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Statistical inference for the shape parameter change-point estimator in negative associated gamma distribution

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Abstract
In this paper, the change-point estimator for the shape parameter is proposed in a negative associated gamma random variable sequence. Suppose that $X_1, \ldots, X_n$ are negative associated random variables satisfying that $X_1, \ldots, X_{[n\tau_0]}$ are identically distributed with $\Gamma(x; \nu_1, \lambda)$, and that $X_{[n\tau_0]+1}, \ldots, X_n$ are identically distributed with $\Gamma(x; \nu_2, \lambda)$; the change point τ_0 is unknown. The weak and strong consistency, and the weak and strong convergence rate of the change-point estimator, are given by the CUSUM method. Furthermore, the O_P convergence rate of the change-point estimator is presented under the local alternative hypothesis condition.

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1 Introduction

The gamma distribution occurs frequently in a variety of applications, especially in reliability, in survival analysis and in modeling income distributions. The density of a gamma-distributed random variable X with a shape parameter ν and a scale parameter λ is given by

$$f(x; \nu, \lambda) = \frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x} I \quad (x > 0), \tag{1}$$

where $I(\cdot)$ is the indicator function, $\Gamma(\cdot)$ is a Γ function with $\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\,dx$.

The family of gamma distributions includes the chi-squared distribution, exponential distribution and Erlang distribution. For example, the gamma distribution is an Erlang distribution with a positive integer ν. When the shape parameter $\nu = 1$, the gamma distribution is an exponential distribution with parameter λ; when $\lambda = \frac{1}{2}$, the gamma distribution is a chi-squared distribution, with 2ν degrees of freedom. The shape parameter is especially of interest in reliability theory because the gamma distribution is either a decreasing failure rate (DFR), a constant or an increasing failure rate (IFR) according to whether the shape parameter is negative, zero or positive. The shape parameter also plays an important role in renewal theory when modeling arrival times of events.

As for the gamma distribution parameter change-point problems, Kander and Zacks [1] proposed a statistic for testing a change in the one-parameter exponential family; Hsu [2] considered a change point for the scale parameter of gamma random variables, assuming that the shape parameter was constant; Diaz [3] posed the Bayesian test regarding the scale parameter change point for the independent gamma variables; Gupta and Ramanayake [4], Ramanayake and Gupta [5] discussed a linear trend change for the exponential distribution; Ramanayake [6] proposed some tests for detecting a change in the shape parameter of gamma distributions assuming that λ is constant. The strong consistency and convergence rate of the change-point estimator have been investigated by applying the moving averages method (Tan *et al.* [7]), assuming that there is at most one change point.

Change-point analysis is widely used in fields such as quality control, economics and finance, biostatistics and so on (see Page [8]; Bai and Perron [9]; Braun *et al.* [10]; Chen *et al.* [11]). Change-point problems have also received considerable attention due to the wide variety of applications and recent developments in computational methods. There is a considerable body of literature on change-point analysis that assume that the random variables being considered are independent.

Let $X_1, X_2, \ldots, X_n$ be a negative associated sequence that satisfies the conditions that $X_1, \ldots, X_{[n\tau_0]}$ have the common distribution $\Gamma(x; \nu_1, \lambda)$, and that $X_{[n\tau_0]+1}, \ldots, X_n$ have the common distribution $\Gamma(x; \nu_2, \lambda)$, where τ_0 is an unknown parameter called the change point; ν_1, ν_2 are the shape parameters before and after change, respectively. In this paper, we assume that the scale parameter does not change, but the shape parameter is susceptible to change at an unknown time $[n\tau_0]$ in the sequence. Noticing that $\lambda X \sim \Gamma(x; \nu, 1)$ and its distribution is not related to the scale parameter, logarithm transformations may be made for $\{X_i, i = 1, \ldots, n\}$ as follows. Let

$$Y_i = \ln \lambda X_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

It can be shown that the mean of Y_1 is $\mu_1 = EY_1 = \Psi(\nu_1)$ and the mean of $Y_{[n\tau_0]+1}$ is $\mu_2 = EY_{[n\tau_0]+1} = \Psi(\nu_2)$, where $\Psi(\nu)$ is the derivation of $\ln \Gamma(\nu)$; that is,

$$\Psi(\nu) = \frac{d[\ln(\Gamma(\nu))]}{d\nu} = \frac{\Gamma'(\nu)}{\Gamma(\nu)}.$$

$\Psi(\nu)$ can be expressed, as in [12, p.16], by

$$\Psi(\nu) = -\gamma + \int_0^{+\infty} \frac{e^{-t} - e^{-\nu t}}{1 - e^{-t}}\, dt,$$

where γ is the Euler-Mascheroni constant, that is, $\gamma = -\int_0^{+\infty} e^{-x} \ln x\, dx$. Since $\Psi'(\nu) = \int_0^{+\infty} \frac{t}{1-e^{-t}} e^{-\nu t}\, dt > 0$, hence $\Psi(\nu)$ is an increasing function in $(0, +\infty)$.

Define

$$\begin{aligned} U_k &= \sum_{i=1}^{k} Y_i - \frac{k}{n} \sum_{i=1}^{n} Y_i, \\ \rho_0 &= \nu_1 - \nu_2, \qquad \mu_0 = \mu_1 - \mu_2. \end{aligned} \tag{3}$$

Since

$$U_k = \sum_{i=1}^{k} \ln \lambda X_i - \frac{k}{n} \sum_{i=1}^{n} \ln \lambda X_i = \sum_{i=1}^{k} \ln X_i - \frac{k}{n} \sum_{i=1}^{n} \ln X_i,$$

are not related to the scale parameter λ, then if we know in advance or by test that there is a change in the shape parameter, we may define the estimator of the change point τ_0 as

$$\widehat{\tau} = \frac{1}{n} \min\Big\{k : |U_k| = \max_{1\le j\le n} |U_j|\Big\}. \tag{4}$$

For convenience, throughout this paper, $c, c_1, \ldots$ represent a constant which is independent of n and may take different values in different expressions.

The paper is arranged as follows. In Section 2, the change-point estimator $\widehat{\tau}$ is proposed based on the CUSUM method by an appropriate logarithm transformation for $\{X_i, i = 1, \ldots, n\}$, and its constancy and convergence rate are investigated. The proofs of theorems are given in Section 3.

2 Main results

Theorem 1 *Assume that $X_1, X_2, \ldots, X_n$ is a negative associated random variable sequence satisfying the conditions that $X_1, \ldots, X_{[n\tau_0]}$ are identically distributed with $\Gamma(x; \nu_1, \lambda)$, and $X_{[n\tau_0]+1}, \ldots, X_n$ are identically distributed with $\Gamma(x; \nu_2, \lambda)$. Let*

$$k_0 = [n\tau_0], \qquad \widehat{k} = [n\widehat{\tau}], \qquad k = [n\tau] \quad \textit{for some } 0 < \tau < 1, \tag{5}$$

where $[A]$ denotes the integer part of a number A. If the $\rho_0 = \nu_1 - \nu_2$ is a non-zero constant, then $\widehat{\tau}$ is a consistent estimator of τ_0 and

$$|\widehat{\tau} - \tau_0| = o_P\big(n^{-\frac{1}{2}} l(n)\big), \tag{6}$$

where $l(n)$ is a slowly varying function with $\lim_{n\to\infty} l(n) = +\infty$.

Theorem 2 *Assume that the conditions of Theorem 1 hold, then $\widehat{\tau}$ is a strong consistent estimator of τ_0, and*

$$|\widehat{\tau} - \tau_0| = o\big(n^{-\frac{1}{2}+\delta}\big), \quad \textit{a.s. for some } 0 < \delta < \frac{1}{2}. \tag{7}$$

Next, we will study the O_P convergence rate of $\widehat{\tau}$ under the local alternative hypothesis; that is, ρ_0 is not a constant independent of n, but it depends on n and is denoted by ρ_n. Noticing that if ρ_n is large, the change-point estimation is usually quite precise. In practice it may be more important to construct confidence intervals for τ_0 when ρ_n is small. We hence assume that $\rho_n \longrightarrow 0$ as $n \longrightarrow \infty$. It can be seen that the results obtained in the above theorems cannot be applied here, and we need to establish stronger results than those obtained in the above theorems.

Notice that $\mu_i = \Psi(\nu_i)$, $i = 1, 2$. Then, by the mean theorem, μ_n (under the local alternative hypothesis, denoting μ_0 as μ_n) can be expressed as

$$\mu_n = \mu_1 - \mu_2 = \Psi'(\tilde{\nu})(\nu_1 - \nu_2) = \Psi'(\tilde{\nu})\rho_n,$$

where $\tilde{\nu}$ lies between ν_1 and ν_2. Hence, with some added conditions, μ_n is equal to ρ_n in practice.

Theorem 3 *Assume that $X_1, X_2, \ldots, X_n$ is a negative associated random variable sequence, and $X_1, \ldots, X_{[n\tau_0]}$ are identically distributed by $\Gamma(x; \nu_1, \lambda)$, and $X_{[n\tau_0]+1}, \ldots, X_n$ are identically distributed by $\Gamma(x; \nu_2, \lambda)$. If μ_n satisfies*

$$\mu_n \longrightarrow 0, \qquad \sqrt{n}\mu_n \longrightarrow \infty, \tag{8}$$

then

$$|\widehat{\tau} - \tau_0| = O_P\left(\frac{1}{n\mu_n^2}\right). \tag{9}$$

Remark 1 Theorems 1 and 2 give the weak and strong consistency and convergence rates for the change-point estimator $\widehat{\tau}$ of the shape parameter in a gamma distribution. In Theorem 3, the O_P convergence rate of the change-point estimator $\widehat{\tau}$ of the shape parameter is proposed under the local alternative condition, and it is one of the necessary conditions for studying the limiting distribution of $\widehat{\tau}$. Having this O_P value, we can study the limiting distribution of $\widehat{\tau}$. This will be the subject of a future paper.

3 Proof of the theorem

To prove the above theorems, we first consider the following lemmas.

Lemma 1 *Let $A_1, A_2, \ldots, A_m$ be disjoint subsets of $\{1, 2, \ldots, n\}$, and let $a_i = \sharp(A_i)$ be the number of elements in A_i, $i = 1, 2, \ldots, m$. Assume that $Z_1, Z_2, \ldots, Z_n$ are negative associated variables, then if*

$$f_i : R^{a_i} \to R, \quad i = 1, 2, \ldots, m \tag{10}$$

are the increasing (or decreasing) positive functions for every element, then $f_1(Z_j, j \in A_1), \ldots, f_m(Z_j, j \in A_m)$ are the negative associated variables.

Proof See Joag-Dev and Proschan [13]. □

Lemma 2 *Let $\{Z_j, j \in N\}$ be a negative associated sequence with zero mean satisfying $\beta_p = \sup_{j \in N} E|Z_j|^p < \infty$ for some $p \geq 2$. Denoting $S_{a,k} = \sum_{j=0}^{k-1} Z_{a+j}$, then there exist constants $C_p, K_p \geq 1$ related to p, for all $a, n \in N$, such that*

$$E|S_{1,n}|^p \leq C_p n^{\frac{p}{2}-1} \sum_{j=1}^{n} E|Z_j|^p; \qquad E\Bigl(\max_{1 \leq k \leq n} |S_{a,k}|\Bigr)^p \leq K_p \beta_p n^{\frac{p}{2}}. \tag{11}$$

Proof See Su, Zhao and Wang [14]. □

Lemma 3 *Let $\{Z_n, n \geq 1\}$ be a negative associated sequence, if $\{b_k, k \geq 1\}$ is an increasing number serial, then $\forall \varepsilon > 0$ and $m \leq n$, we have*

$$P\left(\max_{1 \leq k \leq n} \left|\frac{1}{b_k} \sum_{i=1}^{k} (X_i - EX_i)\right| \geq \varepsilon\right) \leq \frac{8}{\varepsilon^2} \sum_{j=1}^{n} \frac{\operatorname{Var}(X_j)}{b_j^2}; \tag{12}$$

$$P\left(\max_{m \leq k \leq n} \left|\frac{1}{b_k} \sum_{i=1}^{k} (X_i - EX_i)\right| \geq \varepsilon\right) \leq \frac{4}{\varepsilon^2}\left(\sum_{j=1}^{m} \frac{\operatorname{Var}(X_j)}{b_m^2} + 8 \sum_{j=m+1}^{n} \frac{\operatorname{Var}(X_j)}{b_j^2}\right). \tag{13}$$

Proof See Hu and Zhang [15]. □

Proof of Theorem 1 Noticing that $\ln(\lambda x)$ is an increasing positive function and $X_1, \dots, X_n$ are negative associated sequences, we have from Lemma 1 that $\{Y_i, i = 1, \dots, n\}$ are negative associated sequences. Without loss of generality, assuming that $\nu_1 > \nu_2$, by the increasing character of $\Psi(\nu)$ in $(0, \infty)$, we know that $\mu_0 = \mu_1 - \mu_2 > 0$. By simple computation, it can be shown that

$$EU_{k_0} = \frac{k_0(n-k_0)}{n}(\mu_1 - \mu_2) = n\tau_0(1-\tau_0)\mu_0, \tag{14}$$

and

$$EU_k = \begin{cases} \frac{k(n-k_0)}{n}\mu_0, & k \le k_0, \\ \frac{(n-k)k_0}{n}\mu_0, & k > k_0, \end{cases} = \begin{cases} n\tau(1-\tau_0)\mu_0, & k \le k_0, \\ n(1-\tau)\tau_0\mu_0, & k > k_0. \end{cases} \tag{15}$$

Hence,

$$\begin{aligned} |EU_{k_0}| - |EU_k| &= \begin{cases} n(1-\tau_0)(\tau_0 - \tau)\mu_0, & k \le k_0, \\ n\tau_0(\tau - \tau_0)\mu_0, & k > k_0, \end{cases} \\ &\ge n\tau^*\mu_0|\tau - \tau_0|, \end{aligned} \tag{16}$$

where $\tau^* = \min\{\tau_0, 1-\tau_0\}$.

From the triangle inequality, it can easily be shown that

$$|U_k| - |U_{k_0}| \le 2\max_{1\le k\le n}|U_k - EU_k| + |EU_k| - |EU_{k_0}|,$$

namely,

$$|EU_{k_0}| - |EU_k| \le 2\max_{1\le k\le n}|U_k - E\mu_k| + |U_{k_0}| - |U_k|.$$

Noticing that $|U_{k_0}| \le |U_{\widehat{k}}|$, hence we have

$$|EU_{k_0}| - |EU_{\widehat{k}}| \le 2\max_{1\le k\le n}|U_k - EU_k|. \tag{17}$$

Let $Y_i^* = Y_i - EY_i$, then by (16) and (17), it follows that

$$\begin{aligned} n\tau^*\mu_0|\tau - \tau_0| &\le 2\max_{1\le k\le n}|U_k - EU_k| = 2\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^* - \frac{k}{n}\sum_{i=1}^{n} Y_i^*\right| \\ &= 2\max_{1\le k\le n}\left|\frac{n-k}{n}\sum_{i=1}^{k} Y_i^* - \frac{k}{n}\sum_{i=k+1}^{n} Y_i^*\right| \\ &\le 2\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^*\right| + 2\max_{1\le k\le n}\left|\sum_{i=k+1}^{n} Y_i^*\right|. \end{aligned} \tag{18}$$

Hence,

$$\begin{aligned} P\big(g_1(n)|\widehat{\tau}-\tau_0|>\varepsilon\big) &= P\bigg(|\widehat{\tau}-\tau_0|>\frac{\varepsilon}{g_1(n)}\bigg) \\ &\le P\Bigg(\frac{2}{n\tau^*\mu_0}\Bigg\{\max_{1\le k\le n}\Bigg|\sum_{i=1}^{k}Y_i^*\Bigg| + \max_{1\le k\le n}\Bigg|\sum_{i=k+1}^{n}Y_i^*\Bigg|\Bigg\} > \frac{\varepsilon}{g_1(n)}\Bigg) \\ &\le P\Bigg(\max_{1\le k\le n}\Bigg|\sum_{i=1}^{k}Y_i^*\Bigg| > \frac{n\tau^*\mu_0}{4g_1(n)}\varepsilon\Bigg) + P\Bigg(\max_{1\le k\le n}\Bigg|\sum_{i=k+1}^{n}Y_i^*\Bigg| > \frac{n\tau^*\mu_0}{4g_1(n)}\varepsilon\Bigg) \\ &\mathrel{\widehat{=}} A_1 + A_2. \end{aligned} \tag{19}$$

Since $Y_1, Y_2, \ldots, Y_n$ are the negative associated variables, by the Markov inequality and Lemma 2, $\forall r > 2$, we have

$$\begin{aligned} A_1 &\le E\Bigg(\max_{1\le k\le n}\Bigg|\sum_{i=1}^{k}Y_i^*\Bigg|\Bigg)^r \Big/ \bigg(\frac{n\tau^*\mu_0\varepsilon}{4g_1(n)}\bigg)^r = \frac{4^r}{(\varepsilon\tau^*\mu_0)^r}\frac{g_1^r(n)}{n^r}E\Bigg(\max_{1\le k\le n}\Bigg|\sum_{i=1}^{k}Y_i^*\Bigg|\Bigg)^r \\ &\le \frac{4^r}{(\varepsilon\tau^*\mu_0)^r}\frac{g_1^r(n)}{n^r}K_r\beta_r n^{\frac{r}{2}} \le \frac{4^r K_r\beta_r}{(\varepsilon\tau^*\mu_0)^r}\bigg(\frac{n^{\frac{1}{2}}}{g_1(n)}\bigg)^{-r}, \end{aligned} \tag{20}$$

where $\beta_r = \max\{E|Y_1^*|^r, E|Y_{[n\tau_0]+1}^*|^r\}$ is a constant independent of n. Similar arguments give that

$$A_2 \le \frac{4^r K_r\beta_r}{(\varepsilon\tau^*\mu_0)^r}\bigg(\frac{n^{\frac{1}{2}}}{g_1(n)}\bigg)^{-r}. \tag{21}$$

Hence, if we choose $g_1(n) = n^{\frac{1}{2}}l^{-1}(n)$, where $l(n)$ is a slowly varying function satisfying $\lim_{n\to\infty} l(n) = +\infty$, then combining (19)-(21) we have, as $n \to \infty$,

$$P\big(g_1(n)|\widehat{\tau}-\tau_0|>\varepsilon\big) \longrightarrow 0,$$

that is, $\widehat{\tau}$ is the weak consistent estimator of τ_0, and

$$|\widehat{\tau}-\tau_0| = o_P\big(n^{-\frac{1}{2}}l(n)\big). \qquad \square$$

Proof of Theorem 2 From (19) to (21) we have, for $\forall\varepsilon > 0$,

$$\sum_{n=1}^{\infty}P\big(g_2(n)|\widehat{\tau}-\tau_0|>\varepsilon\big) \le \sum_{n=1}^{\infty}\frac{4^r K_r\beta_r}{(\varepsilon\tau^*\mu_0)^r}\bigg(\frac{n^{\frac{1}{2}}}{g_2(n)}\bigg)^{-r}$$

if we choose $g_2(n) = n^{\frac{1}{2}-\delta}$ for some $0 < \delta < \frac{1}{2}$. Let $r > \frac{1}{\delta}$, then we have, as $n \to \infty$,

$$\sum_{n=1}^{\infty}P\big(g_2(n)|\widehat{\tau}-\tau_0|>\varepsilon\big) < \infty. \tag{22}$$

By the *Borel-Cantelli* lemma, we obtain that $\widehat{\tau}$ is the strongly consistent estimator of τ, and

$$|\widehat{\tau}-\tau_0| = o\big(n^{-\frac{1}{2}+\delta}\big) \quad \text{a.s. for some } 0 < \delta < \frac{1}{2}. \qquad \square$$

Proof of Theorem 3 To this end, we choose a value $0 < \theta < \frac{1}{2}$ such that $\tau \in (\theta, 1-\theta)$. By (8) and (19)-(21) ($g_1(n) = 1$), it is easily found that $\widehat{\tau}$ is a consistent estimator of τ_0. Therefore, for every $\varepsilon > 0$, $P(\widehat{\tau} \notin (\theta, 1-\theta)) < \varepsilon$. Thus, we now have only to examine the behavior of U_k over those k for which $n\theta \le k \le n(1-\theta)$. To prove $|\widehat{\tau} - \tau_0| = O_P(\frac{1}{n\mu_n^2})$, we shall prove that

$$P\left(|\widehat{\tau} - \tau_0| > \frac{M}{n\mu_0^2}\right) \longrightarrow 0, \tag{23}$$

when $M \longrightarrow \infty$. For every $M > 0$, define

$$D_{n,M} = \left\{k : n\theta \le k \le n(1-\theta), |k - k_0| > \frac{M}{\mu_n^2}\right\}.$$

Then we have

$$\begin{aligned} P\left(|\widehat{\tau} - \tau_0| > \frac{M}{n\mu_0^2}\right) &\le P\big(\widehat{\tau} \notin (\theta, 1-\theta)\big) + P\left(|\widehat{\tau} - \tau_0| > \frac{M}{n\mu_n^2}, \hat{\tau} \in (\theta, 1-\theta)\right) \\ &\le \varepsilon + P\Big(\sup_{k\in D_{n,M}} |U_k| \ge |U_{k_0}|\Big). \end{aligned} \tag{24}$$

Since

$$\begin{aligned} &P\Big(\sup_{k\in D_{n,M}} |U_k| \ge |U_{k_0}|\Big) \\ &\quad= P\Big(\sup_{k\in D_{n,M}} |U_k| \ge U_{k_0}, U_{k_0} \ge 0\Big) + P\Big(\sup_{k\in D_{n,M}} |U_k| \ge -U_{k_0}, U_{k_0} < 0\Big) \\ &\quad= P\Big(\sup_{k\in D_{n,M}} |U_k| - U_{k_0} \ge 0, U_{k_0} \ge 0\Big) + P\Big(\sup_{k\in D_{n,M}} |U_k| + U_{k_0} \ge 0, U_{k_0} < 0\Big) \\ &\quad\le P\Big(\sup_{k\in D_{n,M}} (U_k - U_{k_0}) \ge 0\Big) + P\Big(\inf_{k\in D_{n,M}} (U_k + U_{k_0}) \le 0\Big) \\ &\quad\widehat{=} B_1 + B_2. \end{aligned} \tag{25}$$

Noticing that $U_k + U_{k_0} \le 0$ implies $U_k - EU_k + U_{k_0} - EU_{k_0} \le -EU_k - EU_{k_0} \le -EU_{k_0}$, which in turn implies that

$$U_k - EU_k \le -\frac{1}{2}EU_{k_0} \quad \text{or} \quad U_{k_0} - EU_{k_0} \le -\frac{1}{2}EU_{k_0}. \tag{26}$$

Since $EU_{k_0} > 0$, we have

$$|U_k - EU_k| \ge \frac{1}{2}EU_{k_0} \quad \text{or} \quad |U_{k_0} - EU_{k_0}| \ge \frac{1}{2}EU_{k_0}.$$

Furthermore, we obtain

$$\begin{aligned} B_2 &\le P\left(\sup_{k\in D_{n,M}} |U_k - EU_k| \ge \frac{1}{2}EU_{k_0}\right) + P\left(|U_{k_0} - EU_{k_0}| \ge \frac{1}{2}EU_{k_0}\right) \\ &\widehat{=} D_1 + D_2. \end{aligned} \tag{27}$$

It can be seen from the definition of $D_{n,M}$ that

$$\begin{aligned}
D_1 &\le P\left(\sup_{n\theta\le k\le n(1-\theta)}|U_k - EU_k| \ge \frac{1}{2}EU_{k_0}\right) \\
&= P\left(\sup_{n\theta\le k\le n(1-\theta)}\left|\sum_{i=1}^{k} Y^* - \frac{k}{n}\sum_{i=1}^{n} Y^*\right| \ge \frac{1}{2}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^* - \frac{k}{n}\sum_{i=1}^{n} Y_i^*\right| \ge \frac{1}{2}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) + P\left(\max_{1\le k\le n}\left|\frac{k}{n}\sum_{i=1}^{n} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) + P\left(\left|\sum_{i=1}^{n} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) \\
&\widehat{=}\, E_1 + E_2. \qquad (28)
\end{aligned}$$

Because $Y_1^*, \ldots, Y_n^*$ are negative associated variables, by the Markov inequality, Lemma 2 and (8), $\forall p \ge 2$, we obtain

$$\begin{aligned}
E_1 &= P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le E\left(\max_{k}\left|\sum_{i=1}^{k} Y_i^*\right|^p\right) \Big/ \left(\frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right)^p \\
&\le \frac{4^p}{[n\tau_0(1-\tau_0)\mu_n]^p}K_p\beta_p n^{\frac{p}{2}} \le c_1\frac{1}{(n^{\frac{1}{2}}\mu_n)^p} \longrightarrow 0 \quad \left(\text{as } n\mu_n^2 \longrightarrow \infty\right). \qquad (29)
\end{aligned}$$

Similar arguments give

$$\begin{aligned}
E_2 &= P\left(\left|\sum_{i=1}^{n} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le \frac{4^p}{[n\tau_0(1-\tau_0)\mu_n]^p}C_p n^{\frac{p}{2}-1}\sum_{i=1}^{n} E|Y^*|^p \le c_2\frac{1}{(n^{\frac{1}{2}}\mu_n)^p} \longrightarrow 0 \quad \left(\text{as } n\mu_n^2 \longrightarrow \infty\right). \qquad (30)
\end{aligned}$$

Combining (28)-(30), we see that

$$D_1 \longrightarrow 0 \quad \text{as } n\mu_n^2 \longrightarrow \infty. \qquad (31)$$

Similar arguments as those for D_1 give

$$\begin{aligned}
D_2 &= P\left(\left|\sum_{i=1}^{k_0} Y_i^* - \frac{k_0}{n}\sum_{i=1}^{n} Y_i^*\right| \ge \frac{1}{2}n\tau_0(1-\tau_0)\mu_n\right) \\
&\le P\left(\left|\sum_{i=1}^{k_0} Y_i^*\right| \ge \frac{1}{4}n\tau_0(1-\tau_0)\mu_n\right)
\end{aligned}$$

$$+ P\left(\left|\sum_{i=1}^{n} Y_i^*\right| \geq \frac{1}{4} n\tau_0(1-\tau_0)\mu_n\right)$$

$$\leq c_3 \frac{1}{(n^{\frac{1}{2}}\mu_n)^p} \longrightarrow 0 \quad \text{as } n\mu_n^2 \longrightarrow \infty. \tag{32}$$

Combining (27), (31) and (32), we obtain

$$B_2 \longrightarrow 0. \tag{33}$$

Now consider B_1. Because of symmetry, we only consider the case of $k \leq k_0$. The event $U_k - U_{k_0} \geq 0$ implies that

$$U_k - EU_k - (U_{k_0} - EU_{k_0})$$

$$\geq EU_{k_0} - EU_k$$

$$\iff \quad \left(\sum_{i=1}^{k} Y_i^* - \frac{k}{n}\sum_{i=1}^{n} Y_i^*\right) - \left(\sum_{i=1}^{k_0} Y_i^* - \frac{k_0}{n}\sum_{i=1}^{n} Y_i^*\right) \geq n\tau^*\mu_n|\tau-\tau_0|$$

$$\iff \quad \left(-\sum_{i=k+1}^{k_0} Y_i^* - \frac{k-k_0}{n}\sum_{i=1}^{n} Y_i^*\right) \geq n\tau^*\mu_n|\tau-\tau_0|$$

$$\iff \quad \frac{-1}{k_0-k}\sum_{i=k+1}^{k_0} Y_i^* + \frac{1}{n}\sum_{i=1}^{n} Y_i^* \geq \tau^*\mu_n.$$

Therefore,

$$B_1 \leq P\left(\max_{k\in D_{n,M}}\left|\frac{1}{k_0-k}\sum_{i=k+1}^{k_0} Y_i^* + \frac{1}{n}\sum_{i=1}^{n} Y_i^*\right| \geq \tau^*\mu_n\right)$$

$$\leq P\left(\max_{k\in D_{n,M}}\left|\frac{1}{k_0-k}\sum_{i=k+1}^{k_0} Y_i^*\right| \geq \frac{1}{2}\tau^*\mu_n\right) + P\left(\max_{k\in D_{n,M}}\left|\sum_{i=1}^{n} Y_i^*\right| \geq \frac{1}{2}n\tau^*\mu_n\right)$$

$$\widehat{=} F_1 + F_2. \tag{34}$$

From the Markov inequality, Lemma 2 and (8), we obtain

$$F_2 \leq 2^p \frac{C_P}{(n\tau^*\mu_n)^p} n^{\frac{p}{2}} \leq c_4\left(\frac{1}{n\mu_n^2}\right)^{\frac{p}{2}} \longrightarrow 0. \tag{35}$$

Denoting $\sigma_1^2 = \operatorname{Var}(Y_1^*)$, from Lemma 3 and (8), we obtain

$$F_1 = P\left(\max_{n\theta\leq k\leq k_0-\frac{M}{\mu_n^2}}\left|\frac{1}{k_0-k}\sum_{i=k+1}^{k_0} Y_i^*\right| \geq \frac{1}{2}\tau^*\mu_n\right)$$

$$= P\left(\max_{\frac{M}{\mu_n^2}\leq k_0-k\leq k_0-n\theta}\left|\frac{1}{k_0-k}\sum_{j=1}^{k_0-k} Y_{k_0+1-j}^*\right| \geq \frac{1}{2}\tau^*\mu_n\right)$$

$$= P\left(\max_{\frac{M}{\mu_n^2}\leq t\leq k_0-n\theta}\left|\frac{1}{t}\sum_{j=1}^{t} Y_{k_0+1-j}^*\right| \geq \frac{1}{2}\tau^*\mu_n\right)$$

$$\leq \frac{4*4}{(\tau^*\mu_n)^2}\left\{\sum_{j=1}^{\frac{M}{\mu_n^2}}\frac{\mathrm{Var}(Y^*_{k_0+1-j})}{(\frac{M}{\mu_n^2})^2}+8\sum_{j=\frac{M}{\mu_n^2}+1}^{k_0-n\theta}\frac{\mathrm{Var}(Y^*_{k_0+1-j})}{j^2}\right\}$$

$$\leq 16\frac{\sigma_1^2}{(\tau^*)^2\mu_n^2}\left\{\frac{\mu_n^2}{M}+8\sum_{j=\frac{M}{\mu_n^2}+1}^{k_0-n\theta}\frac{1}{j(j-1)}\right\}$$

$$= 16\frac{\sigma_1^2}{(\tau^*)^2\mu_n^2}\left\{\frac{\mu_n^2}{M}+8\left(\frac{\mu_n^2}{M}-\frac{1}{k_0-n\theta-1}\right)\right\}$$

$$= 16\frac{\sigma_1^2}{(\tau^*)^2}\left(\frac{9}{M}-\frac{8}{(k_0-n\theta-1)\mu_n^2}\right)\longrightarrow 0 \quad \left(\text{as } n\mu_n^2\to\infty, M\to\infty\right). \tag{36}$$

Combining (34)-(36), we have

$$B_1 \longrightarrow 0. \tag{37}$$

From (24), (33) and (37), we know that (23) holds; that is, Theorem 3 is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
C-cT conceived of the study questions which can be done, participated in the proofs and drafted the manuscript. B-qM participated in the proofs and provided the related reference. X-cZ participated in the proof of Theorems and helped to draft the manuscript. All authors read and approved the final manuscript.

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