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α-admissible mappings and related fixed point theorems

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Abstract
In this paper, we prove the existence and uniqueness of a fixed point for certain α-admissible contraction mappings. Our results generalize and extend some well-known results on the topic in the literature. We consider some examples to illustrate the usability of our results.
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Keywords: α-admissible mappings; contractive mappings; fixed point

1 Introduction

Fixed point theory is one of the outstanding subfields of nonlinear functional analysis. It has been used in the research areas of mathematics and nonlinear sciences (see, *e.g.*, [1–8]). In 1922 Banach [10] proved that in a complete metric space every contraction has a unique fixed point. In the proof of this theorem, he not only showed the existence and uniqueness of a fixed point, but also provided a method (generally, iterative) for constructing the fixed point. This property of the Banach theorem differentiates it from other fixed point theorems. Therefore, the Banach fixed point theorem has attracted great attention of authors since then (see, *e.g.*, [11–48]). On the other hand, the fixed point technique suggested by Banach attracted many researchers to solve various concrete problems.

2 Main results

In an attempt to generalize the Banach contraction principle, many researchers extended the following result in certain directions.

Theorem 1 (See, *e.g.*, [9, 37, 38]) *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$d(fx,fy) \le \beta\big(d(x,y)\big)d(x,y)$$

for all $x,y \in X$. Then f has a unique fixed point.

Definition 2 (See, *e.g.*, [40]) Let $f : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$. We say that f is an α-admissible mapping if

$$\alpha(x,y) \ge 1 \quad \text{implies} \quad \alpha(fx,fy) \ge 1, \quad x,y \in X.$$

Example 3 (*cf.* [40]) Let $X = \mathbb{R}$. Define $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$fx = \begin{cases} \ln|x| & \text{if } x \neq 0, \\ 3 & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha(x,y) = \begin{cases} 3 & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is α-admissible.

Our first result is the following.

Theorem 4 *Let (X,d) be a complete metric space and $f : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$\bigl(d(fx,fy) + \ell\bigr)^{\alpha(x,fx)\alpha(y,fy)} \leq \beta\bigl(d(x,y)\bigr)d(x,y) + \ell \tag{2.1}$$

for all $x,y \in X$ where $\ell \geq 1$. Suppose that either
(a) *f is continuous, or*
(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n,x_{n+1}) \geq 1$ for all n, then $\alpha(x,fx) \geq 1$.*
If there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$, then f has a fixed point.

Proof Let $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0,fx_0) \geq 1$, we deduce that $\alpha(x_1,x_2) = \alpha(fx_0,f^2x_0) \geq 1$. By continuing this process, we get $\alpha(x_n,fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. By the inequality (2.1), we have

$$\begin{aligned} d(fx_{n-1},fx_n) + \ell &\leq \bigl(d(fx_{n-1},fx_n) + \ell\bigr)^{\alpha(x_{n-1},fx_{n-1})\alpha(x_n,fx_n)} \\ &\leq \beta\bigl(d(x_{n-1},x_n)\bigr)d(x_{n-1},x_n) + \ell, \end{aligned}$$

then

$$d(x_n,x_{n+1}) \leq \beta\bigl(d(x_{n-1},x_n)\bigr)d(x_{n-1},x_n), \tag{2.2}$$

which implies $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$. It follows that the sequence $\{d(x_n,x_{n+1})\}$ is decreasing. Thus, there exists $d \in \mathbb{R}_+$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = d$. We will prove that $d = 0$. From (2.2) we have

$$\frac{d(x_n,x_{n+1})}{d(x_{n-1},x_n)} \leq \beta\bigl(d(x_{n-1},x_n)\bigr) \leq 1,$$

which implies $\lim_{n\to\infty} \beta(d(x_{n-1},x_n)) = 1$. Using the property of the function β, we conclude that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.3}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k, we have

$$n(k) > m(k) > k, \qquad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

By the triangle inequality, we derive that

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \varepsilon + d(x_{m(k)-1}, x_{m(k)}), \end{aligned}$$

$k \in \mathbb{N}$. Taking the limit as $k \to +\infty$ in the above inequality and using (2.3), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.4}$$

Again, by the triangle inequality, we find that

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}).$$

Taking the limit as $k \to +\infty$, together with (2.3) and (2.4), we deduce that

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.5}$$

From (2.1), (2.4) and (2.5) we have

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) + \ell &\leq \bigl(d(x_{n(k)+1}, x_{m(k)+1}) + \ell\bigr)^{\alpha(x_{n(k)}, fx_{n(k)})\alpha(x_{m(k)}, fx_{m(k)})} \\ &= \bigl(d(fx_{n(k)}, fx_{m(k)}) + \ell\bigr)^{\alpha(x_{n(k)}, fx_{n(k)})\alpha(x_{m(k)}, fx_{m(k)})} \\ &\leq \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr) d(x_{n(k)}, x_{m(k)}) + \ell. \end{aligned}$$

Hence,

$$\frac{d(x_{n(k)+1}, x_{m(k)+1})}{d(x_{n(k)}, x_{m(k)})} \leq \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr) \leq 1.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr) = 1.$$

That is, $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = 0 < \varepsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$. First, we suppose that f is continuous. Since f is continuous, then we have

$$fz = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So, z is a fixed point of f. Next, we suppose that (b) holds. Then $\alpha(z,fz) \geq 1$. Now, by (2.1) we have

$$\begin{aligned} d(fz,x_{n+1}) + \ell &\leq \bigl(d(fz,fx_n) + \ell\bigr)^{\alpha(z,fz)\alpha(x_n,fx_n)} \\ &\leq \beta\bigl(d(z,x_n)\bigr)d(z,x_n) + \ell. \end{aligned}$$

That is, $d(fz,x_{n+1}) \leq \beta(d(z,x_n))d(z,x_n)$, and so we get

$$d(fz,z) \leq d(fz,x_{n+1}) + d(z,x_{n+1}) \leq \beta\bigl(d(z,x_n)\bigr)d(z,x_n) + d(z,x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we get $d(fz,z) = 0$, that is, $z = fz$. □

Example 5 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{x+1} & \text{if } x \in [0,1], \\ 2x & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\alpha : X \times X \to [0,+\infty)$ and $\beta : [0,\infty) \to [0,1]$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(t) = \frac{1}{1+t}.$$

We prove that Theorem 4 can be applied to f, but Theorem 1 cannot be applied to f.

Clearly, (X,d) is a complete metric space. We show that f is an α-admissible mapping. Let $x,y \in X$, if $\alpha(x,y) \geq 1$, then $x,y \in [0,1]$. On the other hand, for all $x \in [0,1]$, we have $fx \leq 1$. It follows that $\alpha(fx,fy) \geq 1$. Thus the assertion holds. In reason of the above arguments, $\alpha(0,f0) \geq 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies that $\alpha(x,fx) \geq 1$.

Let $x,y \in [0,1]$ and $y \geq x$. We get

$$\begin{aligned} \bigl(d(fx,fy) + \ell\bigr)^{\alpha(x,fx)\alpha(y,fy)} &= fy - fx + \ell = \frac{y}{y+1} - \frac{x}{x+1} + \ell \\ &= \frac{y-x}{(1+x)(1+y)} + \ell \\ &\leq \frac{y-x}{1+y-x} + \ell = \beta\bigl(d(x,y)\bigr)d(x,y) + \ell. \end{aligned}$$

Otherwise, $\alpha(x,fx)\alpha(y,fy) = 0$ and so

$$\bigl(d(fx,fy) + \ell\bigr)^{\alpha(x,fx)\alpha(y,fy)} = 1 \leq \beta\bigl(d(x,y)\bigr)d(x,y) + \ell,$$

then the condition of Theorem 4 holds. Hence, f has a fixed point. Let $x = 2$ and $y = 3$. Then

$$d(f2,f3) = 2 > \frac{1}{2} = \frac{1}{1+|2-3|}|2-3| = \beta\bigl(d(2,3)\bigr)d(2,3),$$

that is, the contractive condition of Theorem 1 does not hold for this example.

Theorem 6 *Let (X,d) be a complete metric space and $f : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$\bigl(\alpha(x,fx)\alpha(y,fy)+1\bigr)^{d(fx,fy)} \le 2^{\beta(d(x,y))d(x,y)} \tag{2.6}$$

for all $x,y \in X$. Suppose that either

(a) *f is continuous, or*

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n,x_{n+1}) \ge 1$ for all n, then $\alpha(x,fx) \ge 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge 1$, then f has a fixed point.

Proof Let $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge 1$. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. As in Theorem 4, we conclude that $\alpha(x_n,fx_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Due to (2.6) we have

$$\begin{aligned} 2^{d(fx_{n-1},fx_n)} &\le \bigl(\alpha(x_{n-1},fx_{n-1})\alpha(x_n,fx_n)+1\bigr)^{d(fx_{n-1},fx_n)} \\ &\le 2^{\beta(d(x_{n-1},x_n))d(x_{n-1},x_n)}, \end{aligned}$$

which yields that

$$d(x_n,x_{n+1}) \le \beta\bigl(d(x_{n-1},x_n)\bigr)d(x_{n-1},x_n). \tag{2.7}$$

So, we conclude that $d(x_n,x_{n+1}) \le d(x_{n-1},x_n)$. It follows that the sequence $d_n := d(x_n,x_{n+1})$ is decreasing. Thus, there exists $d \in \mathbb{R}_+$ such that $d_n \to d$ as $n \to \infty$. We claim that $d = 0$. Suppose, to the contrary, that $d > 0$. Considering (2.7), we obtain

$$\frac{d(x_n,x_{n+1})}{d(x_{n-1},x_n)} \le \beta\bigl(d(x_{n-1},x_n)\bigr) \le 1,$$

which implies $\lim_{n\to\infty} \beta(d(x_{n-1},x_n)) = 1$. Hence, $d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_{n-1},x_n) = 0$, which is a contradiction. Hence, we derive that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

We prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k,

$$n(k) > m(k) > k, \qquad d(x_{n(k)},x_{m(k)}) \ge \varepsilon \quad \text{and} \quad d(x_{n(k)},x_{m(k)-1}) < \varepsilon.$$

Following the related lines in the proof of Theorem 4, we get

$$\lim_{k\to+\infty} d(x_{n(k)},x_{m(k)}) = \varepsilon \tag{2.8}$$

and

$$\lim_{k\to+\infty} d(x_{n(k)+1},x_{m(k)+1}) = \varepsilon. \tag{2.9}$$

Now, from (2.6), (2.8) and (2.9), we have

$$\begin{aligned}2^{d(x_{n(k)+1},x_{m(k)+1})} &\le \left(\alpha(x_{n(k)},fx_{n(k)})\alpha(x_{m(k)},fx_{m(k)})+1\right)^{d(x_{n(k)+1},x_{m(k)+1})}\\ &= \left(\alpha(x_{n(k)},fx_{n(k)})\alpha(x_{m(k)},fx_{m(k)})+1\right)^{d(fx_{n(k)},fx_{m(k)})}\\ &\le 2^{\beta(d(x_{n(k)},x_{m(k)}))d(x_{n(k)},x_{m(k)})}.\end{aligned}$$

Hence,

$$\frac{d(x_{n(k)+1},x_{m(k)+1})}{d(x_{n(k)},x_{m(k)})} \le \beta\left(d(x_{n(k)},x_{m(k)})\right) \le 1.$$

By taking limit as $k \to \infty$, we get

$$\lim_{n\to\infty} \beta\left(d(x_{n(k)},x_{m(k)})\right) = 1.$$

That is, $\lim_{k\to\infty} d(x_{n(k)},x_{m(k)}) = 0 < \varepsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$. First of all, we suppose that f is continuous. We obtain that

$$fz = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = z$$

due to the continuity of f. Thus, we derived that z is a fixed point of f.

Next, we suppose that (b) holds. Then, $\alpha(z,fz) \ge 1$. Now, by (2.6) we have

$$\begin{aligned}2^{d(fz,x_{n+1})} &\le \left(\alpha(z,fz)\alpha(x_n,fx_n)+1\right)^{d(fz,fx_n)}\\ &\le 2^{\beta(d(z,x_n))d(z,x_n)}.\end{aligned}$$

That is, $d(fz,x_{n+1}) \le \beta(d(z,x_n))d(z,x_n)$, and so we get

$$d(fz,z) \le d(fz,x_{n+1}) + d(z,x_{n+1}) \le \beta\left(d(z,x_n)\right)d(z,x_n) + d(z,x_{n+1}).$$

By taking the limit as $n \to \infty$, we get $d(fz,z) = 0$, *i.e.*, $z = fz$. □

Example 7 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{8}x^2 & \text{if } x \in [0,1], \\ \ln x & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\alpha : X \times X \to [0,+\infty)$ and $\beta : [0,\infty) \to [0,1]$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(t) = \frac{1}{4}.$$

We prove that Theorem 6 can be applied to f, but Theorem 1 cannot be applied to f.

By a similar method to that in the proof of Example 5, we can show that f is an α-admissible mapping and $\alpha(x_n, fx_n) \geq 1$, $x_n \to x$ as $n \to +\infty$ implies that $\alpha(x, fx) \geq 1$. Clearly, $\alpha(0, f0) \geq 1$.

Let $x, y \in [0,1]$. Then

$$\begin{aligned}\bigl(\alpha(x,fx)\alpha(y,fy)+1\bigr)^{d(fx,fy)} &= 2^{|fx-fy|} = 2^{\frac{1}{8}|x-y||x+y|}\\ &\leq 2^{\frac{1}{4}|x-y|} = 2^{\beta(d(x,y))d(x,y)}.\end{aligned}$$

Otherwise, $\alpha(x,fx)\alpha(y,fy) = 0$, and so

$$\begin{aligned}\bigl(\alpha(x,fx)\alpha(y,fy)+1\bigr)^{d(fx,fy)} &\leq \bigl(\alpha(x,fx)\alpha(y,fy)+1\bigr)^{d(fx,fy)}\\ &= 1^{d(fx,fy)} = 2^0 \leq 2^{\beta(d(x,y))d(x,y)},\end{aligned}$$

then the contractive condition of Theorem 6 holds and f has a fixed point. Let $x = 2$ and $y = 4$; then

$$d(f2, f4) = \ln 2 > \frac{1}{2} = \frac{1}{4}|2-4| = \beta\bigl(d(2,4)\bigr)d(2,4).$$

That is, the contractive condition of Theorem 1 does not hold for this example.

Theorem 8 *Let (X,d) be a complete metric space and $f : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$\alpha(x,fx)\alpha(y,fy)d(fx,fy) \leq \beta\bigl(d(x,y)\bigr)d(x,y) \tag{2.10}$$

for all $x, y \in X$. Suppose that either

(a) *f is continuous, or*

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x,fx) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof Let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for f and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. As in Theorem 4, we conclude that $\alpha(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by (2.10) we have

$$\alpha(x_{n-1}, fx_{n-1})\alpha(x_n, fx_n)d(fx_{n-1}, fx_n) \leq \beta\bigl(d(x_{n-1}, x_n)\bigr)d(x_{n-1}, x_n),$$

then

$$d(x_n, x_{n+1}) \leq \beta\bigl(d(x_{n-1}, x_n)\bigr)d(x_{n-1}, x_n). \tag{2.11}$$

It yields that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. It follows that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing. Consequently, there exists $d \in \mathbb{R}_+$ such that $d(x_n, x_{n+1}) \to d$ as $n \to \infty$. Regarding (2.11), we observe that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta\bigl(d(x_{n-1}, x_n)\bigr) \leq 1.$$

Thus, we find that $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$ by the property of the function β. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Next, we will show that the sequence $\{x_n\}$ is Cauchy. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k,

$$n(k) > m(k) > k, \qquad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Again, by following the lines of the proof of Theorem 4, we derive that

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon \tag{2.12}$$

and

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.13}$$

Combining (2.10), (2.12) and (2.13), we have

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) &\leq \alpha(x_{n(k)}, fx_{n(k)})\alpha(x_{m(k)}, fx_{m(k)})d(x_{n(k)+1}, x_{m(k)+1}) \\ &= \alpha(x_{n(k)}, fx_{n(k)})\alpha(x_{m(k)}, fx_{m(k)})d(fx_{n(k)}, fx_{m(k)}) \\ &\leq \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr)d(x_{n(k)}, x_{m(k)}). \end{aligned}$$

Hence,

$$\frac{d(x_{n(k)+1}, x_{m(k)+1})}{d(x_{n(k)}, x_{m(k)})} \leq \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr) \leq 1.$$

By taking limit as $k \to \infty$, we get

$$\lim_{n\to\infty} \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr) = 1.$$

That is, $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$.

First, suppose that f is continuous. Since f is continuous, then we have

$$fz = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = z.$$

So, z is a fixed point of f.

We suppose that (b) holds. Then $\alpha(z, fz) \geq 1$. Now, by (2.10) we have

$$\begin{aligned} d(fz, x_{n+1}) &\leq \alpha(z, fz)\alpha(x_n, fx_n)d(fz, fx_n) \\ &\leq \beta\bigl(d(z, x_n)\bigr)d(z, x_n). \end{aligned}$$

That is, $d(fz,x_{n+1}) \le \beta(d(z,x_n))d(z,x_n)$, and so we get

$$d(fz,z) \le d(fz,x_{n+1}) + d(z,x_{n+1}) \le \beta\bigl(d(z,x_n)\bigr)d(z,x_n) + d(z,x_{n+1}).$$

Letting $n \to \infty$ in the above inequality, we get $d(fz,z) = 0$, *i.e.*, $z = fz$. □

Example 9 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{4}(1-x^2) & \text{if } x \in [0,1], \\ 3x & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\alpha : X \times X \to [0,+\infty)$ and $\beta : [0,\infty) \to [0,1]$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(t) = \frac{1}{2}.$$

We prove that Theorem 8 can be applied to f (here, a fixed point is $u = \sqrt{5}-2$), but Theorem 1 cannot be applied to f.

By a similar method to that in the proof of Example 5, we can show that f is an α-admissible mapping and $\alpha(x_n,fx_n) \ge 1$, $x_n \to x$ as $n \to +\infty$ implies that $\alpha(x,fx) \ge 1$. Clearly, $\alpha(0,f0) \ge 1$.

Let $x,y \in [0,1]$. Then

$$\alpha(x,fx)\alpha(y,fy)d(fx,fy) = |fx - fy| = \frac{1}{4}|x-y||x+y| \le \frac{1}{2}|x-y| = \beta\bigl(d(x,y)\bigr)d(x,y).$$

Otherwise, $\alpha(x,fx)\alpha(y,fy) = 0$, and so

$$\alpha(x,fx)\alpha(y,fy)d(fx,fy) = 0 \le \beta\bigl(d(x,y)\bigr)d(x,y),$$

then the conditions of Theorem 8 hold and f has a fixed point. Let $x = 3$ and $y = 4$; then

$$d(f3,f4) = 3 > \frac{1}{2} = \frac{1}{2}|3-4| = \beta\bigl(d(3,4)\bigr)d(3,4).$$

That is, the contractive condition of Theorem 1 does not hold for this example.

Theorem 10 *Assume that all the hypotheses of Theorems* 4, 6 *and* 8 *hold. Adding the following condition*:

(c) *if* $x = fx$ *then* $\alpha(x,fx) \ge 1$,

we obtain the uniqueness of the fixed point of f.

Proof Suppose that z and z^* are two fixed points of f such that $z \ne z^*$. Then $\alpha(z,fz) \ge 1$ and $\alpha(z^*,fz^*) \ge 1$.

For Theorem 4 we have

$$d\bigl(fz,fz^*\bigr) + \ell \le \bigl(d\bigl(fz,fz^*\bigr) + \ell\bigr)^{\alpha(z,fz)\alpha(z^*,fz^*)} \le \beta\bigl(d\bigl(z,z^*\bigr)\bigr)d\bigl(z,z^*\bigr) + \ell.$$

For Theorem 6 we have

$$(2)^{d(fz,fz^*)} \leq \left(\alpha(z,fz)\alpha\left(z^*,fz^*\right)+1\right)^{d(fz,fz^*)} \leq (2)^{\beta(d(z,z^*))d(z,z^*)}.$$

For Theorem 8 we have

$$d\left(fz,fz^*\right) \leq \alpha(z,fz)\alpha\left(z^*,fz^*\right)d\left(fz,fz^*\right) \leq \beta\left(d\left(z,z^*\right)\right)d\left(z,z^*\right).$$

Hence, all the three inequalities separately imply that $\beta(d(z,z^*)) = 1$. Thus $d(z,z^*) = 0$, *i.e.*, $z = z^*$ as required. □

Remark 11 By utilizing the technique of Samet *et al.* [40], we can obtain corresponding coupled fixed point results from our Theorems 4, 6 and 8.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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