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A sum analogous to the high-dimensional Kloosterman sums and its upper bound estimate

Yijun Li[1,2] and Di Han[1*]

*Correspondence: handi515@.com
[1]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China
Full list of author information is available at the end of the article

Abstract
The main purpose of this paper is, using the properties of Gauss sums and the estimate for the generalized exponential sums, to study the upper bound estimate problem of one kind sums analogous to the high-dimensional Kloosterman sums and to give some interesting mean value formula and an upper bound estimate for it.

MSC: 11L05

Keywords: a sum analogous to the high-dimensional Kloosterman sums; Gauss sums; upper bound estimate; mean value

1 Introduction

For any integer $q \geq 3$, the high-dimensional Kloosterman sums $K(c_1, c_2, \ldots, c_k, m; q)$ are defined as follows:

$$K(c_1, c_2, \ldots, c_k, m; q) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' e\left(\frac{c_1 a_1 + \cdots + c_k a_k + m\overline{a}_1 \cdots \overline{a}_k}{q}\right),$$

where $e(x) = e^{2\pi i x}$, $\sum_{a_i=1}^{\prime q}$ denotes the summation over all integers $1 \leq a_i \leq q$ such that $(a_i, q) = 1$, c_i and m are integers with $(m, q) = 1$, $\overline{a}_i$ denotes the solution of the congruent equation $x \cdot a_i \equiv 1 \bmod q$ $(i = 1, 2, \ldots, k)$.

There are several results on the properties of the Kloosterman sums $K(c_1, c_2, \ldots, c_k, m; q)$. For example, see [1, 2] and [3]. Related works can also be found in [4–8] and [9].

In this paper, we consider a sum analogous to the high-dimensional Kloosterman sums as follows:

$$S(c_1, c_2, \ldots, c_k, m, \chi; q) = \sum_{a_1=1}^{q}{}' \cdots \sum_{a_k=1}^{q}{}' \chi(c_1 a_1 + \cdots + c_k a_k + m\overline{a}_1 \cdots \overline{a}_k), \tag{1.1}$$

where χ is a Dirichlet character mod$q$.

If $k = 1$ and $q = p$ (an odd prime), then for any integer a with $(a, p) = 1$, applying the Fermat little theorem, one can deduce $a^{p-2} \equiv \overline{a} \bmod p$. So, the sum (1.1) becomes

$$\sum_{a=1}^{p-1} \chi(ca + m\overline{a}).$$

It is a special case of the general polynomial character sums

$$\sum_{a=N+1}^{N+M} \chi\bigl(f(a)\bigr),$$

where M and N are any positive integers and $f(x)$ is a polynomial. Let χ be a qth-order character mod$p$. If $f(x)$ is not a perfect qth power mod$p$, then from Weil's classical work (see [10]), we can deduce the estimate

$$\sum_{x=N+1}^{N+M} \chi\bigl(f(x)\bigr) \ll p^{\frac{1}{2}} \ln p,$$

where '$\ll$' constant depends only on the degree of $f(x)$. Some related results can also be found in [11–13] and [14].

Now we are concerned with the upper bound estimate problem of (1.1). Regarding this contents, it seems that nobody has yet studied it, at least we have not seen any related result before. The problem is interesting because it can reflect some new properties of character sums. The main purpose of this paper is, using the analytic methods and the properties of Gauss sums, to study this problem and give a sharp upper bound estimate for (1.1). That is, we prove the following conclusions.

Theorem 1 *Let p be an odd prime, let k be a positive integer with $k \geq 2$, and let χ be any non-principal character* mod$p$. *Then for any integers $c_1, c_2, \ldots, c_k$ and m with $(c_1 c_2 \cdots c_k m, p) = 1$, we have the identity*

$$\bigl|S(c_1, c_2, \ldots, c_k, m, \chi; p)\bigr| = \begin{cases} p^{\frac{k}{2}} & \text{if } (k+1, p-1) = 1, \\ 0 & \text{if } (k+1) | (p-1) \text{ and } \chi^{\frac{p-1}{k+1}} \neq \chi_0, \end{cases}$$

where χ_0 denotes the principal character mod$p$.

Theorem 2 *Let p be an odd prime, let k be a positive integer with $k \geq 2$, and let χ be any non-principal character* mod$p$. *Then for any integers $c_1, c_2, \ldots, c_k$ and m with $(c_1 c_2 \cdots c_k m, p) = 1$, we have the estimate*

$$\bigl|S(c_1, c_2, \ldots, c_k, m, \chi; p)\bigr| \leq (k+1) \cdot p^{\frac{k}{2}}.$$

Theorem 3 *Let p and q be two odd primes, let r be any qth non-residue* mod$p$. *Then for any integers $c_1, c_2, \ldots, c_{q-1}$ with $(c_1 c_2 \cdots c_{q-1}, p) = 1$, we have the identity*

$$\sum_{i=0}^{q-1} \left[\sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1 a_1 + c_2 a_2 + \cdots + c_{q-1} a_{q-1} + r^i \overline{a}_1 \overline{a}_2 \cdots \overline{a}_{q-1}}{p} \right) \right]^2$$
$$= \begin{cases} q^2 \cdot p^{q-1} & \text{if } q | (p-1), \\ q \cdot p^{q-1} & \text{if } (q, p-1) = 1. \end{cases}$$

If $p \equiv 1$ mod 4, then the above formula also holds for $q = 2$, where $(\frac{*}{p})$ denotes the Legendre symbol.

Taking $p = 4m + 1$ and $q = 2$ in Theorem 3, note that $2|(p-1)$, we may immediately deduce the following.

Corollary *Let p be an odd prime with* $p \equiv 1 \bmod 4$, *then we have the identity*

$$p = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + r\overline{a}}{p}\right)\right)^2 + \left(\sum_{b=1}^{\frac{p-1}{2}} \left(\frac{b + s\overline{b}}{p}\right)\right)^2,$$

where r and s are any two integers such that $(\frac{r}{p}) \cdot (\frac{s}{p}) = -1$.

This gives another proof for a classical work in elementary number theory (*i.e.*, see [15] Theorems 4-11): For any prime p with $p \equiv 1 \bmod 4$, there exist two positive integers x and y such that $p = x^2 + y^2$.

2 Several lemmas

To complete the proof of our theorems, we need the following basic lemmas.

Lemma 1 *Let p be an odd prime, let* χ *be any non-principal character* mod$p$, *and let k be any positive integer such that* $(k, p-1) = 1$ *or* $k|p-1$. *Then for any integer m with* $(m,p) = 1$, *we have the identity*

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = \begin{cases} \overline{\chi}^r(m) \cdot \tau(\chi^r) & \text{if } (k, p-1) = 1, \\ 0 & \text{if } k|(p-1) \text{ and } \chi^{\frac{p-1}{k}} \neq \chi_0, \\ \overline{\chi}_1(m) \cdot \sum_{i=0}^{k-1} \overline{\chi}_k^i(m) \tau(\chi_1 \chi_k^i) & \text{if } k|(p-1) \text{ and } \chi^{\frac{p-1}{k}} = \chi_0, \end{cases}$$

where $r \cdot k \equiv 1 \bmod (p-1)$, χ_0 *denotes the principal character* mod$p$, χ_k *denotes any k-order character* mod$p$ *and* $\chi_1^k = \chi$.

Proof If $(k, p-1) = 1$, then there exists one integer r with $(r, p-1) = 1$ such that $r \cdot k \equiv 1 \bmod (p-1)$. This time, for any integer a with $(a,p) = 1$, we have $a^{rk} \equiv a \bmod p$. If a passes through a reduced residue system mod$p$, then a^r also passes through a reduced residue system mod$p$. Therefore, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi(a^r) e\left(\frac{ma^{rk}}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi^r(a) e\left(\frac{ma}{p}\right) = \overline{\chi}^r(m) \cdot \tau(\chi^r). \end{aligned} \tag{2.1}$$

If $k > 1$ and $k|(p-1)$ with $\chi^{\frac{p-1}{k}} \neq \chi_0$, then there must exist an integer n with $(n,p) = 1$ such that $\chi^{\frac{p-1}{k}}(n) \neq 1$. For this n, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi\left(a \cdot n^{\frac{p-1}{k}}\right) e\left(\frac{m(a \cdot n^{\frac{p-1}{k}})^k}{p}\right) \\ &= \chi\left(n^{\frac{p-1}{k}}\right) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k n^{p-1}}{p}\right) = \chi^{\frac{p-1}{k}}(n) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) \end{aligned}$$

or

$$\big(1 - \chi^{\frac{p-1}{k}}(n)\big) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = 0.$$

Since $\chi^{\frac{p-1}{k}}(n) \neq 1$, from the above identity, we have

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) = 0. \tag{2.2}$$

If $\chi^{\frac{p-1}{k}} = \chi_0$, then χ must be a kth character $\bmod p$, so there exists one character $\chi_1 \bmod p$ such that $\chi = \chi_1^k$. Let χ_k be a k-order character $\bmod p$ (*i.e.*, $\chi_k^k = \chi_0$), then for any integer a with $(a,p) = 1$, note that

$$1 + \chi_k(a) + \chi_k^2(a) + \cdots + \chi_k^{k-1}(a) = \begin{cases} k & \text{if } a \text{ is a } k\text{th residue } \bmod p, \\ 0, & \text{otherwise.} \end{cases}$$

From the properties of Gauss sums, we have

$$\begin{aligned} \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right) &= \sum_{a=1}^{p-1} \chi_1^k(a) e\left(\frac{ma^k}{p}\right) = \sum_{a=1}^{p-1} \chi_1\big(a^k\big) e\left(\frac{ma^k}{p}\right) \\ &= \sum_{a=1}^{p-1} \chi_1(a)\big(1 + \chi_k(a) + \chi_k^2(a) + \cdots + \chi_k^{k-1}(a)\big) e\left(\frac{ma}{p}\right) \\ &= \overline{\chi}_1(m) \cdot \sum_{i=0}^{k-1} \overline{\chi}_k^i(m) \tau\big(\chi_1 \chi_k^i\big). \end{aligned} \tag{2.3}$$

Now Lemma 1 follows from (2.1), (2.2) and (2.3). □

Lemma 2 *Let p and q be two odd primes with $q|(p-1)$, and let χ_q be any q-order character* $\bmod p$. *Then for any integers $c_1, c_2, \ldots, c_{q-1}$ and m with $(mc_1c_2 \cdots c_{q-1}, p) = 1$, we have the identities*

$$\text{(I)} \quad \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{q-1}=1}^{p-1} \left(\frac{c_1a_1 + c_2a_2 + \cdots + c_{q-1}a_{q-1} + m\overline{a}_1\overline{a}_2 \cdots \overline{a}_{q-1}}{p}\right) = \left(\frac{n}{p}\right) \cdot \tau^{q-1}(\chi_2) \left(\sum_{i=0}^{q-1} \chi_q^i(n) \frac{\tau^q(\chi_2\chi_q^i)}{\tau^q(\chi_2)}\right);$$

$$\text{(II)} \quad \sum_{a=1}^{p-1} \left(\frac{a + m\overline{a}}{p}\right) = \begin{cases} \dfrac{\overline{\chi}_1(m)}{\tau(\chi_2)}\big(\tau^2(\chi_1) + (\dfrac{m}{p})\tau^2(\overline{\chi}_1)\big) & \text{if } p \equiv 1 \bmod 4, \\ 0 & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

where $(\frac{*}{p}) = \chi_2$ denotes the Legendre symbol, $n = mc_1c_2 \cdots c_{q-1}$ and $\chi_1^2 = \chi_2$.

Proof If q is an odd prime, then $\chi_2^q = \chi_2$ and $\overline{\chi}_2 = \chi_2$, so applying (2.3) and the properties of Gauss sums, we have

$$\begin{aligned}
&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-1}=1}^{p-1}\left(\frac{c_1a_1+c_2a_2+\cdots+c_{q-1}a_{q-1}+m\overline{a}_1\overline{a}_2\cdots\overline{a}_{q-1}}{p}\right)\\
&\quad=\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-1}=1}^{p-1}\left(\frac{a_1+a_2+\cdots+a_{q-1}+n\overline{a}_1\overline{a}_2\cdots\overline{a}_{q-1}}{p}\right)\\
&\quad=\frac{1}{\tau(\chi_2)}\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-1}=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a_1+\cdots+a_{q-1})+bn\overline{a}_1\cdots\overline{a}_{q-1}}{p}\right)\\
&\quad=\frac{1}{\tau(\chi_2)}\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-2}=1}^{p-1}\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b(1+a_1+\cdots+a_{q-2})}{p}\right)\\
&\qquad\times\sum_{a_{q-1}=1}^{p-1}\chi_2(a_{q-1})e\left(\frac{bn\overline{a}_{q-1}^q\overline{a}_1\cdots\overline{a}_{q-2}}{p}\right)\\
&\quad=\frac{1}{\tau(\chi_2)}\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-2}=1}^{p-1}\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b(1+a_1+\cdots+a_{q-2})}{p}\right)\\
&\qquad\times\sum_{c=1}^{p-1}\chi_2(c)\bigl(1+\chi_q(c)+\cdots+\chi_q^{q-1}(c)\bigr)e\left(\frac{bnc\overline{a}_1\cdots\overline{a}_{q-2}}{p}\right)\\
&\quad=\left(\frac{n}{p}\right)\cdot\tau^{q-1}(\chi_2)\left(\sum_{i=0}^{q-1}\chi_q^i(n)\frac{\tau^q(\chi_2\chi_q^i)}{\tau^q(\chi_2)}\right).
\end{aligned}$$

This proves formula (I).

To prove formula (II), note that if $p\equiv 3 \bmod 4$, then χ_2 must be an odd character $\bmod p$ (*i.e.*, $\chi_2(-1)=-1$) so that

$$\begin{aligned}
\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)&=\sum_{a=1}^{p-1}\left(\frac{p-a+m\overline{p-a}}{p}\right)\\
&=\left(\frac{-1}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)=-\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)
\end{aligned}$$

or

$$\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)=0.$$

If $p\equiv 1 \bmod 4$, then there exists one character $\chi_1 \bmod p$ such that $\chi_1^2=\chi_2$. Note that $\chi_1^3=\overline{\chi}_1$ and $\chi_2\overline{\chi}_1=\chi_1$; from the properties of Gauss sums, we have

$$\begin{aligned}
\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)&=\frac{1}{\tau(\chi_2)}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a+m\overline{a})}{p}\right)\\
&=\frac{1}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)\sum_{a=1}^{p-1}\chi_2(\overline{a})e\left(\frac{b+mb\overline{a}^2}{p}\right)
\end{aligned}$$

$$= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi_1^2(a) e\left(\frac{mba^2}{p}\right)$$
$$= \frac{1}{\tau(\chi_2)} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi_1(a)\bigl(1+\chi_2(a)\bigr) e\left(\frac{mba}{p}\right)$$
$$= \frac{\overline{\chi}_1(m)}{\tau(\chi_2)} \left(\tau^2(\chi_1) + \left(\frac{m}{p}\right)\tau^2(\overline{\chi}_1)\right).$$

This proves Lemma 2. □

3 Proof of the theorems

In this section, we complete the proof of our theorems. First we prove Theorems 1 and 2. Let $n = mc_1c_2\cdots c_k$, $(k+1, p-1) = d$. If $d = 1$, we can assume $r(k+1) \equiv 1 \bmod (p-1)$, then from Lemma 1, the properties of a reduced residue system $\bmod p$ and Gauss sums, we have

$$\begin{aligned}
&S(c_1, c_2, \ldots, c_k, m, \chi; p) \\
&= \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi(a_1 + \cdots + a_k + mc_1c_2\cdots c_k \cdot \overline{a}_1 \cdots \overline{a}_k) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b(a_1 + \cdots + a_k + n\overline{a}_1 \cdots \overline{a}_k)}{p}\right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{a_1=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}(b\overline{a}_1) e\left(\frac{b(1 + \cdots + a_k) + nb\overline{a}_1^{k+1}\overline{a}_2 \cdots \overline{a}_k}{p}\right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b + b(a_2 + \cdots + a_k)}{p}\right) \\
&\quad \times \sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\overline{a}_1^{k+1}\overline{a}_2 \cdots \overline{a}_k}{p}\right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b + b(a_2 + \cdots + a_k)}{p}\right) \\
&\quad \times \sum_{a_1=1}^{p-1} \overline{\chi}^r(a_1) e\left(\frac{nba_1\overline{a}_2 \cdots \overline{a}_k}{p}\right) \\
&= \frac{\tau(\overline{\chi}^r)}{\tau(\overline{\chi})} \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b}{p}\right) \sum_{a_2=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi^r(nb\overline{a}_2 \cdots \overline{a}_k) e\left(\frac{b(a_2 + \cdots + a_k)}{p}\right) \\
&= \chi^r(n) \cdot \frac{\tau^{k+1}(\overline{\chi}^r)}{\tau(\overline{\chi})} \\
&= \chi^r(mc_1c_2\cdots c_k) \cdot \frac{\tau^{k+1}(\overline{\chi}^r)}{\tau(\overline{\chi})}. \qquad (3.1)
\end{aligned}$$

If $d > 1$ and $\chi^{\frac{p-1}{d}} \neq \chi_0$, then from the method of proving (2.2), we have

$$\sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\overline{a}_1^{k+1}\overline{a}_2 \cdots \overline{a}_k}{p}\right) = 0.$$

From this identity and the method of proving (3.1), we may immediately deduce that if $\chi^{\frac{p-1}{d}} \neq \chi_0$, then

$$S(c_1, c_2, \ldots, c_k, m, \chi; p) = 0. \tag{3.2}$$

If $d > 1$ and $\chi^{\frac{p-1}{d}} = \chi_0$, then χ must be a dth character mod$p$, so there exists a character χ_1 mod p such that $\chi = \chi_1^d$. Let χ_d be a d-order character mod$p$, then we have

$$\begin{aligned}
&\sum_{a_1=1}^{p-1} \chi(a_1) e\left(\frac{nb\overline{a}_1^{k+1}\overline{a}_2\cdots\overline{a}_k}{p}\right) \\
&\quad = \sum_{a=1}^{p-1} \chi_1(a^d) e\left(\frac{nb\overline{a}^{d\cdot\frac{k+1}{d}}\overline{a}_2\cdots\overline{a}_k}{p}\right) \\
&\quad = \sum_{a=1}^{p-1} \overline{\chi}_1(a)\left(1 + \chi_d(a) + \chi_d^2(a) + \cdots + \chi_d^{d-1}(a)\right) e\left(\frac{nba^{\frac{k+1}{d}}\overline{a}_2\cdots\overline{a}_k}{p}\right) \\
&\quad = \sum_{i=0}^{d-1}\sum_{a=1}^{p-1} \overline{\chi}_1(a)\chi_d^i(a) e\left(\frac{nba^{\frac{k+1}{d}}\overline{a}_2\cdots\overline{a}_k}{p}\right).
\end{aligned} \tag{3.3}$$

Let $(\frac{k+1}{d}, p-1) = d_1$, then repeat the process of proving (2.1), (2.2) and (2.3). Combining (3.1), (3.2) and (3.3), we may immediately deduce the estimate

$$\left|S(c_1, c_2, \ldots, c_k, m, \chi; p)\right| \leq (k+1)\cdot p^{\frac{k}{2}}. \tag{3.4}$$

Now note that $|\tau(\chi)| = |\tau(\chi^r)| = \sqrt{p}$, Theorems 1 and 2 follow from (3.1), (3.2) and (3.4).

Now we prove Theorem 3. If $q \geq 3$, we separate q into two cases $(q, p-1) = 1$ and $(q, p-1) = q$. If $(q, p-1) = q$, then note that for any qth non-residue r mod p, we have

$$\sum_{h=0}^{q-1} \chi_q^l(r^h) = \begin{cases} q & \text{if } q|l, \\ 0 & \text{if } (q, l) = 1. \end{cases}$$

From (I) of Lemma 2, we can deduce that

$$\begin{aligned}
&\sum_{i=0}^{q-1}\left[\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-1}=1}^{p-1}\left(\frac{c_1a_1 + c_2a_2 + \cdots + c_{q-1}a_{q-1} + r^i\overline{a}_1\overline{a}_2\cdots\overline{a}_{q-1}}{p}\right)\right]^2 \\
&\quad = p^{q-1}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{h=0}^{q-1}\chi_q^{i+j}(c_1c_2\cdots c_{q-1}r^h)\frac{\tau^q(\chi_2\chi_q^i)}{\tau^q(\chi_2)}\cdot\frac{\tau^q(\chi_2\chi_q^j)}{\tau^q(\chi_2)} \\
&\quad = q\cdot p^{q-1}\sum_{i=0}^{q-1}\sum_{\substack{j=0 \\ i+j=0,q}}^{q-1}\frac{\tau^q(\chi_2\chi_q^i)}{\tau^q(\chi_2)}\cdot\frac{\tau^q(\overline{\chi}_2\overline{\chi}_q^i)}{\tau^q(\chi_2)} \\
&\quad = q^2\cdot p^{q-1}.
\end{aligned}$$

If $(q, p-1) = 1$, then from the method of proving (2.1) and the properties of Gauss sums, we can deduce that

$$\sum_{i=0}^{q-1}\left[\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{q-1}=1}^{p-1}\left(\frac{c_1a_1+c_2a_2+\cdots+c_{q-1}a_{q-1}+r^i\overline{a}_1\overline{a}_2\cdots\overline{a}_{q-1}}{p}\right)\right]^2$$
$$=\sum_{i=0}^{q-1}p^{q-1}=q\cdot p^{q-1}.$$

If $q = 2$ and $p \equiv 1 \bmod 4$, then applying (II) of Lemma 2, we have

$$\left(\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)\right)^2 = p\cdot\left|1+\left(\frac{m}{p}\right)\cdot\frac{\tau^2(\overline{\chi}_1)}{\tau^2(\chi_1)}\right|^2$$
$$= 2p+\left(\frac{m}{p}\right)\cdot\frac{\tau^2(\overline{\chi}_1)}{\tau^2(\chi_1)}+\left(\frac{m}{p}\right)\cdot\frac{\tau^2(\chi_1)}{\tau^2(\overline{\chi}_1)}. \tag{3.5}$$

Therefore, from (3.5) we can deduce that

$$\left(\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)\right)^2+\left(\sum_{a=1}^{p-1}\left(\frac{a+r\overline{a}}{p}\right)\right)^2=4p.$$

This proves Theorem 3.

To prove the corollary, note that

$$\sum_{a=1}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)=\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+m\overline{a}}{p}\right)+\sum_{a=\frac{p+1}{2}}^{p-1}\left(\frac{a+m\overline{a}}{p}\right)=2\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+m\overline{a}}{p}\right).$$

From (3.5) we may immediately deduce the identity

$$p=\left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2+\left(\sum_{b=1}^{\frac{p-1}{2}}\left(\frac{b+s\overline{b}}{p}\right)\right)^2,$$

where r and s are any two integers such that $(\frac{r}{p})\cdot(\frac{s}{p})=-1$.

This completes the proof of our corollary.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL carried out the upper bound estimate problem of one kind sums analogous to the high-dimensional Kloosterman sums. DH participated in the research and summary of the study.

Author details
[1]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China. [2]Department of Mathematics, Xi'an Shiyou University, Xi'an, Shaanxi, P.R. China.

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