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Symplectic diffeomorphisms with inverse shadowing

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Abstract

Let f be a symplectic diffeomorphism of a closed C^∞ $2n$-dimensional Riemannian manifold M. In this paper, we prove the equivalence between the following conditions:

(a) f belongs to the C^1-interior of the set of symplectic diffeomorphisms satisfying the inverse shadowing property with respect to the continuous methods,

(b) f belongs to the C^1-interior of the set of symplectic diffeomorphisms satisfying the orbital inverse shadowing property with respect to the continuous methods,

(c) f is Anosov.

This result extends Bessa and Rocha's result (Appl. Math. Lett. 25:163-165, 2012).

MSC: 37C15; 37C50

Keywords: topological stability; inverse shadowing; orbital inverse shadowing; Anosov; symplectic diffeomorphism

1 Introduction

Let M be a closed C^∞ $2n$-dimensional manifold with Riemannian structure and endowed with a symplectic form ω, and let $\mathrm{Diff}_\omega(M)$ be the set of symplectomorphisms, that is, of diffeomorphisms f defined on M and such that

$$\omega_x(v_1, v_2) = \omega_{f(x)}\big(D_xf(v_1), D_xf(v_2)\big),$$

for $x \in M$ and $v_1, v_2 \in T_xM$. Consider this space endowed with the C^1 Whitney topology. It is well known that $\mathrm{Diff}_\omega(M)$ is a subset of all C^1 volume-preserving diffeomorphisms. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. By the theorem of Darboux [1, Theorem 1.8], there is an atlas $\{\varphi_i^j : U_i \to \mathbb{R}^{2n}\}$, where U_i is an open set of M satisfying $\varphi_i^*\omega_0 = \omega$ with $\omega_0 = \sum_{i=0}^n dy_i \wedge dy_{n+i}$.

The notion of the inverse shadowing property which is a 'dual' notion of the shadowing property. Inverse shadowing property was introduced by Corless and Pilyugin in [2], and the qualitative theory of dynamical systems with the property was developed by various authors (see [2–4]).

Now, we introduce the inverse shadowing property and some results for the inverse shadowing. Let $M^{\mathbb{Z}}$ be the space of all two-sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in M$, endowed with the product topology. Let $f : M \to M$ be a symplectic diffeomorphism. For a fixed $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping

$\varphi : M \to \Phi_f(\delta) \subset M^{\mathbb{Z}}$ is said to be a δ-*method* for f if $\varphi(x)_0 = x$, and each $\varphi(x)$ is a δ-pseudo orbit of f through x, where $\varphi(x)_0$ denotes the 0th component of $\varphi(x)$. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k\in\mathbb{Z}}$. The set of all δ-methods for f will be denoted by $\mathcal{T}_0(f,\delta)$. Say that φ is *continuous* δ-*method* for f if φ is continuous. The set of all continuous δ-methods for f will be denoted by $\mathcal{T}_c(f,\delta)$. If $g : M \to M$ is a homeomorphism with $d_0(f,g) < \delta$ then g induces a continuous δ-method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

Let $\mathcal{T}_h(f,\delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by a homeomorphism $g : M \to M$ with $d_0(f,g) < \delta$, where d_0 is the usual C^0-metric. Let $\mathcal{T}_d(f,\delta)$ denote by the set of all continuous δ-methods φ_g for f, which are induced by $g \in \mathrm{Diff}_\omega(M)$ with $d_1(f,g) < \delta$. Then clearly we know that

$$\mathcal{T}_d(f) \subset \mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f),$$

$\mathcal{T}_\alpha(f) = \bigcup_{\delta>0} \mathcal{T}_\alpha(f,\delta)$, $\alpha = 0,c,h,d$. We say that f has the *inverse shadowing property* with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0,c,h,d$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f,\delta)$, and for a point $x \in M$ there is a point $y \in M$ such that

$$d\big(f^k(x), \varphi_g(y)_k\big) < \epsilon, \quad k \in \mathbb{Z}.$$

We say that f has the *orbital inverse shadowing property* with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0,c,h,d$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(f,\delta)$, and for a point $x \in M$ there is a point $y \in M$ such that

$$d_H\big(\overline{\mathcal{O}_f(x)}, \overline{\mathcal{O}_{\varphi_g}(y)}\big) < \epsilon,$$

where d_H is the Hausdorff metric, and $\overline{A}$ is the closure of A. We denote by $\mathcal{IS}_{\omega,\alpha}(M)$ the set of symplectic diffeomorphisms on M with the inverse shadowing property with respect to the class $\mathcal{T}_\alpha$ and denote by $\mathcal{OIS}_{\omega,\alpha}(M)$ the set of symplectic diffeomorphisms on M with the orbital inverse shadowing property respect to the class $\mathcal{T}_\alpha$, where $\alpha = a,c,h,d$. Let $\mathrm{int}\,\mathcal{IS}_{\omega,\alpha}(M)$ be the C^1-interior of the set of symplectic diffeomorphisms on M with the inverse shadowing property respect to the class $\mathcal{T}_\alpha$, and let $\mathrm{int}\,\mathcal{OIS}_{\omega,\alpha}(M)$ be the C^1-interior of the set of symplectic diffeomorphisms on M with the orbital inverse shadowing property respect to the class $\mathcal{T}_\alpha$, where $\alpha = a,c,h,d$. Note that f has the inverse shadowing property if and only if f^n has the inverse shadowing property, for all $n \in \mathbb{Z}$. Lee [3] showed that a diffeomorphism belongs to the C^1-interior of the set of diffeomorphisms having the inverse shadowing property with respect to the $\mathcal{T}_d(f)$ if and only if it is structurally stable. Pilyugin [4] proved that a diffeomorphism belongs to the C^1-interior of the set of f diffeomorphisms having the inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ if and only if it is structurally stable.

The notion of topological stability was introduced by Walters [5], and he showed that every Anosov diffeomorphism is topological stable. In [6], Nitecki proved that if f satisfies both Axiom A and the strong transversality condition, then it is topological stable. We say that f is *topological stable* if for any $\epsilon > 0$, there is $\delta > 0$ such that for any $g \in \mathrm{Diff}(M)$, δ-C^0-closed to f, there is a continuous map $h : M \to M$ satisfying $h \circ g = f \circ h$ and $d(f(x),x) < \epsilon$

for all $x \in M$. Moriyasu [7] showed that the C^1-interior of the set of all topologically stable diffeomorphisms is characterized as the set of C^1-structurally stable. Very recently, Bessa and Rocha [8] proved that if a symplectic diffeomorphism belongs to the C^1-interior of the set of topologically stable, then the diffeomorphism is Anosov.

Remark 1.1 By the definition of the inverse shadowing, we have the following implication: topological stability $\Rightarrow$ inverse shadowing property with respect to the continuous method $\mathcal{T}_d \Rightarrow$ orbital inverse shadowing property with respect to the continuous method $\mathcal{T}_d$.

From the above remark, we know that our result is a slight generalization of the main theorem in [8]. In this paper, we omit the phrase 'with respect to the class $\mathcal{T}_d$' for simplicity. So, we say that f has the inverse shadowing property means that f has the inverse shadowing property with respect to the class $\mathcal{T}_d$.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \le C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. If $\Lambda = M$ then f is Anosov. We define the set $\mathcal{F}_\omega(M)$ as the set of diffeomorphisms $f \in \mathrm{Diff}_\omega(M)$ which have a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\omega(M)$ such that if for any $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic. Then we can see the following.

Lemma 1.2 [6] *If $f \in \mathcal{F}_\omega(M)$, then f is Anosov.*

Note that $\mathcal{F}_\omega(M) \subset \mathcal{F}(M)$ (see [9, Corollary 1.2]). By a result of Newhouse [10], if the symplectic diffeomorphisms is not Anosov then 1-elliptic points can be created by an arbitrary small C^1-perturbations of the symplectic diffeomorphism.

In this paper, we investigate the cases when a symplectic diffeomorphism f is in C^1-interior inverse shadowing property with respect to the class $\mathcal{T}_d(f)$, then it is Anosov. Let $\mathrm{int}\,\mathcal{IS}_\omega(M)$ be denoted the set of symplectic diffeomorphisms in $\mathrm{Diff}_\omega(M)$ satisfying the inverse shadowing property with respect to the class $\mathcal{T}_d$, and let $\mathrm{int}\,\mathcal{OIS}_\omega(M)$ be denoted the set of symplectic diffeomorphisms in $\mathrm{Diff}_\omega(M)$ satisfying the orbital inverse shadowing property with respect to the class $\mathcal{T}_d$ when we mention the inverse shadowing property (resp. orbital inverse shadowing property); that is, the 'inverse shadowing property (resp. orbital inverse shadowing property)' implies the 'inverse shadowing property (resp. orbital inverse shadowing property) with respect to the class $\mathcal{T}_d$'. Now we are in position to state the theorem of our paper.

Theorem 1.3 *Let $f \in \mathrm{Diff}_\omega(M)$. Then*

$$\mathrm{int}\,\mathcal{IS}_\omega(M) = \mathrm{int}\,\mathcal{OIS}_\omega(M) = \mathcal{AN}_\omega(M),$$

where $\mathcal{AN}_\omega(M)$ is the set of Anosov symplectic diffeomorphisms in $\mathrm{Diff}_\omega(M)$.

2 Proof of Theorem 1.3

Let M be as before, and let $f \in \mathrm{Diff}_\omega(M)$. Then the following is symplectic version of Franks' lemma.

Lemma 2.1 [11, Lemma 5.1] *Let $f \in \mathrm{Diff}_\omega(M)$ and $\mathcal{U}(f)$ be given. Then there are $\delta_0 > 0$ and $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_n\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_n\}$ and symplectic maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - Dg(x_i)\| < \delta_0$ for all $1 \le i \le n$, there are $\epsilon_0 > 0$ and $\widetilde{g} \in \mathcal{U}(f)$ such that*

(a) $\widetilde{g}(x) = g(x)$ *if* $x \in M \setminus U$,

(b) $\widetilde{g}(x) = \varphi_{g(x_i)} \circ L_i \circ \varphi_{x_i}^{-1}(x)$ *if* $x \in B_{\epsilon_0}(x_i)$,

where $B_{\epsilon_0}(x_i)$ is the ϵ_0-neighborhood of x_i.

A *periodic point* for f is a point $p \in M$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the minimum period of p. We say that a periodic point p is *elliptic* if $D_pf^{\pi(p)}$ has one nonreal eigenvalue of norm one. We say that a periodic point p is a *k-elliptic periodic point* if for a periodic point p of period $\pi(p)$ the tangent map $D_pf^{\pi(p)}$ has exactly $2k$ simple nonreal eigenvalues of norm 1 and the other ones have norm different from 1. In dimension 2, then 1-elliptic periodic points are actually elliptic. We say that p is *hyperbolic* if $Df^{\pi(p)}$ has no norm one eigenvalue. The following facts are enough to prove Theorem 1.3 by Lemma 1.2.

Lemma 2.2 *If $f \in \mathrm{Diff}_\omega(M)$ has the orbital inverse shadowing property, then f is not the identity map.*

Proof Suppose, by contradiction, that f is the identity map. Take $\epsilon = 1/4$, and let $0 < \delta < \epsilon$ be the number of the orbital inverse shadowing property of f. Since f is the identity map, we defined $g \in \mathrm{Diff}_\omega(M)$ by $g(x) = (x_1 + \delta/2, x_2, \ldots, x_{2n})$, for $x = (x_1, x_2, \ldots, x_{2n}) \in M$. Then $g \in \mathcal{T}_d(f)$. Since f has the orbital inverse shadowing property, there is $y \in M$ such that for any $x \in M$,

$$d(x,y) < \epsilon \quad \text{and} \quad d_H\big(\overline{\mathcal{O}_f(y)}, \overline{\mathcal{O}_g(x)}\big) < \epsilon.$$

Since f is the identity map and g is an increasing map, there is $k \in \mathbb{N}$ such that $d(y, g^k(x)) > \epsilon$. Thus, by the definition of the Hausdorff metric,

$$d_H\big(\overline{\mathcal{O}_f(y)}, \overline{\mathcal{O}_g(x)}\big) > \epsilon.$$

This is a contradiction. □

Lemma 2.3 *If $f \in \mathrm{int}\,\mathcal{IS}_\omega(M)$, then every periodic point of f is hyperbolic.*

Proof Let $f \in \mathrm{int}\,\mathcal{IS}_\omega(M)$, and let $\mathcal{U}_0(f)$ be a C^1-neighborhood of f. Suppose that there is a $g \in \mathcal{U}_0(f)$ such that g have a periodic elliptic point p. To simplify, we may assume that $g(p) = p$. Then D_pg has n pairs of nonreal eigenvalues, that is, $|a_i| = |\overline{a}_i| = 1$, $i = 1, \ldots, n$ with $T_pM = E_p^{c_1} \oplus \cdots \oplus E_p^{c_n}$ and $\dim E_p^{c_i} = 2$, $i = 1, \ldots, n$. By Lemma 2.1, there are $\alpha > 0$ and $g_1 \in \mathcal{U}(f)$ such that

$$g_1(x) = \begin{cases} \varphi_{g(p)} \circ D_pg \circ \varphi_p^{-1}(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Then $g(p) = g_1(p) = p$.

First, we consider the case $E_p^{c_1}(\alpha)$ other case is similar. Since p is nonhyperbolic for g_1, by our construction, we may assume that there is $l > 0$ such that $D_p g_1^l(v) = v$ for any $v \in E_p^{c_1}(\alpha) \cap \varphi_p^{-1}(B_\alpha(p))$. Take $v \in E_p^{c_1}(\alpha)$ such that $\|v\| = \alpha/4$. Then we can find a small arc $\mathcal{I}_p = \varphi_p(\{tv : -\alpha/4 \le t \le \alpha/4\}) \subset B_\alpha(p)$ such that

(i) $g_1^i(\mathcal{I}_p) \cap g_1^j(\mathcal{I}_p) = \emptyset$ if $0 \le i \ne j \le l-1$, and

(ii) $g_1^l(\mathcal{I}_p) = \mathcal{I}_p$, that is, $g_1^l|_{\mathcal{I}_p}$ is the identity map.

Then we can choose $0 < \eta < \alpha/4$ sufficiently small such that $B_\eta(g_1^i(\mathcal{I}_p)) \cap B_\eta(g_1^j(\mathcal{I}_p)) = \emptyset$ for all $1 \le i \ne j \le l-1$. Take $\epsilon = \eta/4$, and let $0 < \delta < \epsilon$ be the number of the definition of the inverse shadowing property of g_1 for ϵ. For the $\delta > 0$, we can define $\mathcal{T}_d(g_1)$-method as follows: Let $\psi \in \mathrm{Diff}_\omega(M)$ be such that p is a hyperbolic periodic point for ψ with $\psi(p) = p$ and $d(g_1, \psi) < \delta$. Then $\psi \in \mathcal{T}_d(g_1)$. To simplicity, we may assume that $g_1^l = g_1$. Take $y \in B_{4\epsilon}(p) \cap \mathcal{I}_p$ such that $d(y,p) = 2\epsilon$, and $g_1^n(y) = y$ for all $n \in \mathbb{Z}$. Since g_1 has the inverse shadowing property, we can see that for any $z \in M$

$$d\big(g_1^n(y), \varphi_\psi(z)_n\big) = d\big(g_1^n(y), \psi^n(z)\big) < \epsilon$$

for all $n \in \mathbb{Z}$. For any $z \in B_\epsilon(p)$, if $z = p$, then since $g_1|_{\mathcal{I}_p}$ is the identity map, it is clear that g_1 does not have the inverse shadowing property. If $z \ne p$, then since p is a hyperbolic periodic point for ψ, there is $k \in \mathbb{Z}$ such that

$$d\big(g_1^k(y), \psi^k(z)\big) = d\big(y, \psi^k(z)\big) > \epsilon.$$

This is a contradiction.

Finally, we may assume that there are m_i (the minimum numbers) such that $D_p g_1^{m_i}(v) = v$ for any $v \in E_p^{c_i}(\alpha) \cap \varphi_p^{-1}(B_\alpha(p))$, $i = 2, \ldots, n$. Let $K = \mathrm{lcm}\{m_i : i = 2, \ldots, n\}$. Here, lcm is the lowest common multiple.

To simplify, we assume that $g_2 = g_1^K$. Since $D_p g_1^{m_i}(v) = v$ for any $v \in E_p^{c_i}(\alpha) \cap \varphi_p^{-1}(B_\alpha(p))$, $i = 2, \ldots, n$, by the above argument, there exists a small arc $\mathcal{I}_p \subset B_\alpha(p)$ such that $g_2^l|_{\mathcal{I}_p}$ is the identity map, for some $l > 0$. Then we can find $\psi \in \mathcal{T}_d(g_2)$ such that p is a hyperbolic periodic point for ψ with $\psi(p) = p$, $d(g_2, \psi) < \delta$. By the inverse shadowing property for g_2, there exists $y \in B_{4\epsilon}(p) \cap \mathcal{I}_p$ such that

$$d(y,p) = 2\epsilon \quad \text{and} \quad g_2^{ln}(y) = y$$

for all $n \in \mathbb{Z}$. Then there exists $j \in \mathbb{Z}$ such that

$$d\big(g_2^j(y), \psi^j(z)\big) = d\big(y, \psi^j(z)\big) > \epsilon.$$

This is a contradiction. Thus, every periodic point of $f \in \mathrm{int}\,\mathcal{IS}_\omega(M)$ is hyperbolic. □

Lemma 2.4 *If $f \in \mathrm{int}\,\mathcal{OIS}_\omega(M)$, then every periodic point of f is hyperbolic.*

Proof Let $f \in \mathrm{int}\,\mathcal{OIS}_\omega(M)$. Then as in the proof of Lemma 2.3 and Lemma 2.2, we can obtain a contradiction. □

Proof of Theorem 1.3 Let $f \in \mathrm{int}\,\mathcal{IS}_\omega(M)$, and let $f \in \mathrm{int}\,\mathcal{OIS}_\omega(M)$. Suppose that $f \notin \mathcal{F}_\omega(M)$. Then there is $g \in \mathcal{U}_0(f) \subset \mathcal{U}(f)$ such that g have a periodic elliptic point p. By

Lemma 2.3 and Lemma 2.4, g does not have a periodic elliptic point. This is a contradiction. Thus, if $f \in \operatorname{int}\mathcal{IS}_\omega(M)$ or $f \in \operatorname{int}\mathcal{OIS}_\omega(M)$ then $f \in \mathcal{F}_\omega(M)$. By Lemma 1.2, f is Anosov. □

Competing interests
The author declares that they have no competing interests.

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