frac{|y|^{\beta/p}}{|x|^{(-\beta-2\lambda+2)/q}}\right)dx\right]^p \\
&\quad\le \int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\frac{|x|^{(1-p)(\beta+2\lambda-2)}}{|y|^{\beta}}f^p(x)\,dx \\
&\qquad\times\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\frac{|y|^{(q-1)\beta}}{|x|^{(-\beta-2\lambda+2)}}\,dx\right)^{p-1} \\
&\quad= C_\lambda^{p-1}|y|^{p(\beta-1)+1}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\frac{|x|^{(1-p)(\beta+2\lambda-2)}}{|y|^{\beta}}f^p(x)\,dx. \qquad (7)
\end{aligned}$$

Then by the Fubini theorem, it follows that

$$\begin{aligned}
J &\le C_\lambda^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\frac{|x|^{(1-p)(\beta+2\lambda-2)}}{|y|^{\beta}}f^p(x)\,dx\right]dy \\
&= C_\lambda^{p-1}\int_{-\infty}^{\infty}\omega(x)|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx \\
&= C_\lambda^{p}\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx.
\end{aligned}$$

The lemma is proved. □

3 Main results

Theorem 3.1 *Let p and q be conjugate parameters with $p>1$, and let $a,c,\lambda>0$, $b\ge 0$, $1-2\lambda<\beta<1$, $0<\alpha_1<\alpha_2<\pi$ and $b^2\max\{\cos^2\alpha_1,\cos^2(\pi-\alpha_2)\}<ac$, and $f,g\ge 0$, satisfy $0<\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx<\infty$ and $0<\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy<\infty$. Then*

$$\begin{aligned}
I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}f(x)g(y)\,dx\,dy \\
&< C_\lambda\left(\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy\right)^{1/q}, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
J &= \int_{-\infty}^{\infty}|y|^{p(1-\beta)-1}\left(\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}f(x)\,dx\right)^p dy \\
&< C_\lambda^{p}\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx, \qquad (9)
\end{aligned}$$

where the constant factors C_λ and C_λ^p are the best possible and C_λ is defined in Lemma 2.2. Inequalities (8) and (9) are equivalent.

Proof If (7) takes the form of the equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then there exist constants A and B such that they are not all zero, and

$$A\frac{|x|^{(1-p)(\beta+2\lambda-2)}}{|y|^{\beta}}f^p(x) = B\frac{|y|^{(q-1)\beta}}{|x|^{(-\beta-2\lambda+2)}} \quad \text{a.e. in } (-\infty,\infty)\times(-\infty,\infty).$$

Hence, there exists a constant K such that

$$A|x|^{p(\beta+2\lambda-2)}f^p(x) = B|y|^{q\beta} = K \quad \text{a.e. in } (-\infty,\infty)\times(-\infty,\infty).$$

We suppose $A \neq 0$ (otherwise $B = A = 0$). Then $|x|^{p(\beta+2\lambda-2)-1}f^p(x) = K/(A|x|)$ a.e. in $(-\infty,\infty)$, which contradicts the fact that $0 < \int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx < \infty$. Hence, (7) takes the form of a strict inequality, so does (6), and we have (9).

By Hölder's inequality [8], we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\left(|y|^{1/q-\beta}\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^{\lambda}}\right\}f(x)\,dx\right) \\ &\quad\times\left(|y|^{\beta-1/q}g(y)\right)dy \\ &\le J^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy\right)^{1/q}. \end{aligned} \tag{10}$$

By (9), we have (8). On the other hand, suppose that (8) is valid. Set

$$g(y) = |y|^{p(1-\beta)-1}\left(\int_{-\infty}^{\infty}\min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^{\lambda}}\right\}f(x)\,dx\right)^{p-1},$$

then it follows $J = \int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy$. By (6), we have $J < \infty$. If $J = 0$, then (9) is obviously valid. If $0 < J < \infty$, then by (8), we obtain

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy = J = I \\ &< C_{\lambda}\left(\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy\right)^{1/q}, \end{aligned}$$

and

$$\begin{aligned} J^{1/p} &= \left(\int_{-\infty}^{\infty}|y|^{q\beta-1}g^q(y)\,dy\right)^{1/p} \\ &< C_{\lambda}\left(\int_{-\infty}^{\infty}|x|^{-p(\beta+2\lambda-2)-1}f^p(x)\,dx\right)^{1/p}. \end{aligned}$$

Hence, we have (9), which is equivalent to (8).

For $\varepsilon > 0$, define functions $\widetilde{f}(x)$, $\widetilde{g}(y)$ as follows:

$$\widetilde{f}(x) := \begin{cases} x^{(\beta+2\lambda-2)-2\varepsilon/p}, & x\in(1,\infty), \\ 0, & x\in[-1,1], \\ (-x)^{(\beta+2\lambda-2)-2\varepsilon/p}, & x\in(-\infty,-1), \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} y^{-\beta-2\varepsilon/q}, & y \in (1,\infty), \\ 0, & y \in [-1,1], \\ (-y)^{-\beta-2\varepsilon/q}, & y \in (-\infty,-1). \end{cases}$$

Then

$$\widetilde{L} := \left(\int_{-\infty}^{\infty} |x|^{-p(\beta+2\lambda-2)-1}\widetilde{f}^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty} |y|^{q\beta-1}\widetilde{g}^q(y)\,dy\right)^{1/q} = \frac{1}{\varepsilon},$$

and

$$\begin{aligned}\widetilde{I} &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min_{i\in\{1,2\}}\left\{\frac{1}{(ax^2+2bxy\cos\alpha_i+cy^2)^\lambda}\right\}\widetilde{f}(x)\widetilde{g}(y)\,dx\,dy \\ &= I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \int_{-\infty}^{-1}(-x)^{(\beta+2\lambda-2)-2\varepsilon/p}\left[\int_{-\infty}^{-1}\frac{(-y)^{-\beta-2\varepsilon/q}}{(ax^2+2bxy\cos\alpha_1+cy^2)^\lambda}\,dy\right]dx, \\ I_2 &:= \int_{-\infty}^{-1}(-x)^{(\beta+2\lambda-2)-2\varepsilon/p}\left[\int_{1}^{\infty}\frac{y^{-\beta-2\varepsilon/q}}{(ax^2+2bxy\cos\alpha_2+cy^2)^\lambda}\,dy\right]dx, \\ I_3 &:= \int_{1}^{\infty}x^{(\beta+2\lambda-2)-2\varepsilon/p}\left[\int_{-\infty}^{-1}\frac{(-y)^{-\beta-2\varepsilon/q}}{(ax^2+2bxy\cos\alpha_2+cy^2)^\lambda}\,dy\right]dx,\end{aligned}$$

and

$$I_4 := \int_{1}^{\infty}x^{(\beta+2\lambda-2)-2\varepsilon/p}\left[\int_{1}^{\infty}\frac{y^{-\beta-2\varepsilon/q}}{(ax^2+2bxy\cos\alpha_1+cy^2)^\lambda}\,dy\right]dx.$$

Taking $u = y/x$, by the Fubini theorem, we obtain

$$\begin{aligned}I_1 = I_4 &= \int_1^\infty x^{-1-2\varepsilon}\int_{1/x}^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du\,dx \\ &= \int_1^\infty x^{-1-2\varepsilon}\left(\int_{1/x}^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du\right)dx \\ &= \int_0^1\left(\int_{1/u}^\infty x^{-1-2\varepsilon}\,dx\right)\frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du \\ &\quad + \frac{1}{2\varepsilon}\int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du \\ &= \frac{1}{2\varepsilon}\left(\int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du\right),\end{aligned}$$

and

$$I_2 = I_3 = \frac{1}{2\varepsilon}\left(\int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du\right).$$

In view of the above results, if the constant factor C_λ in (8) is not the best possible, then there exists a positive number $\widetilde{C}$ with $\widetilde{C} < C_\lambda$ such that

$$\begin{aligned}
&\int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du \\
&\quad + \int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du \\
&\quad = \varepsilon\widetilde{I} < \varepsilon\widetilde{C}\cdot\widetilde{L} = \widetilde{C}.
\end{aligned} \tag{11}$$

By the Fatou lemma and (11), we have

$$\begin{aligned}
C_\lambda &= \int_0^\infty \frac{u^{-\beta}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_0^\infty \frac{u^{-\beta}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du \\
&= \int_0^1 \lim_{\varepsilon\to0^+} \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_1^\infty \lim_{\varepsilon\to0^+} \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du \\
&\quad + \int_0^1 \lim_{\varepsilon\to0^+} \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du + \int_1^\infty \lim_{\varepsilon\to0^+} \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du \\
&\leq \lim_{\varepsilon\to0^+}\left[\int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2+2bu\cos\alpha_1+a)^\lambda}\,du \right.\\
&\quad \left. + \int_0^1 \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du + \int_1^\infty \frac{u^{-\beta-2\varepsilon/q}}{(cu^2-2bu\cos\alpha_2+a)^\lambda}\,du\right] \\
&\leq \widetilde{C},
\end{aligned}$$

which contradicts the fact that $\widetilde{C} < C_\lambda$. Hence, the constant factor C_λ in (8) is the best possible.

If the constant factor in (9) is not the best possible, then by (10), we may get a contradiction that the constant factor in (8) is not the best possible. Thus the theorem is proved. □

Remark 1 Setting $\lambda = a = b = c = 1$ in Theorem 3.1, we have (3) and (4).

Remark 2 Setting $\lambda = 1/2$ in Theorem 3.1, we have the following particular results:

$$\begin{aligned}
&\int_{-\infty}^\infty\int_{-\infty}^\infty \min_{i\in\{1,2\}}\left\{\frac{1}{\sqrt{ax^2+2bxy\cos\alpha_i+cy^2}}\right\} f(x)g(y)\,dx\,dy \\
&\quad < C_{1/2}\left(\int_{-\infty}^\infty |x|^{-p(\beta-1)-1} f^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^\infty |y|^{q\beta-1} g^q(y)\,dy\right)^{1/q},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^\infty |y|^{p(1-\beta)-1}\left(\int_{-\infty}^\infty \min_{i\in\{1,2\}}\left\{\frac{1}{\sqrt{ax^2+2bxy\cos\alpha_i+cy^2}}\right\} f(x)\,dx\right)^p dy \\
&\quad < C_{1/2}^p \int_{-\infty}^\infty |x|^{-p(\beta-1)-1} f^p(x)\,dx.
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematical Analysis, National University of Mongolia, Ulaanbaatar, Mongolia. [2]Institute of Mathematics, National University of Mongolia, P.O. Box 46A/104, Ulaanbaatar, 14201, Mongolia.

Received: 17 May 2012 Accepted: 16 January 2013 Published: 19 April 2013

References
1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, DS, Pečarić, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Boston (1991)
3. Yang, B: On Hardy-Hilbert's integral inequality. J. Math. Anal. Appl. **261**, 295-306 (2001)
4. Zeng, Z, Xie, Z: On a new Hilbert-type integral inequality with the integral in whole plane. J. Inequal. Appl. **2010**, Article ID 256796 (2010)
5. Xin, D, Yang, B: A Hilbert-type integral inequality in the whole plane with the homogeneous kernel of degree −2. J. Inequal. Appl. **2011**, Article ID 401428 (2011)
6. Abramowitz, M, Stegun, IA: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, pp. 807-808. Dover, New York (1972)
7. Azar, LE: Some extension of Hilbert's integral inequality. J. Math. Inequal. **5**(1), 131-140 (2011)
8. Kuang, J: Applied Inequalities. Shangdong Science and Technology Press, Jinan (2004)

doi:10.1186/1029-242X-2013-189
Cite this article as: Adiyasuren and Batbold: **On a generalization of a Hilbert-type integral inequality in the whole plane with a hypergeometric function.** *Journal of Inequalities and Applications* 2013 **2013**:189.