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Probability inequality concerning a chirp-type signal

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Abstract

Probability inequalities of random variables play important roles, especially in the theory of limiting theorems of sums of independent random variables. In this note, we give a proof of the following:

Let $\{\epsilon_t\}$ be a sequence of independent random variables such that $E(\epsilon_t) = 0$, $E(\epsilon_t^2) = v < \infty$ and $E(\epsilon_t^4) = \sigma < \infty$. Then

$$\max_{\substack{0 \le \omega < \pi \\ 0 \le \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{j(\omega t + \alpha t^2)} \right|^2 = O_p(n^{\beta + \frac{7}{8}}).$$

It is shown that this result will be useful in estimating the parameters of a chirp-type statistical model and in establishing the consistency of estimators.

Keywords: probability inequality; independent random variables; chirp-type signal; consistency

1 Introduction

In [1] Whittle (1952) considered the problem of estimating the parameters of a sine wave:

$$X_t = A\cos\omega t + B\sin\omega t + \epsilon_t,$$

where X_t's are the observations and ϵ_t's are independent, identically distributed random variables with mean zero and finite(unknown) variance. Whittle's solution to the problem of estimating the parameters and the proof of consistency of parameters of the above sine wave used arguments which are not mathematically rigorous. In 1973, a rigorous solution to Whittle's problem was given by Walker [2]. In his proof of consistency of parameters, he used the following O_p result:

$$\max_{0 \le \omega \le \pi} \left| \sum_{t=1}^{n} Y_t e^{i\omega t} \right| = O_p(n^{\frac{3}{4}}),$$

where $\{Y_t\}$ is a linear process.

In this paper, we extend the above O_p result to obtain

$$\max_{\substack{0 \le \omega < \pi \\ 0 \le \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 = O_p(n^{\beta + \frac{7}{8}}). \tag{1}$$

![Springer logo]

We came across this problem while attempting to establish the consistency of the parameters of the model

$$X_t = A\cos(\omega t + \alpha t^2) + B\sin(\omega t + \alpha t^2) + \epsilon_t,$$

where X_t's and ϵ_t's are as mentioned above. Models of this type are referred to as 'chirp' models [3, 4] and [5], and they have drawn the attention of many researchers [6, 7] and [8]. Although we tried to find our main result given in (1) in the literature, we were unable to find one. We make use of the following basic results to establish our main result.

2 Basic results

Definition 1 Let $\{X_n\}$ be a sequence of random variables. We say that $X_n = O_p(1)$ if for given $\epsilon > 0$, there exists M such that

$$P\big[|X_n| > M\big] < \epsilon \quad \text{for all } n.$$

Let (a_n) be a sequence of nonzero real numbers. We say that $X_n = O_p(a_n)$ if $X_n/a_n = O_p(1)$.

Lemma 1 *Let $\{X_n\}$ be a sequence of random variables, and let $\{a_n\}$ and $\{b_n\}$ be two sequence of nonzero real numbers. Then*

$$X_n = O_p(b_n) \quad \Rightarrow \quad a_n X_n = O_p(a_n b_n).$$

Lemma 2 *Let $\{X_n\}$ be a sequence of random variables, let $\{a_n\}$ be a sequence of real numbers, and let q be a positive real number. Then*

$$X_n = O_p(a_n) \quad \Rightarrow \quad X_n^q = O_p(a_n^q).$$

Definition 2 Let $\{X_n\}$ be a sequence of random variables. We say that $X_n = o_p(1)$ if $X_n \to 0$ in probability. (So, $X_n = o_p(1)$ if for each $\epsilon \geq 0$, we have $\lim_{n\to\infty} P(|X_n| > \epsilon) = 0$.)

Lemma 3 *Let $\{X_n\}$ be a sequence of random variables, and let $\{a_n\}$ be a sequence of real numbers with $a_n \to 0$ as $n \to \infty$. If $X_n = O_p(1)$, then $a_n X_n \xrightarrow{P} 0$.*

Lemma 4 *Let $\{X_n\}$ be a sequence of random variables, and let β be a positive real number. Then*

$$X_n = O_p\left(\frac{1}{n^\beta}\right) \quad \Rightarrow \quad X_n = o_p(1).$$

Lemma 5 *Let $\{X_n\}$ be a sequence of random variables, and let β be any real number. If $X_n = O_p(n^\beta)$ and $Y_n = O_p(n^\beta)$, then $X_n + Y_n = O_p(n^\beta)$.*

Lemma 6 *Let μ be any real number. Let $\{X_n\}$ be a sequence of random variables such that $E(|X_n|) = O(n^\mu)$. Then $X_n = O(n^\mu)$.*

We now establish two lemmas that we need to prove the main result.

Lemma 7 *Let $a_{t,u}$; $t = 1, 2, \ldots, n$, $u = 1, 2, \ldots, n$ be complex numbers. Then*

$$\sum_{t=1}^{n} \sum_{u=1}^{n} a_{t,u} = \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} [a_{t,t+s} + a_{t+s,t}] + \sum_{t=1}^{n} a_{t,t}.$$

Proof The $\sum_{t=1}^{n} \sum_{u=1}^{n} a_{t,u}$ is the sum of the terms in the table:

$$
\begin{array}{cccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n}.
\end{array}
$$

We can add by summing along the diagonals. So, the sum is equal to

$$\sum_{s=1}^{n-1} \sum_{t=1}^{n-s} a_{t,t+s} + \sum_{p=1}^{n-1} \sum_{u=1}^{n-p} a_{u+p,u} + \sum_{t=1}^{n} a_{t,t}.$$

Therefore

$$\sum_{t=1}^{n} \sum_{u=1}^{n} a_{t,u} = \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} [a_{t,t+s} + a_{t+s,t}] + \sum_{t=1}^{n} a_{t,t}. \qquad \square$$

Lemma 8 *Let b_t; $t = 1, 2, \ldots, n$ be complex numbers. Then*

$$\left| \sum_{t=1}^{n} b_t \right|^2 \le 2 \left| \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} b_t \bar{b}_{t+s} \right| + \sum_{t=1}^{n} |b_t|^2.$$

Proof Since $|\sum_{t=1}^{n} b_t|^2 = \sum_{t=1}^{n} \sum_{u=1}^{n} b_t \bar{b}_u$, by using $a_{t,u} = b_t \bar{b}_u$ in Lemma 7, then

$$\left| \sum_{t=1}^{n} b_t \right|^2 = \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} [b_t \bar{b}_{t+s} + b_{t+s} \bar{b}_t] + \sum_{t=1}^{n} b_t \bar{b}_t$$

$$= \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} 2 \Re b_t \bar{b}_{t+s} + \sum_{t=1}^{n} |b_t|^2$$

$$\le 2 \left| \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} b_t \bar{b}_{t+s} \right| + \sum_{t=1}^{n} |b_t|^2. \qquad \square$$

3 Main result

Theorem 1 *Let $\{\epsilon_t\}$ be a sequence of independent random variables such that $E(\epsilon_t) = 0$, $E(\epsilon_t^2) = v < \infty$ and $E(\epsilon_t^4) = \sigma < \infty$. Then*

$$\max_{\substack{0 \le \omega < \pi \\ 0 \le \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 = O_p\left(n^{\beta + \frac{7}{8}}\right).$$

Proof Letting $b_t = t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)}$ in Lemma 8, we see that

$$\left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \leq 2 \left| \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{i(\omega t + \alpha t^2)} e^{-i(\omega(t+s) + \alpha(t+s)^2)} \right|$$

$$+ \sum_{t=1}^{n} \left| t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2$$

$$= 2 \left| \sum_{s=1}^{n-1} \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-i\omega s} e^{-i\alpha s^2} e^{-2i\alpha ts} \right|$$

$$+ \sum_{t=1}^{n} t^{2\beta} \epsilon_t^2.$$

Using the triangle inequality, we obtain

$$\left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \leq 2 \sum_{s=1}^{n-1} \left| \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right| + \sum_{t=1}^{n} t^{2\beta} \epsilon_t^2. \tag{2}$$

Fix s and consider $\left| \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right|$. By Lemma 8 with $b_t = t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts}$, we obtain

$$\left| \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right|^2$$

$$\leq 2 \left| \sum_{p=1}^{n-s-1} \sum_{t=1}^{n-s-p} t^\beta (t+s)^\beta (t+p)^\beta (t+p+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \epsilon_{t+p} \epsilon_{t+p+s} e^{2i\alpha s(t+p)} \right|$$

$$+ \sum_{t=1}^{n-s} \left| t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right|^2,$$

and it follows from the triangle inequality that

$$\left| \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right|^2$$

$$\leq 2 \sum_{p=1}^{n-s-1} \left| \sum_{t=1}^{n-s-p} t^\beta (t+s)^\beta (t+p)^\beta (t+p+s)^\beta \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| + \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t^2 \epsilon_{t+s}^2.$$

Therefore

$$\left| \sum_{t=1}^{n-s} t^\beta (t+s)^\beta \epsilon_t \epsilon_{t+s} e^{-2i\alpha ts} \right|^2$$

$$\leq \sqrt{2} \left[\sum_{p=1}^{n-s-1} \left| \sum_{t=1}^{n-s-p} t^\beta (t+s)^\beta (t+p)^\beta (t+p+s)^\beta \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| \right.$$

$$\left. + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} \epsilon_t^2 \epsilon_{t+s}^2 \right]^{1/2}.$$

Substituting this in equation (2), we obtain

$$
\left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2
$$

$$
\leq 2 \sum_{s=1}^{n-1} \left[2 \sum_{p=1}^{n-s-1} \left| \sum_{t=1}^{n-s-p} t^{\beta}(t+s)^{\beta}(t+p)^{\beta}(t+p+s)^{\beta} \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| \right.
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta}(t+s)^{2\beta} \epsilon_t^2 \epsilon_{t+s}^2 \right]^{1/2} + \sum_{t=1}^{n} t^{2\beta} \epsilon_t^2 .
$$

It follows that

$$
\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2
$$

$$
\leq 2\sqrt{2} \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left| \sum_{t=1}^{n-s-p} t^{\beta}(t+s)^{\beta}(t+p)^{\beta}(t+p+s)^{\beta} \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| \right.
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta}(t+s)^{2\beta} \epsilon_t^2 \epsilon_{t+s}^2 \right]^{1/2} + \sum_{t=1}^{n} t^{2\beta} \epsilon_t^2 .
$$

Taking the expectation of both sides of the above inequality, we obtain

$$
E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right]
$$

$$
\leq 2\sqrt{2} \sum_{s=1}^{n-1} E\left[\sum_{p=1}^{n-s-1} \left| \sum_{t=1}^{n-s-p} t^{\beta}(t+s)^{\beta}(t+p)^{\beta}(t+p+s)^{\beta} \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| \right.
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta}(t+s)^{2\beta} \epsilon_t^2 \epsilon_{t+s}^2 \right]^{1/2} + \sum_{t=1}^{n} t^{2\beta} E(\epsilon_t^2) .
$$

We claim that

$$
E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right] = O(n^{2\beta + 7/4}) .
$$

We note that since

$$
\sum_{t=1}^{n} t^{2\beta} E(\epsilon_t^2) = \sum_{t=1}^{n} t^{2\beta} \nu = O(n^{2\beta + 1}) \leq O(n^{2\beta + 7/4}),
$$

we can ignore the term $\sum_{t=1}^{n} t^{2\beta} E(\epsilon_t^2)$ provided, and we show the other term is $O(n^{2\beta + 7/4})$.

Taking $x = |z|^{1/2}$ in Schwarz's inequality $E|x| \leq (E|x|^2)^{1/2}$ yields $E(|z|^{1/2}) \leq (E|z|)^{1/2}$. Using this on the right-hand side of the above inequality, we get

$$
E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right]
$$

$$
\leq \sum_{s=1}^{n-1} \left\{ \sum_{p=1}^{n-s-1} E \left| \sum_{t=1}^{n-s-p} t^\beta (t+s)^\beta (t+p)^\beta (t+p+s)^\beta \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s} \right| \right.
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E\left(\epsilon_t^2 \epsilon_{t+s}^2\right) \right\}^{1/2}.
$$

Using Schwarz's inequality again on the first term on the right, we get

$$
E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right]
$$

$$
\leq \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left[E\left\{ \sum_{t=1}^{n-s-p} t^\beta (t+s)^\beta (t+p)^\beta (t+p+s)^\beta \epsilon_t \epsilon_{t+s} \epsilon_{t+p} \epsilon_{t+p+s}^2 \right\}^2 \right]^{1/2} \right.
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E\left(\epsilon_t^2 \epsilon_{t+s}^2\right) \right]^{1/2}
$$

$$
= \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left[E\left(\sum_{t=1}^{n-s-p} t^{2\beta} (t+s)^\beta (t+p)^{2\beta} (t+p+s)^\beta \epsilon_t^2 \epsilon_{t+s}^2 \epsilon_{t+p}^2 \epsilon_{t+p+s}^2 \right) \right. \right.
$$

$$
+ E\left(2 \sum_{\substack{u,v=1 \\ u<v}}^{n-s-p} u^\beta (u+s)^\beta (u+p)^\beta (u+p+s)^\beta v^\beta (v+s)^\beta (v+p)^\beta (v+p+s)^\beta \right.
$$

$$
\left. \left. \left. \epsilon_u \epsilon_{u+s} \epsilon_{u+p} \epsilon_{u+p+s} \epsilon_v \epsilon_{v+s} \epsilon_{v+p} \epsilon_{v+p+s} \right) \right]^{1/2} + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E\left(\epsilon_t^2 \epsilon_{t+s}^2\right) \right]^{1/2}.
$$

So,

$$
E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right]
$$

$$
\leq \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left[\sum_{t=1}^{n-s-p} t^{2\beta} (t+s)^{2\beta} (t+p)^{2\beta} (t+p+s)^{2\beta} E\left(\epsilon_t^2 \epsilon_{t+s}^2 \epsilon_{t+p}^2 \epsilon_{t+p+s}^2\right) \right. \right.
$$

$$
+ 2 \left(\sum_{\substack{u,v=1 \\ u<v}}^{n-s-p} u^\beta (u+s)^\beta (u+p)^\beta (u+p+s)^\beta v^\beta (v+s)^\beta (v+p)^\beta (v+p+s)^\beta \right.
$$

$$
\left. \left. E(\epsilon_u \epsilon_{u+s} \epsilon_{u+p} \epsilon_{u+p+s} \epsilon_v \epsilon_{v+s} \epsilon_{v+p} \epsilon_{v+p+s}) \right) \right]^{1/2}
$$

$$
\left. + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E\left(\epsilon_t^2 \epsilon_{t+s}^2\right) \right]^{1/2}.
$$

Consider $E(\epsilon_u \epsilon_{u+s} \epsilon_{u+p} \epsilon_{u+p+s} \epsilon_v \epsilon_{v+s} \epsilon_{v+p} \epsilon_{v+p+s})$. Since $u \leq v$, $v + p + s$ is the unique largest subscript, the ϵ_t's are independent and $E(\epsilon_t) = 0$, we have

$$E(\epsilon_u \epsilon_{u+s} \epsilon_{u+p} \epsilon_{u+p+s} \epsilon_v \epsilon_{v+s} \epsilon_{v+p} \epsilon_{v+p+s}) = E(\epsilon_u \epsilon_{u+s} \epsilon_{u+p} \epsilon_{u+p+s} \epsilon_v \epsilon_{v+s} \epsilon_{v+p}) E(\epsilon_{v+p+s}) = 0.$$

Therefore

$$E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right]$$

$$\leq \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left[\sum_{t=1}^{n-s-p} t^{2\beta} (t+s)^{2\beta} (t+p)^{2\beta} (t+p+s)^{2\beta} E(\epsilon_t^2 \epsilon_{t+s}^2 \epsilon_{t+p}^2 \epsilon_{t+p+s}^2) \right]^{1/2} \right.$$

$$\left. + \sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E(\epsilon_t^2 \epsilon_{t+s}^2) \right]^{1/2}.$$

Now consider the term $E(\epsilon_t^2 \epsilon_{t+s}^2)$. Since $E(\epsilon_t^2) = \nu \leq \infty$, and the ϵ_t's are independent,

$$\sum_{t=1}^{n-s} t^{2\beta} (t+s)^{2\beta} E(\epsilon_t^2 \epsilon_{t+s}^2) \leq K n^{4\beta+1}.$$

Similarly, since $E(\epsilon_t^2) = \nu \leq \infty$, $E(\epsilon_t^4) = \sigma \leq \infty$, and the ϵ_t's are independent,

$$\sum_{t=1}^{n-s-p} t^{2\beta} (t+s)^{2\beta} (t+p)^{2\beta} (t+p+s)^{2\beta} E(\epsilon_t^2 \epsilon_{t+s}^2 \epsilon_{t+p}^2 \epsilon_{t+p+s}^2) = K n^{8\beta+1}.$$

Thus

$$E\left[\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 \right] \leq \sum_{s=1}^{n-1} \left[\sum_{p=1}^{n-s-1} \left[K n^{8\beta+1} \right]^{1/2} + K n^{4\beta+1} \right]^{1/2}$$

$$= \sum_{s=1}^{n-1} \left[n K n^{4\beta+1/2} + K n^{4\beta+1} \right]^{1/2}$$

$$= \sum_{s=1}^{n-1} \left[K n^{4\beta+3/2} \right]^{1/2}$$

$$= n K n^{2\beta+3/4}$$

$$= O(n^{2\beta+7/4}).$$

It follows by virtue of Lemma 6 that

$$\max_{\substack{0 \leq \omega < \pi \\ 0 \leq \alpha < \pi}} \left| \sum_{t=1}^{n} t^{\beta} \epsilon_t e^{i(\omega t + \alpha t^2)} \right|^2 = O_p(n^{2\beta+7/4}).$$

Using Lemma 3, we obtain

$$\max_{\substack{0 \le \omega < \pi \\ 0 \le \alpha < \pi}} \left| \sum_{t=1}^{n} t^\beta \epsilon_t e^{i(\omega t + \alpha t^2)} \right| = O_p\big(n^{\beta+7/8}\big),$$

which completes the proof of the theorem. □

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