Plubtieng and Seangwattana *Journal of Inequalities and Applications* 2013, **2013**:120
http://www.journalofinequalitiesandapplications.com/content/2013/1/120

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Generalizations of the strong Ekeland variational principle with a generalized distance in complete metric spaces

Somyot Plubtieng* and Thidaporn Seangwattana

*Correspondence:
Somyotp@nu.ac.th
Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand

Abstract

In this paper, we prove a generalization of the strong Ekeland variational principle for a generalized distance (*i.e.*, *u*-distance) on complete metric spaces. The result present in this paper extends and improves the corresponding result of Georgiev (J. Math. Anal. Appl. 131:1-21, 1988) and Suzuki (J. Math. Anal. Appl. 320:788-794, 2006).

Keywords: *u*-distance; complete metric space; Ekeland's variational principle

1 Introduction

In 1974, Ekeland [1] proved the following, which is called *the Ekeland variational principle (for short, EVP)*.

Theorem 1.1 [1] *Let (X, d) be a complete metric space with metric d and f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous bounded from below. Then for $u \in X$ and $\lambda > 0$, there exists $v \in X$ such that*

(P) $f(v) \leq f(u) - \lambda d(u, v)$;

(Q) $f(w) > f(v) - \lambda d(v, w)$ *for every* $w \neq v$.

Later, Takahashi [2] showed that this principle is equivalent to the Caristis fixed point theorem and nonconvex minimization theorem. In 1988, Georgiev [3] proved the following generalization of Theorem 1.1, which is called *the strong Ekeland variational principle*.

Theorem 1.2 [3] *Let X be a complete metric space with metric d and $f : X \to (-\infty, +\infty]$ be proper lower semicontinuous bounded from below. Then, for all $u \in X$, $\lambda > 0$ and $\delta > 0$, there exists $v \in X$ satisfying the following*:

(P)′ $f(v) < f(u) - \lambda d(u, v) + \delta$;

(Q) $f(w) > f(v) - \lambda d(v, w)$ *for every* $w \in X \setminus \{v\}$;

(R) *if a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_{n\to\infty}(f(x_n) + \lambda d(v, x_n)) = f(v)$, *then* $\{x_n\}$ *converges to* v.

On the other hand, Kada *et al.* [4] introduced the concept of w-distance defined on a metric space and extended the Ekeland variational principle, the Kirk-Caristi fixed point theorem and the minimization theorem for w-distance. Recently, Suzuki [5, 6] introduced a more general concept than w-distance, which is called τ-distance, and established the strong Ekeland variational principle for τ-distance. Very recently, Ume [7] introduced a more generalized concept than τ-distance, which is called u-distance, and proved a new minimization and a new fixed point theorem by using u-distance on a complete metric space.

In this paper, we prove the strong Ekeland variational principle for u-distance on a complete metric space. The results of this paper extend and generalize some results in Georgiev [3], Suzuki [5], Ansari [9] and Park [10].

2 Preliminaries

Throughout the paper, we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+ = [0,\infty)$. Let us recall the following well-known definition of a u-distance.

Definition 2.1 ([8] and [7]) Let X be a complete metric space with metric d. Then a function $p : X \times X \to \mathbb{R}_+$ is called a u-distance on X if there exists a function $\theta : X \times X \times [0,\infty) \times [0,\infty) \to \mathbb{R}_+$ such that

(u1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;

(u2) $\theta(x,y,0,0) = 0$, $\theta(x,y,s,t) \geq \min\{s,t\}$ for all $x,y \in X$ and $s,t \in [0,\infty)$, and for any $x \in X$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in [0,\infty)$ and $y \in X$ imply

$$\left|\theta(x,y,s,t) - \theta(x,y,s_0,t_0)\right| < \epsilon;$$

(u3) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\theta(w_n, z_{n,}, p(w_n,x_m), p(z_n,x_m)) : m \geq n\} = 0$ imply $p(y,x) \leq \lim_{n\to\infty} \inf p(y,x_n)$ for all $y \in X$;

(u4) $\lim_{n\to\infty} \sup\{p(x_n,w_m) : m \geq n\} = 0$, $\lim_{n\to\infty} \sup\{p(y_n,z_m) : m \geq n\} = 0$, $\lim_{n\to\infty} \theta(x_n,w_n,s_n,t_n) = 0$ and $\lim_{n\to\infty} \theta(y_n,z_n,s_n,t_n) = 0$ imply $\lim_{n\to\infty} \theta(w_n,z_n,s_n,t_n) = 0$ or $\lim_{n\to\infty} \sup\{p(w_m,x_n) : m \geq n\} = 0$, $\lim_{n\to\infty} \sup\{p(z_m,y_n) : m \geq n\} = 0$, $\lim_{n\to\infty} \theta(x_n,w_n,s_n,t_n) = 0$ and $\lim_{n\to\infty} \theta(y_n,z_n,s_n,t_n) = 0$ imply $\lim_{n\to\infty} \theta(w_n,z_n,s_n,t_n) = 0$;

(u5) $\lim_{n\to\infty} \theta(w_n,z_n,p(w_n,x_n),p(z_n,x_n)) = 0$ and $\lim_{n\to\infty} \theta(w_n,z_n,p(w_n,y_n),p(z_n,y_n)) = 0$ imply $\lim_{n\to\infty} d(x_n,y_n) = 0$ or $\lim_{n\to\infty} \theta(a_n,b_n,p(x_n,a_n),p(x_n,b_n)) = 0$ and $\lim_{n\to\infty} \theta(a_n,b_n,p(y_n,a_n),p(y_n,b_n)) = 0$ imply $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Proposition 2.2 [7] *Let p be a u-distance on a metric space (X,d) and c be a positive real number. Then a function $q : X \times X \to \mathbb{R}_+$ defined by $q(x,y) = c \cdot p(x,y)$ for every $x,y \in X$ is also a u-distance on X.*

Lemma 2.3 [7] *Let (X,d) be a metric space and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.4 [7] *Let (X,d) be a metric space and p be a u-distance on X. Suppose that a sequence $\{x_n\}$ of X satisfies*

$$\lim_{n\to\infty} \sup\left\{p(x_n,x_m) : m > n\right\} = 0$$

or

$$\lim_{n\to\infty} \sup\{p(x_m, x_n) : m > n\} = 0.$$

Then, $\{x_n\}$ is a p-Cauchy sequence and $\{x_n\}$ is a Cauchy sequence.

3 Main theorem

Lemma 3.1 *Let X be a complete metric space and p be a u-distance on X. If a sequence $\{x_n\}$ of X satisfies $\lim_{n\to\infty} p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ of X also satisfies $\lim_{n\to\infty} p(z, y_n) = 0$, then $\lim_{n\to\infty} p(x_n, y_n) = 0$. In particular, for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

Proof Let θ be a function from $X \times X \times [0,\infty) \times [0,\infty)$ into $\mathbb{R}_+$ satisfying (u1)-(u5). From $\lim_n p(z, x_n) = 0$, it follows by (u2) that $\lim_{n\to\infty} \theta(z, z, p(z, x_n), p(z, x_n)) = 0$. Therefore, $\{x_n\}$ is a p-Cauchy sequence. □

Theorem 3.2 *Let X be a complete metric space and T be a mapping from X into itself. Suppose that there exists a u-distance p on X and $r \in [0, 1)$ such that $p(Tx, T^2x) \le r \cdot p(x, Tx)$ for all $x \in X$. Assume that either of the following hold:*

(i) *If $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_{n\to\infty} p(x_{n,Tx_n}) = 0$ and $\lim_{n\to\infty} p(x_n, y) = 0$, then $Ty = y$;*
(ii) *if $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $Ty = y$;*
(iii) *T is continuous.*

Then, there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.

Proof It is the same as the proof of Theorem 1 in [5]. □

Lemma 3.3 *Let X be a complete metric space, p be a u-distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

(1) *$\phi(x, z) \le \phi(x, y) + \phi(y, z)$ for all $x, y, z \in X$;*
(2) *$\phi(x, \cdot) : X \to (-\infty, \infty]$ is lower semicontinuous for any $x \in X$;*
(3) *there exists an x_0 such that $\inf_{y\in X} \phi(x_0, y) > -\infty$; and*
(4) *$\phi(x, y) = -\phi(y, x)$.*

Define $Mx = \{y \in X : \phi(x, y) + p(x, y) \le 0\}$. Let $u \in X$ and $c \in \mathbb{R}_+$ such that $\phi(x, u) < \infty$ for all $x \in X$, $Mu \ne \emptyset$ and $c \ge \phi(x, u) - \inf_{y\in Mu} \phi(u, y)$. Then a function $q : X \times X \to \mathbb{R}_+$ defined by

$$q(x, y) = \begin{cases} \phi(u, x) - \inf_{y\in Mx} \phi(u, y) & \text{if } x \in Mu \text{ and } y \in Mx, \\ c + p(x, y) & \text{if } x \notin Mu \text{ or } y \notin Mx \end{cases}$$

is a u-distance on X.

Proof Let η be a function from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (u2)-(u5) for a u-distance. We note that $\phi(x, y) + \phi(y, z) + p(x, y) + p(y, z) \le 0$ and $\phi(x, z) + p(x, z) \le 0$. Thus, $y \in Mx$ and $z \in My$ imply $z \in Mx$. If $x \in Mu$ and $y \in Mx$, then

$$\begin{aligned} p(x, y) \le \phi(y, x) \le q(x, y) &= \phi(y, x) - \inf_{y\in Mx} \phi(x, y) \\ &\le \phi(x, u) - \inf_{y\in Mu} \phi(x, y) \le c. \end{aligned}$$

Therefore, $p(x,y) \leq q(y,x) \leq c + p(x,y)$ for all $x, y \in X$. To complete the proof, we will show $(\mathrm{u1})_q$, $(\mathrm{u3})_{q,\eta}$, $(\mathrm{u4})_{q,\eta}$ and $(\mathrm{u5})_{q,\eta}$. Let x, y and z be fixed elements in X. In the case $x \in Mu$, $y \in Mx$, $y \in Mu$ and $z \in My$, we have $z \in Mx$ and hence $q(x,z) = q(x,y) \leq q(x,y) + q(y,z)$. In the other case, we note that

$$\begin{aligned} q(x,z) &\leq c + p(x,z) \leq c + p(x,y) + p(y,z) \\ &\leq 2c + p(x,y) + p(y,z) \\ &= q(x,y) + q(y,z). \end{aligned}$$

This shows $(\mathrm{u1})_q$.

We next suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\eta(w_n, z_n, q(w_n, x_m), q(z_n, x_m)) : m \geq n\} = 0$ and fix $w \in X$. Since $\lim_{n\to\infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n\} = 0$, we have $p(w,x) \leq \liminf_{n\to\infty} p(w, x_n)$ for all $y \in X$.

In the case that $w \in Mu$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in Mw$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \phi(w,x) + p(w,x) &\leq \lim_{n\to\infty} \inf \phi(w, x_n) + \lim_{n\to\infty} p(w, x_n) \\ &\leq \lim_{n\to\infty} \inf\bigl(\phi(w, x_n) + p(w, x_n)\bigr) \\ &\leq \lim_{k\to\infty} \inf\bigl(\phi(w, x_{n_k}) + p(w, x_{n_k})\bigr) \\ &\leq 0, \end{aligned}$$

and so $x \in Mu$. Hence

$$q(w,x) = \phi(u,w) - \inf_{x\in Mw} \phi(u,x) = \lim_{k\to\infty} q(w, x_{n_k}) = \lim_{n\to\infty} \inf q(w, x_n).$$

In the other case, we obtain

$$\begin{aligned} q(w,x) &\leq c + p(w,x) \leq \lim_{n\to\infty} \inf\bigl(c + p(w, x_n)\bigr) \\ &= \lim_{n\to\infty} \inf q(w, x_n). \end{aligned}$$

This shows $(\mathrm{u3})_{q,\eta}$. We will show that q satisfies $(\mathrm{u4})_{q,\eta}$.

Case I: Suppose that $\lim_{n\to\infty} \sup\{q(x_n, w_m) : m \geq n\} = 0$, $\lim_{n\to\infty} \sup\{q(y_n, z_m) : m \geq n\} = 0$, $\lim_{n\to\infty} \eta(x_n, w_n, s_n, t_n) = 0$, and $\lim_{n\to\infty} \eta(y_n, z_n, s_n, t_n) = 0$.

In the case $x_n \in Mu$ and $w_m \in Mx_n$, we note that $q(x_n, w_n) = \phi(u, x_n) - \inf_{w_m \in Mx_n} \phi(u, w_m)$. Since $\phi(x_n, w_m) + p(x_n, w_n) \leq 0$, it follows that

$$\begin{aligned} p(x_n, w_m) &\leq -\phi(x_n, w_n) = \phi(w_m, x_n) \\ &\leq \phi(w_m, u) + \phi(u, x_n) \\ &= \phi(u, x_n) - \phi(u, w_m) \\ &\leq \phi(u, x_n) - \inf_{w_m \in Mx_n} \phi(u, w_m) = q(x_n, w_m). \end{aligned}$$

Thus, we have $p(x_n, w_m) \leq q(x_n, w_m)$. This implies that $\sup_{m\geq n} p(x_n, w_n) \leq \sup_{m\geq n} q(x_n, w_m)$. Take $n \to \infty$, so

$$0 \leq \lim_{n\to\infty} \sup p(x_n, w_m) \leq \lim_{n\to\infty} \sup q(x_n, w_m) = 0$$

and therefore $\lim_{n\to\infty} \sup p(x_n, w_m) = 0$.

Similarly, if $y_n \in Mu$ and $z_m \in My_n$, then $\lim_{n\to\infty} \sup p(y_n, z_m) = 0$.

We note that $\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0 = \lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n)$ and hence

$$\lim_{n\to\infty} \eta(w_n, z_n, s_n, t_n) = 0.$$

In the case $x_n \neq Mu$ or $w_m \neq Mx_n$, we note that $p(x_n, w_m) \leq c + p(x_n, w_m) = q(x_n, w_m)$. Thus, we have $p(x_n, w_m) \leq q(x_n, w_m)$. This implies that $\sup_{m\geq n} p(x_n, w_m) \leq \sup_{m\geq n} q(x_n, w_m)$. Taking $n \to \infty$, we obtain

$$0 \leq \lim_{n\to\infty} \sup p(x_n, w_m) \leq \lim_{n\to\infty} \sup q(x_n, w_m) = 0$$

and therefore $\lim_{n\to\infty} \sup p(x_n, w_n) = 0$. Similarly as above, if $y_n \neq Mu$ and $z_m \neq My_n$, then $\lim_{n\to\infty} \sup p(y_n, z_m) = 0$. We note that $\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0 = \lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n)$ and hence $\lim_{n\to\infty} \eta(w_n, z_n, s_n, t_n) = 0$.

Case II: Suppose that $\lim_{n\to\infty} \sup\{q(w_m, x_n) : m \geq n\} = 0$, $\lim_{n\to\infty} \sup\{q(z_m, y_n) : m \geq n\} = 0$, $\lim_{n\to\infty} \eta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n\to\infty} \eta(y_n, z_n, s_n, t_n) = 0$. Similarly as in Case I, we can show that $\lim_{n\to\infty} \eta(w_n, z_n, s_n, t_n) = 0$. This shows $(u4)_{q,\eta}$. We will show that q satisfies $(u5)_{q,\eta}$.

Case I: Suppose that $\lim_{n\to\infty} \eta(w_n, z_n, q(x_n, w_n), q(x_n, z_n)) = 0$ and $\lim_{n\to\infty} \eta(w_n, z_n, q(y_n, w_n), q(y_n, z_n)) = 0$. In the case $x_n \in Mu$ and $w_n, z_n \in Mx_n$, we note that $q(x_n, w_n) = \phi(u, x_n) - \inf_{w_n\in Mx_n} \phi(u, w_n)$ and hence $q(x_n, z_n) = \phi(u, x_n) - \inf_{z_n\in Mx_n} \phi(u, z_n)$. Thus, we have

$$\begin{aligned}
\theta\bigl(w_n, z_n, p(x_n, w_n), p(x_n, z_n)\bigr) &\leq \theta\bigl(w_n, z_n, \phi(z_n, x_n), \phi(z_n, x_n)\bigr) \\
&\leq \theta\bigl(w_n, z_n, \phi(w_n, u) + \phi(u, x_n), \phi(z_n, u) + \phi(u, x_n)\bigr) \\
&= \theta\bigl(w_n, z_n, \phi(u, x_n) - \phi(u, w_n), \phi(u, x_n) - \phi(u, z_n)\bigr) \\
&\leq \theta\Bigl(w_n, z_n, \phi(u, x_n) - \inf_{w_n\in Mx_n} \phi(u, w_n), \phi(u, x_n) \\
&\quad - \inf_{z_n\in Mx_n} \phi(u, z_n)\Bigr) \\
&= \eta\bigl(w_n, z_n, q(x_n, w_n), q(x_n, z_n)\bigr).
\end{aligned}$$

Taking $n \to \infty$, we have

$$0 \leq \lim_{n\to\infty} \theta\bigl(w_n, z_n, p(x_n, w_n), p(x_n, z_n)\bigr) \leq \lim_{n\to\infty} \eta\bigl(w_n, z_n, q(x_n, w_n), q(x_n, z_n)\bigr) = 0.$$

Therefore $\lim_{n\to\infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) = 0$. Similarly, if $y_n \in Mu$ and $z_n, w_n \in My_n$, then $\lim_{n\to\infty} \theta(w_n, z_n, p(y_n, w_n), p(y_n, z_n)) = 0$. In the case $x_n \neq Mu$ or $w_n, z_n \neq Mx_n$, we have $q(x_n, w_n) = c + p(x_n, w_n)$ and $q(x_n, z_n) = c + p(x_n, z_n)$. Since p is a u-distance, we have

$\lim_{n\to\infty} d(x_n, y_n) = 0$. Hence

$$\begin{aligned}\theta\big(w_n, z_n, c + p(x_n, w_n), c + p(x_n, z_n)\big) &\le \theta\big(w_n, z_n, c + p(x_n, w_n), c + p(x_n, z_n)\big)\\ &\le \eta\big(w_n, z_n, q(x_n, w_n), q(x_n, z_n)\big).\end{aligned}$$

Take $n \to \infty$, thus

$$0 \le \lim_{n\to\infty} \theta\big(w_n, z_n, p(x_n, w_n), p(x_n, z_n)\big) \le \lim_{n\to\infty} \eta\big(w_n, z_n, q(x_n, w_n), q(x_n, z_n)\big) = 0.$$

Therefore $\lim_{n\to\infty} \theta(w_n, z_n, p(x_n, w_n), p(x_n, z_n)) = 0$. Similarly, if $y_n \neq Mu$ or $w_n, z_n \neq My_n$, then $\lim_{n\to\infty} \theta(w_n, z_n, p(y_n, w_n), p(y_n, z_n)) = 0$. Since p is a u-distance, we have $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Case II: Suppose that $\lim_{n\to\infty} \eta(w_n, z_n, q(w_n, x_n), q(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(w_n, z_n, q(w_n, y_n), q(z_n, y_n)) = 0$. Similarly as in Case I, we can show that $\lim_{n\to\infty} d(x_n, y_n) = 0$. This shows $(\mathrm{u5})_{q,\eta}$. □

Proposition 3.4 *Let X be a complete metric space, p be a u-distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

(1) $\phi(x, z) \le \phi(x, y) + \phi(y, z)$ *for all* $x, y, z \in X$;
(2) $\phi(x, \cdot) : X \to (-\infty, \infty]$ *is lower semicontinuous for any* $x \in X$;
(3) *there exists an* x_0 *such that* $\inf_{y\in X} \phi(x_0, y) > -\infty$; *and*
(4) $\phi(x, y) = -\phi(y, x)$.

Define $Mx = \{y \in X : \phi(x, y) + p(x, y) \le 0\}$ *for all* $x \in X$. *Then, for each* $u \in X$ *with* $Mu \neq \emptyset$, *there exists* $x_0 \in Mu$ *such that* $Mx_0 \subset \{x_0\}$. *In particular, there exists* $y_0 \in X$ *such that* $My_0 \subset \{y_0\}$.

Proof Let $u \in X$ with $Mu \neq \emptyset$. We have $u_1 \in Mu$ by $\phi(u, u_1) < \infty$. If $Mu = \emptyset$, the assertion holds. Suppose that $Mu_1 \neq \emptyset$ and $Mx \cap (X\{x\}) \neq \emptyset$ for all $x \in Mu_1$. Let $u_2 \in Mu_1$. We know that $\phi(x, y) \le 0$ for all $x \in X$ and $y \in Mx$, we define a mapping $T : X \to X$ as follows: For each $x \in Mu_1$, Tx satisfies $Tx \in Mx$, $Tx \neq x$ and

$$\phi(u_1, Tx) \le \frac{\phi(u_1, x) + \inf_{y\in Mx} \phi(u_1, y)}{2}.$$

For each $x \notin Mu_1$, define $Tx = u_2 \neq x$. We also define a function $q : X \times X \to \mathbb{R}^+$ by

$$q(x, y) = \begin{cases} \phi(u, x) - \inf_{y\in Mx} \phi(u_1, y) & \text{if } x \in Mu_1 \text{ and } y \in Mx,\\ 2\phi(u, u_1) - 2\inf_{w\in Mu_1} \phi(u, w) + 1 + p(x, y) & \textit{if } x \notin Mu_1 \textit{ or } y \notin Mx.\end{cases}$$

By Lemma 3.3, we have q is a u-distance on X. Since $y \in My$ and $z \in My$, it follows by Lemma 3.3 that $z \in Mx$. Hence $Tx \in Mu_1$ and $MTx \subset Mx$ for all $x \in Mu_1$. If $x \in Mu_1$, we obtain

$$\begin{aligned} q\big(Tx, T^2x\big) &= \phi(u_1, Tx) - \inf_{y\in MTx} \phi(u_1, y)\\ &\le \frac{\phi(u_1, x) + \inf_{y\in Mx} \phi(u_1, y)}{2} - \inf_{y\in Mx} \phi(u_1, y)\\ &= \frac{q(x, Tx)}{2}.\end{aligned}$$

If $x \notin Mu_1$,

$$\begin{aligned} q(Tx, T^2x) &= q(u_2, Tu_2) = \phi(u_1, u_2) - \inf_{Tu_2 \in Mu_2} \phi(u_1, Tu_2) \\ &\leq \phi(u, u_1) - \inf_{Tu_1} \phi(u, Tu_1) \\ &\leq \frac{q(x, u_2)}{2} = \frac{q(x, Tx)}{2}. \end{aligned}$$

We will show (i) in Theorem 3.2. Suppose that $\lim_{n\to\infty} \sup\{q(x_n, x_m) : m > n\} = 0$ and $\lim_{n\to\infty} q(x_n, y) = 0$. We may assume $x_n \in Mu_1$ and $y \in Mx_n$ for all $n \in \mathbb{N}$ by the definition of q. Then $y \in Mu_1$ and hence $Ty \in My \subset Mx_n$. By Lemma 2.4 we have $\lim_{n\to\infty} q(x_n, Ty) = \lim_{n\to\infty} q(x_n, y) = 0$ and $Ty = y$. Hence, by Theorem 3.2, T has a fixed point. This is a contradiction. So, there is $x_0 \in Mu_1 \subset Mu$ such that $Mx_0 \subset \{x_0\}$. □

Theorem 3.5 *Let X be a complete metric space, p be a u-distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

(1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ *for all* $x, y, z \in X$;
(2) $\phi(x, \cdot) : X \to (-\infty, \infty]$ *is lower semicontinuous for any* $x \in X$;
(3) *there exists an* x_0 *such that* $\inf_{y\in X} \phi(x_0, y) > -\infty$; *and*
(4) $\phi(x, y) = -\phi(y, x)$.

Then the following hold:

(A) *For each $u \in X$, there exists $v \in X$ such that $\phi(u, v) \leq 0$ and $\phi(v, w) + p(v, w) > 0$ for all $w \in X \setminus \{v\}$*;
(B) *For each $\lambda > 0$ and $u \in X$ with $p(u, u) = 0$, there exists $v \in X$ such that $\phi(u, v) + \lambda p(u, v) \leq 0$ and $\phi(v, w) + \lambda p(v, w) > 0$ for all $w \in X \setminus \{v\}$.*

Proof We will show that (A). For each $x \in X$, we define Mx as in Proposition 3.4. If $Mu = \emptyset$, we have u that satisfies $\phi(u, w) + p(u, w) > 0$ for all $w \in X$ with $w \neq u$. If $Mu \neq \emptyset$ and there exists $v \in Mu$, then it follows by Proposition 3.4 that $Mv \subset \{v\}$. Since $v \in Mu$ implies $\phi(u, v) \leq 0$ and $Mv \subset \{v\}$, this shows that $\phi(v, w) + p(v, w) > 0$ for all $w \in X$ with $w \neq v$.

We will show that (B). By Proposition 2.2, we note that λp is a u-distance. We define $Mx = \{y \in X : \phi(x, y) + \lambda p(x, y) \leq 0\}$ for all $x \in X$. Since $p(u, u) = 0$, we have $Mu \neq \emptyset$, and hence there exists $v \in Mu$ such that $Mv \subseteq \{v\}$ by Proposition 3.4. Therefore v satisfies $\phi(u, v) + \lambda p(u, v) \leq 0$ and $\phi(v, w) + \lambda p(v, w) > 0$ for all $w \in X$ with $w \neq v$. This completes the proof. □

Remark 3.6 By setting $\phi(x, y) = f(y) - f(x)$, where $f : X \to \mathbb{R}$ is lower semicontinuous bounded below, and letting p be a τ-distance in Theorem 3.5, we obtain the Ekeland variational principle proved by Suzuki [5].

Theorem 3.7 *Let X be a complete metric space, p be a u-distance on X and ϕ be a function from $X \times X$ into $(-\infty, \infty]$ satisfying*

(1) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ *for all* $x, y, z \in X$;
(2) $\phi(x, \cdot) : X \to (-\infty, \infty]$ *is lower semicontinuous for any* $x \in X$;
(3) *there exists an* x_0 *such that* $\inf_{y\in X} \phi(x_0, y) > -\infty$; *and*
(4) $\phi(x, y) = -\phi(y, x)$.

Let $u \in X$ with $p(u, u) = 0$. Then $\lambda > 0$ and $\delta > 0$, there exists $v \in X$ satisfying the following:

(i) $\phi(u,v) \leq 0$;
(ii) $\phi(u,v) + \lambda p(u,v) < \delta$;
(iii) $\phi(v,w) + \lambda p(v,w) > 0$ *for all* $w \in X \setminus \{v\}$;
(iv) *if a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n(\phi(v,x_n) + \lambda p(v,x_n)) = 0$, *then* $\{x_n\}$ *is p-Cauchy*, $\lim_n x_n = v$ *and* $p(v,v) = \lim_n p(v,x_n) = 0$.

Proof In the case $\phi(v,u) = \infty$, (i) and (ii) hold for all $v \in X$. We also note that (iii) and (iv) do not depend on $\phi(v,u)$. In the case $\phi(v,u) < \infty$, set $\lambda' \in (0,\lambda)$ satisfying

$$\frac{\lambda - \lambda'}{\lambda'}\Big(\phi(u,v) - \inf_{x\in X}\phi(v,x)\Big) < \delta.$$

By Theorem 3.5(B), there exists $v \in X$ such that $\phi(u,v) + \lambda' p(u,v) \leq 0$ and $\phi(v,w) + \lambda' p(v,w) > 0$ for all $w \in X \setminus \{v\}$. Thus, we have

$$\begin{aligned}
\phi(u,v) &= \left(1 + \frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) - \left(\frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) \\
&\leq \left(1 + \frac{\lambda - \lambda'}{\lambda'}\right)\big(-\lambda' p(u,v)\big) - \left(\frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) \\
&= -\lambda' p(u,v) - \big(\lambda - \lambda'\big)\big(p(u,v)\big) - \left(\frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) \\
&= -\lambda p(u,v) - \left(\frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) \\
&\leq -\lambda p(u,v) + \left(\frac{\lambda - \lambda'}{\lambda'}\right)\phi(u,v) + \left(\frac{\lambda - \lambda'}{\lambda'}\right)\inf_{x\in X}\phi(x,v) \\
&= -\lambda p(u,v) + \left(\frac{\lambda - \lambda'}{\lambda'}\right)\Big(\phi(u,v) - \inf_{x\in X}\phi(v,x)\Big) \\
&< -\lambda p(u,v) + \delta.
\end{aligned}$$

Therefore, $\phi(u,v) + \lambda p(u,v) < \delta$. For $w \in X \setminus \{v\}$, we note that

$$\phi(v,w) > -\lambda' p(v,w) \geq -\lambda p(v,w).$$

So, $\phi(v,w) + \lambda p(v,w) > 0$. Finally, we will show that (iv). Suppose that a sequence $\{x_n\}$ in X satisfies $\lim_n(\phi(v,x_n) + \lambda p(v,x_n)) = 0$. We note that $\phi(v,w) + \lambda' p(v,w) \geq 0$ for all $w \in X$. We have

$$\begin{aligned}
\lim_{n\to\infty}\sup p(v,x_n) &= \lim_{n\to\infty}\sup\left(\frac{\lambda - \lambda'}{\lambda - \lambda'}\right)p(v,x_n) \\
&= \lim_{n\to\infty}\frac{\lambda p(v,x_n) - \lambda' p(v,x_n)}{\lambda - \lambda'} \\
&\leq \lim_{n\to\infty}\frac{\lambda p(v,x_n) - \phi(v,x_n)}{\lambda - \lambda'} \\
&\leq \lim_{n\to\infty}\frac{\lambda p(v,x_n) + \phi(v,x_n)}{\lambda - \lambda'} \\
&= 0.
\end{aligned}$$

By Lemma 3.1, $\{x_n\}$ is a p-Cauchy sequence. From Lemma 2.3, therefore $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\{x_n\}$ converges to some point $x \in X$. From (u3), we have $p(v, x) = 0$ and so

$$\begin{aligned}\phi(v,x) &\le \lim_{n\to\infty} \inf \phi(v,x_n) \\ &\le \lim_{n\to\infty} \bigl(\phi(v,x_n) + \lambda p(v,x_n)\bigr) = 0.\end{aligned}$$

Thus, if $v \neq x$, then we have

$$\phi(v,x) > -\lambda' p(v,x) \ge \phi(v,x).$$

This is a contradiction. Hence, we obtain $v = x$. □

Remark 3.8 By setting $\phi(x,y) = f(y) - f(x)$, where $f: X \to \mathbb{R}$ is lower semicontinuous bounded below. Let p be a τ-distance in Theorem 3.7, we obtain the strong Ekeland variational principle proved by Suzuki [6].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the Thailand Research Fund (TRF) for supporting by permit money of investment under of The Royal Golden Jubilee Ph.D. Program (RGJ-Ph.D.), Thailand.

Received: 28 September 2012 Accepted: 28 February 2013 Published: 21 March 2013

References
1. Ekeland, I: On the variational principle. J. Math. Anal. Appl. **47**, 324-353 (1974)
2. Takahashi, W: Existence theorems generalizing fixed point theorems for multivalued mappings. In: Thera, MA, Baillon, JB (eds.) Fixed Point Theory and Applications. Pitman Res. Notes in Math. Ser., vol. 252, pp. 397-406. Longman, Harlow (1991)
3. Georgiev, PG: The strong Ekeland variational principle, the strong drop theorem and applications. J. Math. Anal. Appl. **131**, 1-21 (1988)
4. Kada, O, Suzuki, T, Takahashi, W: Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Jpn. **44**, 381-391 (1996)
5. Suzuki, T: Generalized distance and existence theorem in complete metric spaces. J. Math. Anal. Appl. **253**, 440-458 (2001)
6. Suzuki, T: The strong Ekeland variational principle. J. Math. Anal. Appl. **320**, 787-794 (2006)
7. Ume, JS: Existence theorem for generalized distance on complete metric spaces. Fixed Point Theory Appl. **2010**, 397150 (2010)
8. Hirunworakit, S, Petrot, N: Some fixed point theorems for contractive multivalued mappings induced by generalized distance in metric spaces. Fixed Point Theory Appl. (2011). doi:10.1186/1687-1812-2011-78
9. Ansari, QH: Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory. J. Math. Anal. Appl. **334**, 561-575 (2007)
10. Park, S: On generalizations of the Ekeland-type variational principle. Nonlinear Anal. **39**, 881-889 (2000)

doi:10.1186/1029-242X-2013-120
Cite this article as: Plubtieng and Seangwattana: **Generalizations of the strong Ekeland variational principle with a generalized distance in complete metric spaces.** *Journal of Inequalities and Applications* 2013 **2013**:120.