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# $f$-Contractive multivalued maps and coincidence points

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**Abstract**

In this paper, we prove a result on the existence of an $f$-orbit for generalized $f$-contractive multivalued maps. Then, we establish main results on the existence of coincidence points and common fixed points for generalized $f$-contractive maps not involving the extended Hausdorff metric and the continuity condition. Our results either generalize or improve a number of metric fixed point results.

**MSC:** 47H10; 47H09; 54H25

**Keywords:** metric space; fixed point; multivalued contractive map; coincidence point

## 1 Introduction

Let $(X,d)$ be a metric space. Let $2^X$, $\mathrm{Cl}(X)$ and $\mathrm{CB}(X)$ denote the collection of nonempty subsets of $X$, nonempty closed subsets of $X$, and nonempty closed bounded subsets of $X$, respectively. Let $H$ be the Hausdorff metric with respect to $d$, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

for every $A,B \in \mathrm{CB}(X)$, where $d(x,B) = \inf_{y\in B} d(x,y)$.

Let $f : X \to X$ be a single-valued map, and let $T : X \to 2^X$ be a multivalued map. A point $x \in X$ is called a fixed point of $T$ if $x \in T(x)$, and the set of fixed points of $T$ is denoted by $\mathrm{Fix}(T)$. A point $x \in X$ is called a coincidence point of $f$ and $T$ if $f(x) \in T(x)$. We denote by $C(f \cap T)$ the set of coincidence points of $f$ and $T$.

We say a sequence $\{x_n\}$ in $X$ is an $f$-orbit of $T$ at $x_0 \in X$ if $fx_n \in Tx_{n-1}$ for all $n \geq 1$. We say that $f$ and $T$ weakly commute if $fTx \subset Tfx$ for all $x \in X$. Clearly, commuting maps $f$ and $T$ weakly commute.

A multivalued map $T : X \to \mathrm{CB}(X)$ is called

(i) *contraction* [1] if for a fixed constant $\lambda \in (0,1)$ and for each $x,y \in X$,

$$H\bigl(T(x),T(y)\bigr) \leq \lambda d(x,y).$$

(ii) *$f$-contraction* [2] if for a fixed constant $\lambda \in (0,1)$ and for each $x,y \in X$,

$$H\bigl(T(x),T(y)\bigr) \leq \lambda d\bigl(f(x),f(y)\bigr).$$

Using the concept of Hausdorff metric, Nadler [1] established the following fixed point result for multivalued contraction maps, which in turn is a generalization of the well-known Banach contraction principle.

**Theorem 1.1** [1] *Let $(X,d)$ be a complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a contraction map. Then* $\mathrm{Fix}(T) \neq \emptyset$.

This result has been generalized in many directions. Kaneko [2] extended the corresponding results of Jungck [3], Nadler [1] and others as follows.

**Theorem 1.2** [2] *Let $(X,d)$ be a complete metric space, and let $T : X \to \mathrm{CB}(X)$ be a multivalued $f$-contraction map which commutes with a continuous map $f$. Then $C(f \cap T) \neq \emptyset$.*

This result has been generalized in different directions. For example, see [4–10].

On the other hand, Kada *et al.* [11] introduced the concept of $w$-distance on a metric space as follows:

Let $(X,d)$ be a metric space. A function $\omega : X \times X \to [0,\infty)$ is called a *$w$-distance* on $X$ if it satisfies the following for each $x,y,z \in X$:

($w_1$) $\omega(x,z) \leq \omega(x,y) + \omega(y,z)$;
($w_2$) a map $\omega(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
($w_3$) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z,x) \leq \delta$ and $\omega(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

Note that, in general, for $x,y \in X$, $\omega(x,y) \neq \omega(y,x)$ and not either of the implications $\omega(x,y) = 0 \Leftrightarrow x = y$ necessarily hold. We say the $w$-distance $\omega$ on $X$ is a $w_0$-distance if $x = y$ implies $\omega(x,y) = 0$. Clearly, the metric $d$ is a $w$-distance on $X$. Let $(Y, \|\cdot\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \to [0,\infty)$ defined by $\omega_1(x,y) = \|y\|$ and $\omega_2(x,y) = \|x\| + \|y\|$ for all $x,y \in Y$ are $w$-distances [11]. Many other examples and properties of the $w$-distance can be found in [11, 12].

The following useful lemma concerning a $w$-distance is given in [11].

**Lemma 1.1** [11] *Let $(X,d)$ be a metric space, and let $\omega$ be a $w$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero. Then, for the $w$-distance $\omega$ on $X$, the following hold for every $x,y,z \in X$:*

(a) *if $\omega(x_n,y) \leq \alpha_n$ and $\omega(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $\omega(x,y) = 0$ and $\omega(x,z) = 0$, then $y = z$;*
(b) *if $\omega(x_n,y_n) \leq \alpha_n$ and $\omega(x_n,z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$;*
(c) *if $\omega(x_n,x_m) \leq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(d) *if $\omega(y,x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

For $x \in X$ and $A \in 2^X$, we denote, $\omega(x,A) = \inf_{y\in A} \omega(x,y)$. Now, let $T : X \to \mathrm{Cl}(X)$ be a multivalued map, and let $f : X \to X$ be a single-valued map. We say

(iii) $T$ is *$w$-contractive* [12] if there exist a $w$-distance $\omega$ on $X$ and $\lambda \in (0,1)$ such that for any $x,y \in X$ and $u \in T(x)$, there is $v \in T(y)$ with

$$\omega(u,v) \leq \lambda\omega(x,y).$$

(iv) $T$ is *generalized $f$-contractive* if there exist a $w_0$-distance $\omega$ on $X$ and $\lambda \in (0,1)$ such that for any $x, y \in X$, $u \in T(x)$, there is $v \in T(y)$ with

$$\omega(u,v) \leq \lambda M_f(x,y),$$

where

$$M_f(x,y) = \max\Big\{\omega\big(f(x),f(y)\big), \omega\big(f(x),T(x)\big), \omega\big(f(y),T(y)\big), \frac{1}{2}\big[\omega\big(f(x),T(y)\big) + \omega\big(f(y),T(x)\big)\big]\Big\}.$$

Using the concept of $w$-distance, Suzuki and Takahashi [12] improved Nadler's fixed point result as follows.

**Theorem 1.3** *Let $(X,d)$ be a complete metric space. Then for each $w$-contractive map $T : X \to \mathrm{Cl}(X)$, the set $\mathrm{Fix}(T) \neq \emptyset$.*

This result has been generalized by many authors, for example, see [13–16]. In this paper, first we establish a lemma with respect to a $w$-distance, which is an improved version of the lemma given in [3], and then we prove a key lemma on the existence of an $f$-orbit for generalized $f$-contractive maps. Finally, we present our main results on the existence of coincidence points and common fixed points for generalized $f$-contractive maps not involving the extended Hausdorff metric. As a consequence, we obtain a fixed point result. Our results either generalize or improve a number of known results.

## 2 Results

Using the concept of $w$-distance, first we improve a corresponding result of Jungck [3] as follows.

**Lemma 2.1** *Let $(X,d)$ be a complete metric space with a $w$-distance $\omega$. If there exist a sequence $\{x_n\}$ in $X$ and a constant $\lambda$, $0 < \lambda < 1$, such that for all $n \in \mathbb{N}$,*

$$\omega(x_n, x_{n+1}) \leq \lambda\omega(x_{n-1}, x_n),$$

*then the sequence $\{x_n\}$ converges in $X$.*

*Proof* It is enough to show that $\{x_n\}$ is a Cauchy sequence in $X$. Note that for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\omega(x_n, x_{n+1}) &\leq \lambda\omega(x_{n-1}, x_n)\\ &\leq \lambda^2\omega(x_{n-2}, x_{n-1})\\ &\vdots\\ &\leq \lambda^n\omega(x_0, x_1).\end{aligned}$$

Thus

$$\omega(x_n, x_{n+1}) \leq \lambda^n \omega(x_0, x_1).$$

Consequently, for $m \geq n$, we get

$$\begin{aligned}\omega(x_n, x_m) &\leq \omega(x_n, x_{n+1}) + \omega(x_{n+1}, x_{n+2}) + \cdots + \omega(x_{m-1}, x_m) \\ &\leq \lambda^n \omega(x_0, x_1) + \lambda^{n+1} \omega(x_0, x_1) + \cdots + \lambda^{m-1} \omega(x_0, x_1),\end{aligned}$$

and thus

$$\omega(x_n, x_m) \leq \frac{\lambda^n}{1-\lambda} \omega(x_0, x_1).$$

Since $0 < \lambda < 1$, we have $\lambda^n \to 0$ as $n \to \infty$. And thus by Lemma 1.1, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, the sequence $\{x_n\}$ converges to a point in $X$. □

The following lemma is crucial for our main results.

**Lemma 2.2** *Let $(X, d)$ be a complete metric space, and let $T : X \to \mathrm{Cl}(X)$ be a generalized $f$-contractive map such that $T(X) \subset f(X)$. Then there exists an $f$-orbit $\{x_n\}$ of $T$ at $x_0 \in X$ such that $\{f(x_n)\}$ converges in $X$.*

*Proof* Let $x_0 \in X$ and choose $y_0 \in T(x_0)$. Since $T(x_0) \subset f(X)$, then there exists $x_1 \in X$ such that $f(x_1) = y_0 \in T(x_0)$, and thus, by the definition of $T$, there exists $y_1 \in T(x_1)$ such that

$$\omega\bigl(f(x_1), y_1\bigr) \leq \lambda M_f(x_0, x_1),$$

where $0 < \lambda < 1$. Since $T(x_1) \subset f(X)$, there exists $x_2 \in X$ such that $f(x_2) = y_1 \in T(x_1)$. Thus

$$\omega\bigl(f(x_1), f(x_2)\bigr) \leq \lambda M_f(x_0, x_1).$$

Similarly, using the definition of $T$ and the fact that $T(X) \subset f(X)$, there exists $x_3 \in X$ such that $f(x_3) \in T(x_2)$ and

$$\omega\bigl(f(x_2), f(x_3)\bigr) \leq \lambda M_f(x_1, x_2).$$

Continuing this process, we get a sequence $\{x_n\}$ in $X$ such that for all $n$, $f(x_{n+1}) \in T(x_n)$ and

$$\omega\bigl(f(x_n), f(x_{n+1})\bigr) \leq \lambda M_f(x_{n-1}, x_n),$$

that is,

$$\begin{aligned}\omega\bigl(f(x_n), f(x_{n+1})\bigr) \leq \lambda \max\Bigl\{&\omega\bigl(f(x_{n-1}), f(x_n)\bigr), \omega\bigl(f(x_{n-1}), T(x_{n-1})\bigr), \omega\bigl(f(x_n), T(x_n)\bigr), \\ &\frac{1}{2}\bigl[\omega\bigl(f(x_{n-1}), T(x_n)\bigr) + \omega\bigl(f(x_n), T(x_{n-1})\bigr)\bigr]\Bigr\}.\end{aligned}$$

Note that

$$\begin{aligned}\omega\big(f(x_n),f(x_{n+1})\big) &\le \lambda \max\Big\{\omega\big(f(x_{n-1}),f(x_n)\big),\omega\big(f(x_{n-1}),f(x_n)\big),\omega\big(f(x_n),f(x_{n+1})\big),\\ &\qquad \frac{1}{2}\big[\omega\big(f(x_{n-1}),f(x_{n+1})\big)+\omega\big(f(x_n),f(x_n)\big)\big]\Big\}\\ &= \lambda \max\Big\{\omega\big(f(x_{n-1}),f(x_n)\big),\omega\big(f(x_n),f(x_{n+1})\big),\frac{1}{2}\big[\omega\big(f(x_{n-1}),f(x_{n+1})\big)\big]\Big\},\end{aligned}$$

and we get

$$\omega\big(f(x_n),f(x_{n+1})\big) \le \lambda \max\Big\{\omega\big(f(x_{n-1}),f(x_n)\big),\frac{1}{2}\big[\omega\big(f(x_{n-1}),f(x_{n+1})\big)\big]\Big\}.$$

Also, note that

$$\begin{aligned}\omega\big(f(x_n),f(x_{n+1})\big) &\le \lambda \max\Big\{\omega\big(f(x_{n-1}),f(x_n)\big),\frac{1}{2}\big[\omega\big(f(x_{n-1}),f(x_n)\big)+\omega\big(f(x_n),f(x_{n+1})\big)\big]\Big\}\\ &\le \lambda \max\big\{\big[\omega\big(f(x_{n-1}),f(x_n)\big),\omega\big(f(x_n),f(x_{n+1})\big)\big]\big\}.\end{aligned}$$

Thus, for each $n \in \mathbb{N}$, we get

$$\omega\big(f(x_n),f(x_{n+1})\big) \le \lambda\omega\big(f(x_{n-1}),f(x_n)\big). \tag{1}$$

Since the sequence $\{f(x_n)\}$ is in the complete metric space $X$ satisfying the inequality (1), it follows from Lemma 2.1 that $\{f(x_n)\}$ converges in $X$. □

**Remark 2.1** Since for each $n \in \mathbb{N}$ we have

$$\omega\big(f(x_n),f(x_{n+1})\big) \le \lambda\omega\big(f(x_{n-1}),f(x_n)\big),$$

following the proof of Lemma 2.1, we obtain the following two useful inequalities.

$$\omega\big(f(x_n),f(x_{n+1})\big) \le \lambda^n\omega\big(f(x_0),f(x_1)\big) \tag{2}$$

and for $m \ge n$

$$\omega\big(f(x_n),f(x_m)\big) \le \frac{\lambda^n}{1-\lambda}\omega\big(f(x_0),f(x_1)\big). \tag{3}$$

Without using the extended Hausdorff metric and continuity conditions, we prove a coincidence result which improves many known results including Theorem 1.2 due to [2], Theorem 3.3 in [17] and Theorem 2 in [7].

**Theorem 2.1** *Suppose that all the hypotheses of Lemma* 2.2 *hold. Furthermore, if for every* $y \in X$ *with* $f(y) \notin T(y)$

$$\inf\big\{\omega\big(f(x),y\big)+\omega\big(f(x),T(x)\big) : x \in X\big\} > 0.$$

*Then* $C(f \cap T) \neq \emptyset$.

*Proof* By Lemma 2.2, there exists an $f$-orbit $\{x_n\}$ of $T$ at $x_0 \in X$ such that $\{f(x_n)\}$ converges in $X$. Also note that for each $n \in \mathbb{N}$, we have

$$\omega\bigl(f(x_n), f(x_{n+1})\bigr) \leq \lambda\omega\bigl(f(x_{n-1}), f(x_n)\bigr),$$

where $0 < \lambda < 1$. Let $f(x_n) \to y \in X$. Now since $\omega(f(x_n), \cdot)$ is lower semicontinuous, from Remark 2.1 (2), we have

$$\begin{aligned}\omega\bigl(f(x_n), y\bigr) &\leq \lim_{m\to\infty} \inf \omega\bigl(f(x_n), f(x_m)\bigr) \\ &\leq \frac{\lambda^n}{1-\lambda}\omega\bigl(f(x_0), f(x_1)\bigr).\end{aligned}$$

Since $\lambda < 1$, we get $\omega(f(x_n), y) \to 0$ as $n \to \infty$. Assume that $f(y) \notin T(y)$, then from the hypothesis and Remark 2.1, we get

$$\begin{aligned}0 &< \inf\bigl\{\omega\bigl(f(x), y\bigr) + \omega\bigl(f(x), T(x)\bigr) : x \in X\bigr\} \\ &\leq \inf\bigl\{\omega\bigl(f(x_n), y\bigr) + \omega\bigl(f(x_n), T(x_n)\bigr) : n \in \mathbb{N}\bigr\} \\ &\leq \inf\bigl\{\omega\bigl(f(x_n), y\bigr) + \omega\bigl(f(x_n), f(x_{n+1})\bigr) : n \in \mathbb{N}\bigr\} \\ &\leq \inf\left\{\frac{\lambda^n}{1-\lambda}\omega\bigl(f(x_o), f(x_1)\bigr) + \lambda^n\omega\bigl(f(x_0), f(x_1)\bigr) : n \in \mathbb{N}\right\} \\ &= \left\{\frac{2-\lambda}{1-\lambda}\right\}\omega\bigl(f(x_0), f(x_1)\bigr) \inf\bigl\{\lambda^n : n \in \mathbb{N}\bigr\} = 0,\end{aligned}$$

which is impossible, and thus $f(y) \in T(y)$, that is, $y$ is a coincidence point of $f$ and $T$. □

If we take $f = I$ (an identity map on $X$) in Theorem 2.1, we obtain the following improved version of the corresponding fixed point results in [12, 17, 18].

**Corollary 2.1** *Let $(X, d)$ be a complete metric space, let $\omega$ be a w-distance on $X$, and let $T : X \to \mathrm{Cl}(X)$ be a multivalued map satisfying the following*:

(I) *for fixed $\lambda \in (0, 1)$, for each $x, y \in X$ and $u \in T(x)$, there exists $v \in T(y)$ such that*

$$\omega(u, v) \leq \lambda M_\omega(x, y),$$

*where*

$$M_\omega(x, y) = \max\left\{\omega(x, y), \omega\bigl(x, T(x)\bigr), \omega\bigl(y, T(y)\bigr), \frac{1}{2}\bigl[\omega\bigl(x, T(y)\bigr) + \omega\bigl(y, T(x)\bigr)\bigr]\right\},$$

(II) $\inf\{\omega(x, y) + \omega(x, T(x)) : x \in X\} > 0$.

*Then* $\mathrm{Fix}(T) \neq \emptyset$.

Finally, we obtain a common fixed point result.

**Theorem 2.2** *Suppose that all the hypotheses of Theorem* 2.1 *hold. Further, if the maps $f$ and $T$ commute weakly and satisfy the condition that $f(x) \neq f^2(x)$, which implies $f(x) \notin T(x)$, then $f$ and $T$ have a common fixed point.*

*Proof* From Theorem 2.1 we have $f(y) \in T(y)$, and thus we get $f(y) = f^2(y)$. Note that

$$f(y) = f\big(f(y)\big) \in f\big(T(y)\big) \subseteq T\big(f(y)\big),$$

that is, $f(y)$ is a fixed point of $T$. Also note that $f(y)$ is a fixed point of $f$ and thus $f(y)$ is a common fixed point of $T$ and $f$. □

**Competing interests**
The author declares that he has no competing interests.

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