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On a new application of almost increasing sequences

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Abstract

In (Bor in Int. J. Math. Math. Sci. 17:479-482, 1994), Bor has proved the main theorem dealing with $|\bar{N},p_n|_k$ summability factors of an infinite series. In the present paper, we have generalized this theorem on the $\varphi-|A,p_n|_k$ summability factors under weaker conditions by using an almost increasing sequence instead of a positive non-decreasing sequence.

MSC: 40D15; 40F05; 40G99

Keywords: absolute matrix summability; almost increasing sequences; infinite series

1 Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). We denote by t_n the nth $(C,1)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \; (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (σ_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [2]). The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{4}$$

Let $A = (a_{nv})$ be a normal matrix, *i.e.*, a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence

$s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots. \tag{5}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{6}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, p_n|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty. \tag{7}$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |A, p_n|_k$ summability reduces to $|A, p_n|_k$ summability. Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability. Furthermore, if we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, $\varphi - |A, p_n|_k$ reduces to $|C, 1|_k$ summability. Finally, if we take $\varphi_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we get $|R, p_n|_k$ summability (see [6]).

Before stating the main theorem, we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots \tag{8}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \qquad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots. \tag{9}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{10}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{11}$$

2 Known result

Many works have been done dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series (see [7–22]). Among them, in [21], the following main theorem has been proved.

Theorem A *Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{12}$$

$$\beta_n \to 0 \quad \text{as } n \to \infty, \tag{13}$$

$$|\lambda_n| X_n = O(1) \quad \text{as } n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty \tag{15}$$

are satisfied. Furthermore, if (p_n) is a sequence of positive numbers such that

$$P_n = O(np_n) \quad \text{as } n \to \infty, \tag{16}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(X_m) \quad \text{as } m \to \infty, \tag{17}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \ge 1$.

3 The main result

The aim of this paper is to generalize Theorem A for $\varphi - |A, p_n|_k$ summability under weaker conditions. For this, we need the concept of an almost increasing sequence. A positive sequence (c_n) is said to be almost increasing if there exists a positive increasing sequence (b_n) and two positive constants A and B such that $Ab_n \le c_n \le Bb_n$ (see [23]). Obviously, every increasing sequence is an almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Also, one can find some results dealing with absolute almost convergent sequences (see [24]). So, we are weakening the hypotheses of Theorem A replacing the increasing sequence by an almost increasing sequence. Now, we shall prove the following theorem.

Theorem *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, \ldots, \tag{18}$$

$$a_{n-1,v} \ge a_{nv} \quad \text{for } n \ge v + 1, \tag{19}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{20}$$

$$|\hat{a}_{n,v+1}| = O\big(v|\Delta_v \hat{a}_{nv}|\big). \tag{21}$$

Let (X_n) be an almost increasing sequence and $(\frac{\varphi_n p_n}{P_n})$ be a non-increasing sequence. If conditions (12)-(16) *and*

$$\sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |s_n|^k = O(X_m) \quad \text{as } m \to \infty, \tag{22}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - |A, p_n|_k$, $k \ge 1$.

Remark It should be noted that if we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem A. In this case, conditions (21) and (22) reduce to conditions (16) and (17), respectively. Also, the condition '$(\frac{\varphi_n p_n}{P_n})$ is a non-increasing sequence' and the conditions (18)-(20) are automatically satisfied.

Lemma [22] *Under the conditions on* (X_n), (β_n) *and* (λ_n) *as taken in the statement of the theorem, we have the following*:

$$n\beta_n X_n = O(1) \quad \text{as } n \to \infty, \tag{23}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{24}$$

Proof of the Theorem Let (T_n) denote A-transform of the series $\sum a_n\lambda_n$. Then we have, by (10) and (11),

$$\bar{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv}\lambda_v a_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v)s_v + \hat{a}_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1})s_v + \hat{a}_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} s_v \Delta\lambda_v + a_{nn}\lambda_n s_n \\
&= T_n(1) + T_n(2) + T_n(3).
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|T_n(r)\right|^k < \infty \quad \text{for } r = 1, 2, 3.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and $\acute{k}$, where $1/k + 1/\acute{k} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{k-1} \left|T_n(1)\right|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right| |\lambda_v| |s_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right| |\lambda_v|^k |s_v|^k \right) \times \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{\varphi_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right| |\lambda_v|^k |s_v|^k \right)
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k|s_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^k|s_r|^k + O(1)|\lambda_m|\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k|s_v|^k \\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m \\
&= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m \\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and the lemma. Again, applying Hölder's inequality and using the fact that $v\beta_v = O(\frac{1}{X_v}) = O(1)$ by (23), we get that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|T_n(2)\right|^k &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|s_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|s_v|^k\right)\times\left(\sum_{v=1}^{n-1}v\left|\Delta_v(\hat{a}_{nv})\right|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}v\left|\Delta_v(\hat{a}_{nv})\right|\beta_v|s_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}v\beta_v|s_v|^k\sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m}v\beta_v|s_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m}v\beta_v|s_v|^k\left(\frac{\varphi_v p_v}{P_v}\right)^{k-1}\left(\frac{p_v}{P_v}\right) \\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^k|s_r|^k + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k|s_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\left|\Delta(v\beta_v)\right|X_v + O(1)m\beta_m X_m
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{\nu=1}^{m-1} \nu|\Delta\beta_\nu|X_\nu + O(1)\sum_{\nu=1}^{m-1}\beta_{\nu+1}X_{\nu+1} + O(1)m\beta_m X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and the lemma. Finally, as in $T_n(1)$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{k-1}\left|T_n(3)\right|^k &= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}|a_{nn}\lambda_n s_n|^k \\
&= O(1)\sum_{n=1}^{m}|\lambda_n|\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k |s_n|^k \\
&= O(1) \quad \text{as } m \to \infty.
\end{aligned}
$$

This completes the proof of the theorem. If we take $\varphi_n = \frac{P_n}{p_n}$, then we get a result concerning the $|A,p_n|_k$ summability factors. If we take $a_{n\nu} = \frac{p_\nu}{P_n}$, then we have another result dealing with $|\bar{N},p_n,\varphi_n|_k$ summability. If we take $a_{n\nu} = \frac{p_\nu}{P_n}$ and $p_n = 1$ for all values of n, then we get a result dealing with $|C,1,\varphi_n|_k$ summability. If we take $\varphi_n = n$, $a_{n\nu} = \frac{p_\nu}{P_n}$ and $p_n = 1$ for all values of n, then we get a result for $|C,1|_k$ summability. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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