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Approximate selection theorems with n-connectedness

Hoonjoo Kim*

*Correspondence:
hoonjoo@sehan.ac.kr
Department of Mathematics Education, Sehan University, YoungAm-gun, Chunnam 526-702, Korea

Abstract

We establish new approximate selection theorems for almost lower semicontinuous multimaps with n-connectedness. Our results unify and extend the approximate selection theorems in many published works and are applied to topological semilattices with path-connected intervals.

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1 Introduction

Since Michael [1] constructed continuous ϵ-approximate selections for the lower semicontinuous maps with convex values in Banach spaces, the result has been improved in many ways. It was extended to lower semicontinuous maps with convex values except on a set of topological dimension less than or equal to zero by Michael and Pixley [2] in 1980. And Ben-El-Mechaiekh and Oudadess [3] generalized the theorem in [2] to a class of lower semicontinuous multimaps with nonconvex values in LC-metric spaces, which have generalized convex metric structures introduced by Horvath [4].

Using the concept of n-connectedness, Kim [5] introduced an LD-metric space and extended the result in [3] to LD-metric spaces which are more general than LC-metric spaces.

On the other hand, in LC-spaces, Wu and Li [6] obtained the approximate selection theorems for quasi-lower semicontinuous multimaps which were generalized by the author and Lee [7] to almost lower semicontinuous multimaps in C-spaces.

In this paper, we establish a new approximate selection theorem for almost lower semicontinuous multimaps with D-set values except on a set of topological dimension less than or equal to zero in LD-spaces. The corollary of this gives a correct and simple proof for the result in [8].

We also establish some approximate selection theorems for almost lower semicontinuous multimaps in D-spaces and apply the results to topological semilattices with path connected intervals. Our results unify and extend the approximate selection theorems in [1–3, 5–9].

2 Preliminaries

A *multimap* (or simply a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context. Let $\langle X\rangle$ denote the set of all nonempty finite subsets of a set X.

Let X be a topological space. A *C-structure* on X is given by a map $\Gamma : \langle X\rangle \multimap X$ such that

(1) for all $A \in \langle X\rangle$, $\Gamma_A = \Gamma(A)$ is nonempty and contractible; and
(2) for all $A, B \in \langle X\rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A pair (X, Γ) is then called a *C-space* by Horvath [4] and an *H-space* by Bardaro and Ceppitelli [10]. For examples of a C-space, see [4, 10]. For an (X, Γ), a subset C of X is said to be *Γ-convex* (or a *C-set*) if $A \in \langle C\rangle$ implies $\Gamma_A \subset C$.

For a uniform space X with a uniform structure $\mathcal{U}$, $A \subset X$ and $U \in \mathcal{U}$, the set $U(A)$ is defined to be $\{y \in X : (x, y) \in U$ for some $x \in A\}$ and if $x \in X$, $U(x) = U(\{x\})$.

A C-space (X, Γ) is called an *LC-space* if X is a uniform space and there exists a base $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$, $\{x \in X : C \cap V_i(x) \neq \emptyset\}$ is Γ-convex whenever $C \subset X$ is Γ-convex.

A C-space (X, Γ) is called an *LC-metric space* if X is equipped with a metric d such that for any $\epsilon > 0$, the set $B(C, \epsilon) = \{x \in X : d(x, C) < \epsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and open balls are Γ-convex. For details, see Horvath [4].

A topological space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f : S^k \to X$ for $k \leq n$ has a continuous extension over B^{k+1}, where S^k is the unit sphere and B^{k+1} is the closed unit ball in $\mathbb{R}^{k+1}$. Note that a contractible space is n-connected for every $n \geq 0$.

The following is introduced by Kim [5]. Let X be a topological space. A *D-structure* on X is a map $\mathcal{D} : \langle X\rangle \multimap X$ such that it satisfies the following conditions:

(1) for each $A \in \langle X\rangle$, $\mathcal{D}(A)$ is nonempty and n-connected for all $n \geq 0$;
(2) for each $A, B \in \langle X\rangle$, $A \subset B$ implies $\mathcal{D}(A) \subset \mathcal{D}(B)$.

The pair $(X, \mathcal{D})$ is called a *D-space*; a subset C of X is said to be a *$\mathcal{D}$-set* if $\mathcal{D}(A) \subset C$ for each $A \in \langle C\rangle$.

A D-space $(X, \mathcal{D})$ is called an *LD-space* if X is a uniform space and if there exists a base $\{V_i : i \in I\}$ for the uniform structure such that for each $i \in I$, the set $\{x \in X : C \cap V_i(x) \neq \emptyset\}$ is a $\mathcal{D}$-set whenever $C \subset X$ is a $\mathcal{D}$-set.

A D-space $(X, \mathcal{D})$ is called an *LD-metric space* if X is a metric space such that for each $\epsilon > 0$, $B(C, \epsilon)$ is a $\mathcal{D}$-set whenever $C \subset X$ is a $\mathcal{D}$-set and open balls are $\mathcal{D}$-sets.

Let X be a topological space and $(Y, \mathcal{D})$ be a D-space with a uniformity $\mathcal{U}$. A multimap $F : X \multimap Y$ is called:

(1) *lower semicontinuous* (lsc) at $x \in X$ if for each open set W with $W \cap F(x) \neq \emptyset$, there is a neighborhood $U(x)$ of x such that $F(z) \cap W \neq \emptyset$ for all $z \in U(x)$;
(2) *quasi-lower semicontinuous* (qlsc) at $x \in X$ if for each $V \in \mathcal{U}$, there are $y \in F(x)$ and a neighborhood $U(x)$ of x such that $F(z) \cap V(y) \neq \emptyset$ for all $z \in U(x)$;
(3) *almost lower semicontinuous* (alsc) at $x \in X$ if for each $V \in \mathcal{U}$, there is a neighborhood $U(x)$ of x such that $\bigcap_{z \in U(x)} V(F(z)) \neq \emptyset$.

If F is lsc (qlsc, alsc, resp.) at each $x \in X$, F is called *lsc* (*qlsc, alsc*, resp.). As in [7, Proposition 3], (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

For $V \in \mathcal{U}$, a continuous function $f : X \to Y$ is called a *V-approximate selection* of F if for all $x \in X, f(x) \in V(F(x))$.

Let $(Y, \mathcal{D})$ be an LD-metric space. For $\epsilon > 0, f$ is called an *ϵ-approximate selection* of F if for all $x \in X, f(x) \in B(F(x), \epsilon)$.

Let X be a topological space. If $Z \subset X$, then $\dim_X Z \le 0$ means that $\dim E \le 0$ for every set $E \subset Z$ which is closed in X, where $\dim E$ denotes the covering dimension of E. Note that if $\dim_X Z \le 0$, then any locally finite open covering of Z has a disjoint locally finite open refinement.

3 Approximate selection theorems on $\mathcal{D}$-spaces

As a main tool, we need Proposition 1 of Kim [5].

Proposition 3.1 *Let X be a paracompact topological space and $\mathcal{R}$ be a locally finite open covering of X, $(Y, \mathcal{D})$ be a D-space, and $\eta : \mathcal{R} \to Y$ be a function. Then there exists a continuous function $f : X \to Y$ such that*

$$f(x) \in \mathcal{D}\big(\{\eta(O) : O \in \mathcal{R}, x \in O\}\big)$$

for each $x \in X$.

With Proposition 3.1, we establish the V-approximate selection theorem which is the key result of this paper.

Theorem 3.2 *Let X be a paracompact topological space and Z be a closed subset of X with $\dim_X Z \le 0$. Let $(Y, \mathcal{D})$ be an LD-space with a uniformity $\mathcal{U}$ and $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. If $F : X \multimap Y$ is an alsc multimap such that $F(x)$ is a $\mathcal{D}$-set for all $x \in X \backslash Z$, then F has a V-approximate selection for each $V \in \mathcal{U}$.*

Furthermore, if X is a precompact uniform space or a compact topological space, there is a subset $A \in \langle Y \rangle$ such that $f(X) \subset \mathcal{D}(A)$.

Proof For each $V \in \mathcal{U}$ and $x \in X$, there is a neighborhood $U(x)$ of x such that $\bigcap_{z \in U(x)} V(F(z)) \neq \emptyset$, because F is alsc. Since X is paracompact, the open cover $\{U(x) : x \in X\}$ of X has a locally finite refinement $\{\tilde{U}(x) : x \in X\}$. And since $\dim_X Z \le 0$, the relatively open cover $\{\tilde{U}(x) \cap Z : x \in Z\}$ of Z has a relatively open disjoint refinement $\{W(x) : x \in Z\}$. Z is closed in X so the collection $\mathcal{R} = \{\tilde{U}(x) \cap (W(x) \cup (X \backslash Z)) : x \in X\}$ forms a locally finite open cover of X.

For each $O \in \mathcal{R}$, choose x_o such that $O \subset U(x_o)$ and $y_o \in \bigcap_{z \in U(x_o)} V(F(z))$. Define $\eta : \mathcal{R} \to Y$ by $\eta(O) = y_o$ for all $O \in \mathcal{R}$. Then $\eta(O) \in \bigcap_{z \in U(x_o)} V(F(z)) \subset \bigcap_{z \in O} V(F(z))$, so $\{\eta(O) : O \in \mathcal{R}, x \in O\} \subset V(F(x))$ for all $x \in X$. By Proposition 3.1, there is a continuous function $f : X \to Y$ such that $f(x) \in \mathcal{D}(\{\eta(O) : O \in \mathcal{R}, x \in O\})$.

We now show that $f(x) \in V(F(x))$ for all $x \in X$. If $x \in Z$, there exists a unique $O \in \mathcal{R}$ such that $x \in O$, that is, $\{\eta(O) : O \in \mathcal{R}, x \in O\}$ is a singleton. So, $f(x) \in \mathcal{D}(\{\eta(O) : O \in \mathcal{R}, x \in O\}) = \{\eta(O)\} \subset V(F(x))$. If $x \in X \backslash Z$, since $F(x)$ is a $\mathcal{D}$-set, $\mathcal{D}(\{\eta(O) : O \in \mathcal{R}, x \in O\}) \subset V(F(x))$, that is, $f(x) \in V(F(x))$.

If X is a precompact uniform space or a compact topological space, $\mathcal{R}$ can be chosen finite. Take $A = \{\eta(O) : O \in \mathcal{R}\}$, then $A \in \langle Y \rangle$ and $f(X) \subset \mathcal{D}(A)$. ☐

Remark If $Z = \emptyset$, then the condition '$\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$' can be omitted. In that case, if $(Y, \mathcal{D})$ is an LC-space with a uniformity $\mathcal{U}$ and F is qlsc, then Theorem 3.2 becomes [6, Theorem 3.1].

Proposition 3.3 *Each singleton is a $\mathcal{D}$-set in an LD-metric space $(X, \mathcal{D})$, so $\mathcal{D}(\{x\}) = \{x\}$.*

Proof For each $x \in X$, $\{x\} = \bigcap_{\epsilon>0} B(x, \epsilon)$. Since all open balls and their intersection are $\mathcal{D}$-sets, $\{x\}$ is a $\mathcal{D}$-set. Therefore $\mathcal{D}(\{x\}) \subset \{x\}$, *i.e.*, $\mathcal{D}(\{x\}) = \{x\}$. □

For LD-metric spaces, Theorem 3.2 reduces to the following.

Corollary 3.4 *Let X be a paracompact space, $(Y, \mathcal{D})$ be an LD-metric space, and Z be a closed subset of X with $\dim_X Z \leq 0$. If $F : X \multimap Y$ is an alsc multimap such that $F(x)$ is a $\mathcal{D}$-set for all $x \in X \backslash Z$, then for $\epsilon > 0$, F has an ϵ-approximate selection.*

For LC-metric spaces, Corollary 3.4 reduces to the following.

Corollary 3.5 *Let X be a paracompact space, (Y, Γ) be an LC-metric space, and Z be a closed subset of X with $\dim_X Z \leq 0$. If $F : X \multimap Y$ is an alsc multimap such that $F(x)$ is Γ-convex for all $x \in X \backslash Z$, then for $\epsilon > 0$, F has an ϵ-approximate selection.*

Remark Corollary 3.5 is Theorem 3.2 in [8] which is a partial generalization of Lemma 2 in [3]. In the proof of Lemma 2 in [3] and Theorem 3.2 in [8], for the subset E of Z, it is claimed that $B(F(x), \epsilon)$ is Γ-convex whenever $x \in E$ and $x \in X \backslash E$, but it cannot be analogized from the assumption that $F(x)$ is Γ-convex for all $x \notin Z$.

Theorem 3.3 in [6] shows that if $X = Z$ and (Y, Γ) is a C-space with a uniformity $\mathcal{U}$ and F has a V-approximate selection for each $V \in \mathcal{U}$, then F is qlsc. Using the same pattern of its proof, we also obtain the same result when $(Y, \mathcal{D})$ is a D-space with a uniformity $\mathcal{U}$. Since a qlsc map is alsc, so the inverse of Theorem 3.2 also holds.

Theorem 3.6 *Let X be a paracompact topological space and Z be a closed subset of X with $\dim_X Z \leq 0$. Let $(Y, \mathcal{D})$ be an LD-space with a uniformity $\mathcal{U}$ and $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. And let $F : X \multimap Y$ be a multimap such that $F(x)$ is a $\mathcal{D}$-set for all $x \in X \backslash Z$. Then F is alsc if and only if F has a V-approximate selection for each $V \in \mathcal{U}$.*

The following notion is motivated by Hadžić [11]. Let $(X, \mathcal{D})$ be a D-space with a uniformity $\mathcal{U}$ and K be a nonempty subset of X. We say that K is *of generalized Zima type* whenever for every $V \in \mathcal{U}$, there exists a $V_1 \in \mathcal{U}$ such that for every $N \in \langle K \rangle$ and every $\mathcal{D}$-set M of K, the following implication holds:

$$M \cap V_1(z) \neq \emptyset, \quad \forall z \in N \quad \Longrightarrow \quad M \cap V(u) \neq \emptyset, \quad \forall u \in \mathcal{D}(N).$$

Note that an LD-space $(X, \mathcal{D})$ is of generalized Zima type.

If $Z = \emptyset$, then the LD-space condition of Y can be weakened in Theorem 3.6.

Theorem 3.7 *Let X be a paracompact topological space, $(Y, \mathcal{D})$ be a D-space with a uniformity $\mathcal{U}$, and $F : X \multimap Y$ be a multimap with $\mathcal{D}$-set values such that $F(X)$ is of generalized Zima type. Then F is alsc if and only if F has a V-approximate selection for each $V \in \mathcal{U}$.*

The proof of Theorem 3.7 proceeds in the same fashion as Theorem 2 in [7], except that all Γ-convex sets in a C-space is replaced by $\mathcal{D}$-sets in a $\mathcal{D}$-space.

Let X be a topological space and Y be a uniform space with a uniformity $\mathcal{U}$. The multimaps $F, T : X \multimap Y$ are said to be *topologically separated* if for each $x \in X$, there exist a neighborhood $U(x)$ of x and an element $V \in \mathcal{U}$ such that $F(U(x)) \cap V(T(x)) = \emptyset$.

Theorem 3.8 *Let X be a compact topological space and Z be a closed subset of X with $\dim_X Z \leq 0$. And let $(Y, \mathcal{D})$ be an LD-space with a uniformity $\mathcal{U}$ and $\mathcal{D}(\{y\}) = \{y\}$ for all $y \in Y$. If $F, T : X \multimap Y$ are two multimaps such that*

(1) *F and T are topologically separated*;
(2) *T is upper semicontinuous*; *and*
(3) *F is an alsc multimap such that $F(x)$ is a $\mathcal{D}$-set for all $x \in X \backslash Z$.*

Then, for each $V \in \mathcal{U}$, F has a V-approximate selection $f : X \to Y$ such that

$$f(x) \notin T(x)$$

for all $x \in X$.

Using Theorem 3.2, the proof of Theorem 3.8 proceeds in precisely the same fashion as Theorem 3.6 in [6].

Particular forms 1. Zheng [9, Theorem 2.2]: Y is a locally convex space, $Z = \emptyset$, and F is sub-lower semicontinuous, that is, for each $x \in X$ and each neighborhood V of 0 in Y, there is $z \in F(x)$ and a neighborhood $U(x)$ of x in X such that for each $y \in U(x)$, $z \in F(y) + V$. Note that if Y is a topological vector space, then F is sub-lower semicontinuous if and only if F is qlsc; see [6, Proposition 1.2].

2. Wu and Li [6, Theorem 3.6]: $(Y, \mathcal{D})$ is an LC-space with a uniformity $\mathcal{U}$, $Z = \emptyset$, and F is qlsc.

Proposition 3.9 *Let X be a topological space and Y be a metric space. If a multimap $F : X \multimap Y$ is alsc at $x \in X$, then F is qlsc at $x \in X$.*

Proof For $\epsilon > 0$, there is a neighborhood $U(x)$ of x such that

$$\bigcap_{z \in U(x)} B\big(F(z), \epsilon/2\big) \neq \emptyset.$$

Select any $y \in \bigcap_{z \in U(x)} B(F(z), \epsilon/2)$. For each $z \in U(x)$, choose $y_z \in F(z)$ such that $d(y, y_z) < \epsilon/2$. Note that $y_x \in F(x)$ and $d(y_x, y_z) \leq d(y_x, y) + d(y, y_z) < \epsilon$ for each $z \in U(x)$. Hence $y_z \in B(y_x, \epsilon) \cap F(z)$ for all $z \in U(x)$. □

The following result is a generalization of Theorem 3.7 in [6].

Theorem 3.10 *Let X be a paracompact topological space, $(Y, \mathcal{D})$ be an LD-metric space, and Z be a closed subset of X with $\dim_X Z \leq 0$. If $F, T : X \multimap Y$ are two multimaps such that*

(1) *F and T are topologically separated*;

(2) *T is upper semicontinuous*; *and*

(3) *F is an alsc multimap such that $F(x)$ is a $\mathcal{D}$-set for all $x \in X \setminus Z$.*

Then for each $\epsilon > 0$, F has an ϵ-approximate selection $f : X \to Y$ such that

$$f(x) \notin T(x)$$

for all $x \in X$.

Proof For each fixed $\epsilon > 0$ and each $x \in X$, by (1) and (2), there exists a neighborhood $U(x)$ of x and an $\eta(x) > 0$ such that $\eta(x) < \epsilon$, $F(U(x)) \cap B(T(x), \eta(x)) = \emptyset$, and $T(U(x)) \subset B(T(x), \eta(x)/2)$. Let $\zeta(x) = \eta(x)/2$. For each $y \in U(x)$, we assert $F(y) \cap B(T(y), \zeta(x)) = \emptyset$. Otherwise, there exist points $p \in T(y)$ and $z \in F(y)$ such that $d(p,z) < \zeta(x)$. Because $y \in U(x)$ and $T(y) \subset B(T(x), \zeta(x))$, so $p \in B(T(x), \zeta(x))$. Consequently, there is a point $b \in T(x)$ such that $d(p,b) < \zeta(x)$. Hence $d(b,z) < \eta(x)$, and thus $z \in F(y) \cap B(T(x), \eta(x)) \subset F(U(x)) \cap B(T(x), \eta(x)) = \emptyset$. It is a contradiction.

Let $\delta(x) = \sup\{r : 0 < r < \epsilon \text{ and } F(x) \cap B(T(x), r) = \emptyset\}$. Obviously, $\delta(x) \leq \epsilon$ and for each $y \in U(x)$, $\delta(y) \geq \zeta(x)$. Now, we assert that $F(x) \cap B(T(x), \delta(x)) = \emptyset$. Otherwise, there exist points $y \in F(x)$ and $z \in T(x)$ such that $d(y,z) < \delta(x)$. Consequently, there is a number $r > d(y,z)$ such that $0 < r < \epsilon$ and $F(x) \cap B(T(x), r) = \emptyset$. But $y \in F(x) \cap B(T(x), r)$, it is a contradiction.

By Proposition 3.9, $F : X \multimap Y$ is qlsc, so there exist a point $y_x \in F(x)$ and an open neighborhood $N(x)$ of x in X such that $N(x) \subset U(x)$ and

$$F(z) \cap B\big(y_x, \zeta(x)\big) \neq \emptyset \qquad (*)$$

for all $z \in N(x)$. Since X is paracompact, the open cover $\{N(x) : x \in X\}$ has a locally finite open refinement $\{V(x) : x \in X\}$. Since $\dim_X Z \leq 0$, the relative open cover $\{V(x) \cap Z : x \in Z\}$ of Z has a relatively open disjoint refinement $\{W(x) : x \in Z\}$. Z is closed in X, so the collection $\mathcal{R} = \{V(x) \cap (W(x) \cup (X \setminus Z)) : x \in X\}$ forms a locally finite open cover of X.

For each $O \in \mathcal{R}$, choose a point x_o such that $O \subset V(x_o)$ and define $\eta : \mathcal{R} \to Y$ by $\eta(O) = y_{x_o}$ such that $y_{x_o} \in F(x_o)$ satisfying the condition $(*)$ for all $z \in V(x_o)$. By Proposition 3.1, there is a continuous function $f : X \to Y$ such that $f(x) \in \mathcal{D}(\{\eta(O) : O \in \mathcal{R}, x \in O\})$. For each $x \in X$ and $O \in \mathcal{R}$ such that $x \in O$, by $(*)$, we have $F(x) \cap B(\eta(O), \zeta(x_o)) \neq \emptyset$ and $\delta(x) \geq \zeta(x_o)$ because $x \in O \subset V(x_o) \subset N(x_o)$, so $F(x) \cap B(\eta(O), \delta(x)) \neq \emptyset$.

Note that for $x \in Z$, $f(x) = \eta(O) = y_{x_o}$ since $\{O \in \mathcal{R} : x \in O\}$ is a singleton. Hence $f(x) \in \{y \in Y : F(x) \cap B(y, \delta(x)) \neq \emptyset\}$.

For each $x \in X \setminus Z$,

$$f(x) \in \mathcal{D}\big(\{\eta(O) : O \in \mathcal{R}, x \in O\}\big) \subset \big\{y \in Y : F(x) \cap B\big(y, \delta(x)\big) \neq \emptyset\big\}$$

since $F(x)$ and $B(F(x), \delta(x))$ are $\mathcal{D}$-sets.

So, $F(x) \cap B(f(x), \delta(x)) \neq \emptyset$ for all $x \in X$ and hence

$$F(x) \cap B\big(f(x), \epsilon\big) \neq \emptyset \quad \text{and} \quad f(x) \notin T(x). \qquad \square$$

4 Approximate selection theorems on topological ordered spaces

A *semilattice* is a partially ordered set X, with the partial ordering denoted by $\leq$, for which any pair (x, x') of elements has a least upper bound. Any nonempty set $A \in \langle X \rangle$ has a least upper bound, denoted by sup A. In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable, but, in the case where $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called *an order interval*.

The following is due to Horvath and Ciscar [12]: Let $(X, \leq)$ be a semilattice such that for each $A \in \langle X \rangle$, $\Delta(A)$ is defined by $\bigcup_{a \in A}[a, \sup A]$. Then

(1) $\Delta(A)$ is well defined;

(2) $A \subset \Delta(A)$;

(3) if $A \subset B$, then $\Delta(A) \subset \Delta(B)$.

A subset $E \subset X$ is said to be *convex* if, for any subset $A \in \langle E \rangle$, we have $\Delta(A) \subset E$.

If X is a topological semilattice with path-connected intervals, then for any $A \in \langle X \rangle$ and $n \geq 0$, $\Delta(A)$ is n-connected by [12, Lemma 1], that is, $(X, \leq, \Delta)$ is a D-space.

Note that $\Delta(\{x\}) = \{x\}$ for all $x \in X$.

In this section, we assume that $(Y, \leq, \Delta)$ is a topological semilattice with path-connected intervals. From the results of Section 3, we obtain the following theorems.

Theorem 4.1 *Let X be a paracompact topological space, Z be a closed subset of X with $\dim_X Z \leq 0$, and $\mathcal{U}$ be a uniformity of $(Y, \leq, \Delta)$ such that for each $V \in \mathcal{U}$, the set $\{y \in Y : C \cap V(y) \neq \emptyset\}$ is convex whenever C is a convex subset of Y. And let $F \subseteq X \times Y$ be a binary relation such that $F(x)$ is convex for all $x \in X \setminus Z$. Then F is alsc if and only if F has a V-approximate selection for each $V \in \mathcal{U}$.*

Theorem 4.2 *Let X be a paracompact topological space and $\mathcal{U}$ be a uniformity of $(Y, \leq, \Delta)$. And let $F \subseteq X \times Y$ be a binary relation with convex values and $F(X)$ be of generalized Zima type. Then F is alsc if and only if F has a V-approximate selection for each $V \in \mathcal{U}$.*

Theorem 4.3 *Let X be a compact topological space and Z be a closed subset of X with $\dim_X Z \leq 0$. And let $\mathcal{U}$ be a uniformity of $(Y, \leq, \Delta)$ such that for each $V \in \mathcal{U}$, the set $\{x \in X : C \cap V(x) \neq \emptyset\}$ is convex whenever $C \subset X$ is convex. If $F, T \subseteq X \times Y$ are two binary relations such that*

(1) *F and T are topologically separated*;

(2) *T is upper semicontinuous*; *and*

(3) *F is an alsc relation such that $F(x)$ is convex for all $x \in X \setminus Z$.*

Then for each $V \in \mathcal{U}$, F has a V-approximate selection $f : X \to Y$ such that

$$f(x) \notin T(x)$$

for all $x \in X$.

Theorem 4.4 *Let X be a paracompact topological space and Z be a closed subset of X with $\dim_X Z \leq 0$. Assume that $(Y, \leq, \Delta)$ is a metric space and has the following properties*:

(i) *for any $\epsilon > 0$, the set $B(C, \epsilon)$ is convex whenever C is a convex subset of Y*; *and*

(ii) *open balls are convex.*

If $F, T \subseteq X \times Y$ are two binary relations such that

(1) *F and T are topologically separated*;

(2) *T is upper semicontinuous*; *and*

(3) *F is an alsc relation such that $F(x)$ is convex for all $x \in X \setminus Z$.*

Then for each $\epsilon > 0$, F *has an* ϵ*-approximate selection* $f : X \to Y$ *such that*

$$f(x) \notin T(x)$$

for all $x \in X$.

Competing interests
The author declares that she has no competing interests.

Abbreviations
alsc: almost lower semicontinuity; lsc: lower semicontinuity; qlsc: quasi-lower semicontinuity.

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