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Convergence analysis of an iterative scheme for Lipschitzian hemicontractive mappings in Hilbert spaces

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Abstract
In this paper, we establish strong convergence for the iterative scheme introduced by Sahu and Petruşel associated with Lipschitzian hemicontractive mappings in Hilbert spaces.
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1 Introduction

Let H be a Hilbert space, and let $T : H \to H$ be a mapping. The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in H.$$

If $L = 1$, then T is called *nonexpansive* and if $0 \le L < 1$, then T is called *contractive*.

The mapping $T : H \to H$ is said to be *pseudocontractive* (see, for example, [1, 2]) if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in H \tag{1.1}$$

and it is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Let $F(T) := \{x \in H : Tx = x\}$, and let K be a nonempty subset of H. A mapping $T : K \to K$ is called *hemicontractive* if $F(T) \neq \emptyset$ and

$$\left\|Tx - x^*\right\|^2 \le \left\|x - x^*\right\|^2 + \|x - Tx\|^2, \quad \forall x \in H, x^* \in F(T).$$

It is easy to see that the class of pseudocontractive mappings with fixed points is a subclass of the class of hemicontractions. For the importance of fixed points of pseudocontractions, the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

Theorem 1.1 *Let K be a compact convex subset of a Hilbert space H, and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping.*

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by the Ishikawa iterative scheme

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Another iterative scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings is the S-iterative scheme introduced by Sahu and Petrușel [4] in 2011.

In this paper, we establish strong convergence for the S-iterative scheme associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

2 Main results

We need the following lemma.

Lemma 2.1 [5] *For all $x, y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds*:

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Now we prove our main results.

Theorem 2.2 *Let K be a compact convex subset of a real Hilbert space H, and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying*

$$\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K. \tag{C}$$

Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying

(iv) $\sum_{n=1}^{\infty} \beta_n = \infty$;
(v) $\lim_{n\to\infty} \beta_n = 0$.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by the S-iterative scheme

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1. \end{cases} \tag{2.1}$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Proof From Schauder's fixed point theorem, $F(T)$ is nonempty since K is a compact convex set and T is continuous. Let $x^* \in F(T)$. Using the fact that T is hemicontractive, we obtain

$$\|Tx_n - x^*\|^2 \le \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2 \tag{2.2}$$

and

$$\|Ty_n - x^*\|^2 \le \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2. \tag{2.3}$$

With the help of (2.1), (2.2) and Lemma 2.1, we obtain the following estimates:

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\|^2 \\
&= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|^2 \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2) \\
&\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - x^*\|^2 + \beta_n^2\|x_n - Tx_n\|^2,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\|y_n - Ty_n\|^2 &= \|(1-\beta_n)x_n + \beta_n Tx_n - Ty_n\|^2 \\
&= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\|^2 \\
&= (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3) we obtain

$$\begin{aligned}
\|Ty_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 + (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
&- \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{2.6}$$

Also, with the help of condition (C) and (2.6), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|Ty_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
&\quad - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&\le \|x_n - x^*\|^2 + \|Tx_n - Ty_n\|^2 - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&\le \|x_n - x^*\|^2 + L^2\|x_n - y_n\|^2 - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - x^*\|^2 + L^2\beta_n^2\|x_n - Tx_n\|^2 - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\
&= \|x_n - x^*\|^2 - \beta_n(1-(2+L^2)\beta_n)\|x_n - Tx_n\|^2.
\end{aligned} \tag{2.7}$$

Now, by $\lim_{n\to\infty}\beta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\beta_n \le \frac{1}{2(2+L^2)}, \tag{2.8}$$

and with the help of (2.8), (2.7) yields

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \frac{1}{2}\beta_n\|x_n - Tx_n\|^2,$$

which implies

$$\frac{1}{2}\beta_n\|x_n - Tx_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

so that

$$\frac{1}{2}\sum_{j=N}^{n}\beta_j\|x_j - Tx_j\|^2 \leq \|x_N - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

The rest of the argument follows exactly as in the proof of theorem of [3]. This completes the proof. □

Theorem 2.3 *Let K be a compact convex subset of a real Hilbert space H, and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying condition* (C). *Let $\{\beta_n\}$ be a sequence in* $[0,1]$ *satisfying conditions* (iv) *and* (v).

Assume that $P_K : H \to K$ is the projection operator of H onto K. Let $\{x_n\}$ be a sequence defined iteratively by

$$\begin{cases} x_{n+1} = P_K(Ty_n), \\ y_n = P_K((1-\beta_n)x_n + \beta_n Tx_n), \quad n \geq 1. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof The operator P_K is nonexpansive (see, *e.g.*, [2]). K is a Chebyshev subset of H so that P_K is a single-valued mapping. Hence, we have the following estimate:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_K(Ty_n) - P_K x^*\|^2 \\ &\leq \|Ty_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \beta_n\bigl(1 - \bigl(2 + L^2\bigr)\beta_n\bigr)\|x_n - Tx_n\|^2. \end{aligned}$$

The set $K = K \cup T(K)$ is compact, and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof. □

Remark 2.4 In Theorem 1.1, putting $\alpha_n = 1$, $0 \leq \alpha_n \leq \beta_n \leq 1$ implies $\beta_n = 1$, which contradicts $\lim_{n\to\infty}\beta_n = 0$. Hence the S-iterative scheme is not the special case of Ishikawa iterative scheme.

Remark 2.5 In Theorems 2.2 and 2.3, condition (C) is not new; it is due to Liu *et al.* [6].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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