Lee and Lee *Journal of Inequalities and Applications* 2013, **2013**:18
http://www.journalofinequalitiesandapplications.com/content/2013/1/18

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Volume preserving diffeomorphisms with orbital shadowing

Keonhee Lee[1] and Manseob Lee[2*]

[*]Correspondence: lmsds@mokwon.ac.kr
[2]Department of Mathematics, Mokwon University, Daejeon, 302-729, Korea
Full list of author information is available at the end of the article

Abstract

Let f be a volume-preserving diffeomorphism of a closed C^∞ Riemannian manifold M. In this paper, we show that the following are equivalent:

(a) f belongs to the C^1-interior of the set of volume-preserving diffeomorphisms with orbital shadowing,

(b) f is Anosov.

MSC: Primary 37C50; secondary 34D10

Keywords: normally hyperbolic; hyperbolic; orbital shadowing; shadowing; Anosov

1 Introduction

A fundamental problem in differentiable dynamical systems is to understand how a robust dynamic property on the underlying manifold would influence the behavior of the tangent map on the tangent bundle. For instance, in [1], Mañé proved that any C^1 structurally stable diffeomorphism is an Axiom A diffeomorphism. And in [2], Palis extended this result to Ω-stable diffeomorphisms. Let M be a closed C^∞ Riemannian manifold endowed with a volume form ω. Let μ denote the Lebesgue measure associated to ω, and let d denote the metric induced on M by the Riemannian structure. Denote by $\mathrm{Diff}_\mu(M)$ the set of diffeomorphisms which preserves the Lebesgue measure μ endowed with the C^1-topology. We know that every volume preserving diffeomorphism satisfying Axiom A is Anosov (for more details, see [3]).

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ $(-\infty \le a < b \le \infty)$ in M is called a δ-*pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. Let $\Lambda \subset M$ be a closed f-invariant set. We say that f has the *shadowing property* on Λ (or Λ is *shadowable*) if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. It is easy to see that f has the shadowing property on Λ if and only if f^n has the shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$. The notion of pseudo-orbits often appears in several methods of the modern theory of dynamical systems. Moreover, the shadowing property plays an important role in the investigation of stability theory. In fact, Pilyugin [4] and Robinson [5] showed that if a diffeomorphism f is structurally stable, then f has the shadowing property. Moreover, Sakai [6] proved that if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property, then f is structurally stable. For each $x \in M$, let $\mathcal{O}_f(x)$ be the orbit of f through x; that is,

$$\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{Z}\}.$$

We say that f has the *orbital shadowing property* on Λ (or Λ is *orbitally shadowable*) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, we can find a point $y \in M$ such that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \quad \text{and} \quad \xi \subset B_\epsilon\big(\mathcal{O}_f(y)\big),$$

where $B_\epsilon(A)$ denotes the ϵ-neighborhood of a set $A \subset M$. f is said to have the *weak shadowing property* on Λ (or Λ is *weakly shadowable*) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there is a point $y \in M$ such that $\xi \subset B_\epsilon(\mathcal{O}_f(y))$. Note that if f has the shadowing property, then f has the orbital shadowing property, but the converse is not true (see [7]). It is easy to see that f has the orbital shadowing property on Λ if and only if f^n has the orbital shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\big\|D_x f^n|_{E^s_x}\big\| \le C\lambda^n \quad \text{and} \quad \big\|D_x f^{-n}|_{E^u_x}\big\| \le C\lambda^{-n}$$

for all $x \in \Lambda$ and $n \ge 0$.

We denote by $\mathcal{F}_\mu(M)$ the set of diffeomorphisms $f \in \mathrm{Diff}_\mu(M)$ which have a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\mu(M)$ such that for any $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic.

Very recently, Arbieto and Catalan [3] proved that every volume preserving diffeomorphism in $\mathcal{F}_\mu(M)$ is Anosov. To prove this, they used Mañé's results in [1, Proposition II.1] and showed that $\overline{P(f)}$ is hyperbolic if $f \in \mathcal{F}_\mu(M)$. Thus, we have the following theorem.

Theorem 1.1 [3, Theorem 1.1] *Every diffeomorphism in $\mathcal{F}_\mu(M)$ is Anosov.*

Let $\mathrm{int}\,\mathcal{OS}_\mu(M)$ denote the C^1-interior of the set of volume preserving diffeomorphisms in $\mathrm{Diff}_\mu(M)$ satisfying the orbital shadowing property. In [7], the authors proved that the C^1-interior of the set of diffeomorphisms having the orbital shadowing property coincides with the set of structurally stable diffeomorphisms. Note that if a diffeomorphism satisfies structurally stable then it is not Anosov in general. But the converse is true. Finally, we prove the following theorem.

Theorem 1.2 *The set $\mathcal{AN}_\mu(M)$ of Anosov diffeomorphisms in* $\mathrm{Diff}_\mu(M)$ *coincides with the C^1-interior of the set of diffeomorphisms in* $\mathrm{Diff}_\mu(M)$ *with orbital shadowing*; *that is,* $\mathcal{AN}_\mu(M) = \mathrm{int}\,\mathcal{OS}_\mu(M)$.

2 Proof of Theorem 1.2

Remark 2.1 Let $f \in \mathrm{Diff}^1_\mu(M)$. From Moser's theorem (see [8]), we can find a smooth conservative change of coordinates $\varphi_x : U(x) \to T_x M$ such that $\varphi_x(x) = 0$, where $U(x)$ is a small neighborhood of $x \in M$.

Recall that f has the *orbital shadowing property* on Λ (Λ is *orbitally shadowable*) if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f, there is $\in M$ such that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \quad \text{and} \quad \xi \subset B_\epsilon\big(\mathcal{O}_f(y)\big).$$

Notice that in this definition, only δ-pseudo orbits of f are contained in Λ, but the shadowing point $y \in M$ is not necessarily contained in Λ. To prove our result, we use Franks' lemma which is proved in [9, Proposition 7.4].

Lemma 2.2 *Let $f \in \mathrm{Diff}^1_\mu(M)$, and $\mathcal{U}$ be a C^1-neighborhood of f in $\mathrm{Diff}^1_\mu(M)$. Then there exist a C^1-neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\epsilon > 0$ such that if $g \in \mathcal{U}_0$, any finite f-invariant set $E = \{x_1, \dots, x_m\}$, any neighborhood U of E, and any volume-preserving linear maps $L_j : T_{x_j}M \to T_{g(x_j)}M$ with $\|L_j - D_{x_j}g\| \le \epsilon$ for all $j = 1, \dots, m$, there is a conservative diffeomorphism $g_1 \in \mathcal{U}$ coinciding with f on E and out of U, and $D_{x_j}g_1 = L_j$ for all $j = 1, \dots, m$.*

We introduce the notion of normally hyperbolic which was founded in [10]. Let $V \subset M$ be an invariant submanifold of $f \in \mathrm{Diff}_\mu(M)$. We say that V is *normally hyperbolic* if there is a splitting $T_V M = TV \oplus N^s \oplus N^u$ such that

- the splitting depends continuously on $x \in V$,
- $D_x f(N^\sigma_x) = N^\sigma_{f(x)}$ ($\sigma = s, u$) for all $x \in V$,
- there are constants $C > 0$ and $0 < \lambda < 1$ such that for every unit vector $x \in T_x V$, $v^s \in N^s_x$ and $v^u \in N^u_x$ ($x \in V$), we have

$$\|D_x f^n(v^s)\| \le C\lambda^n \|D_x f^n(v)\| \quad \text{and} \quad \|D_x f^n(v^u)\| \ge C^{-1}\lambda^{-1} \|D_x f^n(v)\|$$

for all $n \ge 0$.

Proposition 2.3 *If $f \in \operatorname{int} \mathcal{OS}_\mu(M)$, then every periodic point of f is hyperbolic.*

Proof Take $f \in \operatorname{int} \mathcal{OS}_\mu(M)$, and $\mathcal{U}(f)$ is a C^1-neighborhood of $f \in \mathrm{Diff}_\mu(M)$. Let $\epsilon > 0$ and $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be the number and C^1-neighborhood of f corresponding to $\mathcal{U}(f)$ given by Lemma 2.2. To derive a contradiction, we assume that there exists a nonhyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{V}(f)$. To simplify the notation in the proof, we may assume that $g(p) = p$. Then there is at least one eigenvalue λ of $D_p g$ such that $|\lambda| = 1$.

By making use of Lemma 2.2, we linearize g at p using Moser's theorem; that is, by choosing $\alpha > 0$ sufficiently small, we construct g_1 C^1-nearby g such that

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Then $g_1(p) = g(p) = p$.

First, we may assume that $\lambda \in \mathbb{R}$ with $\lambda = 1$. Let v be the associated non-zero eigenvector such that $\|v\| = \alpha/4$. Then we can get a small arc

$$\mathcal{I}_v = \{tv : -1 \le t \le 1\} \subset \varphi_p(B_\alpha(p)).$$

Take $\epsilon = \alpha/8$. Let $0 < \delta < \epsilon$ be a number of the orbital shadowing property of g_1 corresponding to ϵ. Then by our construction of g_1,

$$\varphi_p^{-1}(\mathcal{I}_v) \subset B_\alpha(p).$$

Then it is clear that $\varphi_p^{-1}(\mathcal{I}_v)$ is normally hyperbolic for g_1. Put $\mathcal{J}_p = \varphi_p^{-1}(\mathcal{I}_v)$. Given a constant $\delta > 0$, we construct a δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \mathcal{J}_p$ as follows. For fixed $k \in \mathbb{Z}$, choose distinct points $x_0 = p, x_1, x_2, \ldots, x_k$ in $\mathcal{J}_p$ such that

(a) $d(x_i, x_{i+1}) < \delta$ for $i = 0, 1, \ldots, k-1$,
(b) $d(x_{-i-1}, x_{-i}) < \delta$ for $i = 0, \ldots, k-1$,
(c) $x_0 = x$ and $d(x_{-k}, x_k) > 2\epsilon$.

Now, we define $\xi = \{x_i\}_{i\in\mathbb{Z}}$ by $x_{ki+j} = x_j$ for $i \in \mathbb{Z}$ and $j = -k-1, -k-2, \ldots, -1, 0, 1, \ldots, k-1$. Since g_1 has the orbital shadowing property, $g_1|_{\mathcal{J}_p}$ must have the orbital shadowing property. Thus, we can find a point $y \in M$ such that $\xi \subset B_\epsilon(\mathcal{O}_{g_1}(y))$, and $\mathcal{O}_{g_1}(y) \subset B_\epsilon(\xi)$. For any $v \in \mathcal{I}_v$, $\varphi_p^{-1}(v) \in \mathcal{J}_p \subset B_\alpha(p)$ and

$$g_1\big(\varphi_p^{-1}(v)\big) = \varphi_p^{-1} \circ D_p g \circ \varphi_p\big(\varphi_p^{-1}(v)\big).$$

Then $g_1(\varphi_p^{-1}(v)) = \varphi_p^{-1}(v)$. Thus, $g_1^l(\mathcal{J}_p) = \mathcal{J}_p$ for some $l > 0$. Now, we show that if $\mathcal{J}_p$ is normally hyperbolic for g_1, then the shadowing points belong to $\mathcal{J}_p$. Assume that there is a shadowing point $y \in M \setminus \mathcal{J}_p$. Then by the hyperbolicity, there are $l, k \in \mathbb{Z}$ such that $d(g_1^l(y), x_k) > \epsilon$, where $x_k \in \xi = \{x_i\}_{i\in\mathbb{Z}}$. This is a contradiction since $g_1|_{\mathcal{J}_p}$ has the orbital shadowing property. Thus, if $\mathcal{J}_p$ is normally hyperbolic for g_1, then the shadowing point belongs to $\mathcal{J}_p$. Since $g_1|_{\mathcal{J}_p}$ has the orbital shadowing property, from the above facts, we have $y \in \mathcal{J}_p$. But $g_1^l(\mathcal{J}_p) = \mathcal{J}_p$ and so $g_1^l|_{\mathcal{J}_p}$ is the identity map. Then $g_1^l|_{\mathcal{J}_p}$ does not have the orbital shadowing property. Thus, $g_1|_{\mathcal{J}_p}$ also does not have the orbital shadowing property.

Finally, if $\lambda \in \mathbb{C}$, then to avoid the notational complexity, we may assume that $g(p) = p$. As in the first case, by Lemma 2.2, there are $\alpha > 0$ and $g_1 \in \mathcal{V}(f)$ such that $g_1(p) = g(p) = p$ and

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

With a C^1-small modification of the map $D_p g$, we may suppose that there is $l > 0$ (the minimum number) such that $D_p g^l(v) = v$ for any $v \in \varphi_p(B_\alpha(p)) \subset T_p M$. Then we can go on with the previous argument in order to reach the same contradiction. Thus, every periodic point of $f \in \operatorname{int}\mathcal{OS}_\mu(M)$ is hyperbolic. □

End of the proof of Theorem 1.2 Let $f \in \operatorname{int}\mathcal{OS}_\mu(M)$. By Proposition 2.3, we see that $f \in \mathcal{F}_\mu(M)$. Thus, by Theorem 1.1, f is Anosov. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Author details
[1]Department of Mathematics, Chungnam National University, Daejeon, 305-764, Korea. [2]Department of Mathematics, Mokwon University, Daejeon, 302-729, Korea.

Acknowledgements
We wish to thank the referee for carefully reading of the manuscript and providing us many good suggestions. KL is supported by the National Research Foundation (NRF) of Korea funded by the Korean Government (No. 2011-0015193). ML is supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology, Korea (No. 2011-0007649).

Received: 12 July 2012 Accepted: 26 December 2012 Published: 14 January 2013

References

1. Mãné, R: An ergodic closing lemma. Ann. Math. **116**, 503-540 (1982)
2. Palis, J: On the C^1 Ω-stability conjecture. Publ. Math. Inst. Hautes Études Sci. **66**, 211-215 (1988)
3. Arbieto, A, Catalan, T: Hyperbolicity in the volume preserving scenario. Ergodic Theory Dyn. Syst. (2012). doi:10.1017/etds.2012.111
4. Pilyugin, SY: Shadowing in Dynamical Systems. Lecture Notes in Math., vol. 1706. Springer, Berlin (1999)
5. Robinson, C: Stability theorems and hyperbolicity in dynamical systems. Rocky Mt. J. Math. **7**, 425-437 (1977)
6. Sakai, K: Pseudo orbit tracing property and strong transversality of diffeomorphisms on closed manifolds. Osaka J. Math. **31**, 373-386 (1994)
7. Pilyugin, SY, Rodionova, AA, Sakai, K: Orbital and weak shadowing properties. Discrete Contin. Dyn. Syst. **9**, 287-308 (2003)
8. Moser, J: On the volume elements on a manifold. Trans. Am. Math. Soc. **120**, 286-294 (1965)
9. Bonatti, C, Díaz, LJ, Pujals, ER: A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. Ann. Math. **158**, 355-418 (2003)
10. Hirsh, M, Pugh, C, Shub, M: Invariant Manifolds. Lecture Notes in Math. Springer, Berlin (1977)

doi:10.1186/1029-242X-2013-18
Cite this article as: Lee and Lee: Volume preserving diffeomorphisms with orbital shadowing. *Journal of Inequalities and Applications* 2013 **2013**:18.