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Shearlet approximations to the inverse of a family of linear operators

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Abstract
The Radon transform plays an important role in applied mathematics. It is a fundamental problem to reconstruct images from noisy observations of Radon data. Compared with traditional methods, Colona, Easley and *etc.* apply shearlets to deal with the inverse problem of the Radon transform and receive more effective reconstruction. This paper extends their work to a class of linear operators, which contains Radon, Bessel and Riesz fractional integration transforms as special examples.
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1 Introduction and preliminary

The Radon transform is an important tool in medical imaging. Although $f \in L^1(\mathbb{R}^2)$ can be recovered analytically from the Radon data $Rf(\theta, t)$, the solution is unstable and those data are corrupted by some noise in practice [1]. In order to recover the object f stably and control the amplification of noise in the reconstruction, many methods of regularization were introduced including the Fourier method, singular value decomposition, *etc.* [2]. However, those methods produced a blurred version of the original one.

Curvelets and shearlets were then proposed, which proved to be efficient in dealing with edges [3–7]. In 2002, Candés and Donoho applied curvelets [5] to the inverse problem

$$Y = Rf + \varepsilon W, \tag{1.1}$$

where the recovered function f is compactly supported and twice continuously differentiable away from a smooth edge; W denotes a Wiener sheet; ε is a noisy level. Because curvelets have complicated structure, Colonna, Easley, *etc.* used shearlets to deal with the problem (1.1) in 2010 and received an effective reconstructive algorithm [8].

Note that the Bessel transform and the Riesz fractional integration transform arise in many scientific areas ranging from physical chemistry to extragalactic astronomy. Then this paper considers a more general problem,

$$Y = Kf + \varepsilon W, \tag{1.2}$$

where K stands for a linear operator mapping the Hilbert space $L^2(\mathbb{R}^2)$ to another Hilbert space Y and satisfies

$$\left(K^*Kf\right)^\wedge(\xi) = \left(b + |\xi|^2\right)^{-\alpha}\hat{f}(\xi) \tag{1.3}$$

with $b > 0$, $\alpha > 0$ (K^* is the conjugate operator of K). Here and in what follows, $\hat{f}$ denotes the Fourier transform of f. The next section shows that Radon, Bessel and Riesz fractional integration transforms satisfy the condition (1.3).

The current paper is organized as follows. Section 2 presents three examples for (1.3) and several lemmas. An approximation result is proved in the last section, which contains Theorem 4.2 of [8] as a special case.

At the end of this section, we introduce some basic knowledge of shearlets, which will be used in our discussions. The Fourier transform of a function $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

The classical method extends that definition to $L^2(\mathbb{R}^2)$ functions.

There exist many different constructions for discrete shearlets. We introduce the construction [8] by taking two functions ψ_1, ψ_2 of one variable such that $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$ with their supports $\operatorname{supp}\hat{\psi}_1 \subset [-\frac{1}{2}, -\frac{1}{16}] \cup [\frac{1}{16}, \frac{1}{2}]$, $\operatorname{supp}\hat{\psi}_2 \subset [-1, 1]$ and

$$\sum_{j\geq 0}\left|\hat{\psi}_1\left(2^{-2j}\omega\right)\right|^2 = 1 \quad \left(|\omega| \geq \frac{1}{8}\right), \qquad \sum_{l=-2^j}^{2^j-1}\left|\hat{\psi}_2\left(2^j\omega - l\right)\right|^2 = 1 \quad \left(|\omega| \leq 1\right).$$

Here, $C^\infty(\mathbb{R}^n)$ stands for infinitely many times differentiable functions on the Euclidean space $\mathbb{R}^n$. Then two shearlet functions $\psi^{(0)}$, $\psi^{(1)}$ are defined by

$$\hat{\psi}^{(0)}(\xi) := \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \quad \text{and} \quad \hat{\psi}^{(1)}(\xi) := \hat{\psi}_1(\xi_2)\hat{\psi}_2\left(\frac{\xi_1}{\xi_2}\right)$$

respectively.

To introduce discrete shearlets, we need two shear matrices

$$B_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and two dilation matrices

$$A_0 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Define discrete shearlets $\psi^{(d)}_{j,l,k}(x) := 2^{\frac{3}{2}j}\psi^{(d)}(B^l_dA^j_dx - k)$ for $j \geq 0$, $-2^j \leq l \leq 2^j - 1$, $k \in \mathbb{Z}^2$ and $d = 0, 1$. Then there exists $\hat{\varphi} \in C^\infty_0(\mathbb{R}^2)$ such that

$$\left\{\varphi_{j_0,k}(x), \psi^{(d)}_{j,l,k}(x), j \geq j_0 \geq 0, -2^j \leq l \leq 2^j - 1, k \in \mathbb{Z}^2, d = 0, 1\right\}$$

forms a Parseval frame of $L^2(\mathbb{R}^2)$, where $\varphi_{j_0,k}(x) := 2^{j_0}\varphi(2^{j_0}x - k)$. More precisely, for $f \in L^2(\mathbb{R}^2)$,

$$f(x) = \sum_{k\in\mathbb{Z}^2} \langle f, \varphi_{j_0,k}\rangle \varphi_{j_0,k}(x) + \sum_{d=0}^{1}\sum_{j\geq j_0}\sum_{l=-2^j}^{2^j-1}\sum_{k\in\mathbb{Z}^2} \langle f, \psi_{j,l,k}^{(d)}\rangle \psi_{j,l,k}^{(d)}(x)$$

holds in $L^2(\mathbb{R}^2)$. It should be pointed out that $\psi_{j,l,k}^{(d)}(x)$ are modified for $l = -2^j$ and $2^j - 1$, as seen in [8].

2 Examples and lemmas

In this section, we provide three important examples of a linear operator K satisfying $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$ and present some lemmas which will be used in the next section. To introduce the first one, define a subspace of $L^2(\mathbb{R}^2)$,

$$D(\mathbb{R}^2) := \{f \in L^1(\mathbb{R}^2), f \text{ is bounded}\} \subseteq \left\{f \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} |\xi|^{-1}|\hat{f}(\xi)|^2\, d\xi < +\infty\right\}$$

and a Hilbert space

$$L^2([0,\pi) \times \mathbb{R}) := \left\{f(\theta,t), \int_0^\pi \int_{\mathbb{R}} |f(\theta,t)|^2\, dt\, d\theta < +\infty\right\}$$

with the inner product $\langle f, g\rangle := \int_0^\pi \int_{\mathbb{R}} f(\theta,t)\overline{g(\theta,t)}\, dt\, d\theta$.

Example 2.1 Let $L_{\theta,t} := \{(x,y), x\cos\theta + y\sin\theta = t\} \subseteq \mathbb{R}^2$ and $ds(x,y)$ be the Euclidean measure on the line $L_{\theta,t}$. Then the classical Radon transform R: $D(\mathbb{R}) \to L^2([0,\pi) \times \mathbb{R})$ defined by

$$Rf(\theta,t) = \int_{L_{\theta,t}} f(x,y)\, ds(x,y)$$

satisfies $(R^*Rf)^\wedge(\xi) = |\xi|^{-1}\hat{f}(\xi)$.

Proof By the definition of $D(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} |\xi|^{-1}\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi < +\infty$ for $f, g \in D(\mathbb{R}^2)$. It is easy to see that $\int_0^{2\pi} d\theta \int_0^{+\infty} \hat{f}(\omega\cos\theta, \omega\sin\theta)\overline{\hat{g}(\omega\cos\theta,\omega\sin\theta)}\, d\omega = \int_0^\pi d\theta \int_{\mathbb{R}} \hat{f}(\omega\cos\theta,\omega\sin\theta) \times \overline{\hat{g}(\omega\cos\theta,\omega\sin\theta)}\, d\omega$. This with the Fourier slice theorem ([1, 9]) and the Plancherel formula leads to

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^{-1}\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi &= \int_0^{2\pi} d\theta \int_0^{+\infty} \hat{f}(\omega\cos\theta,\omega\sin\theta)\overline{\hat{g}(\omega\cos\theta,\omega\sin\theta)}\, d\omega \\ &= \int_0^\pi d\theta \int_{\mathbb{R}} (R_\theta f)^\wedge(\omega)\overline{(R_\theta g)^\wedge(\omega)}\, d\omega \\ &= \int_0^\pi d\theta \int_{\mathbb{R}} Rf(\theta,t)\overline{Rg(\theta,t)}\, dt = \langle Rf, Rg\rangle,\end{aligned}$$

where $R_\theta f(t) := Rf(\theta,t)$. Moreover, $\langle (R^*Rf)^\wedge, \hat{g}\rangle = \langle R^*Rf, g\rangle = \langle Rf, Rg\rangle = \langle |\xi|^{-1}\hat{f}(\xi), \hat{g}(\xi)\rangle$ for each $g \in D(\mathbb{R}^2)$.

Because $D(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, one receives the desired conclusion $(R^*Rf)^\wedge(\xi) = |\xi|^{-1}\hat{f}(\xi)$. Here, $R^*Rf \in L^2(\mathbb{R}^2)$ for $f \in D(\mathbb{R}^2)$. In fact, $R^*Rf = 4\pi I_1 f$ by [10], where I_1 is the Riesz fractional integration transform defined by

$$I_1(f)(x) := C_\alpha \int_{\mathbb{R}^2} \frac{f(x-y)}{|y|}\,dy$$

with some normalizing constant C_α [11]. We rewrite $I_1(f)(x) = \int_{|y|\le r} \frac{f(x-y)}{|y|}\,dy + \int_{|y|>r} \frac{f(x-y)}{|y|}\,dy =: J_1 + J_2$. Let $h(y) = \frac{1}{|y|}1_{B(0,1)}(y)$, $h_r(y) = \frac{1}{r^2}h(\frac{y}{r})$, where $B(0,1)$ stands for the unit ball of $\mathbb{R}^2$ and 1_A represents an indicator function on the set A. Then $J_1 = \int_{|y|\le r} \frac{f(x-y)}{|y|}\,dy = r\int_{|y|\le r} h_r(y)f(x-y)\,dy \le rMf(x)$ by Theorem 9 of reference [12, p.59], where Mf is the Hardy-Littlewood maximal function of f.

On the other hand, the Holder inequality implies

$$J_2 \le \|f\|_{\frac{p}{p-1}} \left(\int_{|y|>r} \frac{1}{|y|^p}\,dy\right)^{\frac{1}{p}} \le r^{2-p}\|f\|_{\frac{p}{p-1}}$$

with $p > 3$. Take $r = [Mf(x)]^{-\frac{1}{p-1}}$, one gets $I_1(f)(x) \le [Mf(x)]^{\frac{p-2}{p-1}}(1 + \|f\|_{\frac{p}{p-1}})$ and $\|I_1(f)\|_2 \le (1 + \|f\|_{\frac{p}{p-1}})\|Mf\|^{\frac{p-2}{p-1}}_{\frac{2(p-2)}{p-1}} \lesssim (1 + \|f\|_{\frac{p}{p-1}})\|f\|^{\frac{p-2}{p-1}}_{\frac{2(p-2)}{p-1}} < +\infty$, since $f \in L^{\frac{p}{p-1}}(\mathbb{R}^2) \cap L^{\frac{2(p-2)}{p-1}}(\mathbb{R}^2)$ due to the assumption $f \in D(\mathbb{R}^2)$ and $\frac{2(p-2)}{p-1} > 1$, $\frac{p}{p-1} > 1$.

In order to introduce the next example, we use $f * g$ to denote the convolution of f and g. □

Example 2.2 The Bessel operator $B_\alpha : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by $B_\alpha f = b_\alpha * f$ with $\hat{b}_\alpha(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}}$ and $\alpha > 0$ satisfies

$$\left(B_\alpha^* B_\alpha f\right)^\wedge(\xi) = \left(1 + |\xi|^2\right)^{-\alpha}\hat{f}(\xi).$$

Proof It is known that $b_\alpha(x) \in L^1(\mathbb{R}^2)$ for $\alpha > 0$ [11]. Hence, $(B_\alpha f)^\wedge(\xi) = \hat{b}_\alpha(\xi)\hat{f}(\xi) = (1 + |\xi|^2)^{-\frac{\alpha}{2}}\hat{f}(\xi)$. For $f, g \in L^2(\mathbb{R}^2)$, $\langle (B_\alpha^* B_\alpha f)^\wedge, \hat{g}\rangle = \langle B_\alpha^* B_\alpha f, g\rangle = \langle B_\alpha f, B_\alpha g\rangle = \langle (B_\alpha f)^\wedge, (B_\alpha g)^\wedge\rangle = \langle (1 + |\xi|^2)^{-\alpha}\hat{f}(\xi), \hat{g}(\xi)\rangle$. Thus,

$$\left(B_\alpha^* B_\alpha f\right)^\wedge(\xi) = \left(1 + |\xi|^2\right)^{-\alpha}\hat{f}(\xi).$$ □

To introduce the Riesz fractional integration transform, we define

$$D = \left\{f \in L^2\left(\mathbb{R}^2\right), f \text{ has compact support}\right\} \subseteq L^1\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right).$$

Then $D \subseteq L^s(\mathbb{R}^2)$ $(1 \le s \le 2)$. For $f \in D$ and $0 < \alpha < 1$, the Riesz fractional integration transform is defined by

$$I_\alpha(f)(x) := C_\alpha \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-\alpha}}\,dy \in L^2\left(\mathbb{R}^2\right), \tag{2.1}$$

where C_α is the normalizing constant [11]. In order to show $(I_\alpha^* I_\alpha f)^\wedge(\xi) = |\xi|^{-2\alpha}\hat{f}(\xi)$ for $f \in D$ and $0 < \alpha < 1/2$, we need a lemma ([11], Lemma 2.15).

Lemma 2.1 *Let $S(\mathbb{R}^2)$ be the Schwartz space and $\Psi = \{\psi \in S(\mathbb{R}^2), \frac{\partial^\beta}{\partial x^\beta}\psi(0) = 0, \beta \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$ with $\mathbb{Z}^+$ being the non-negative integer set. Define $\Phi := \{\varphi = \hat{\psi}, \psi \in \Psi\}$. Then with $\alpha > 0$,*

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$$

holds for each $f \in \Phi$.

Example 2.3 The transform I_α defined by (2.1) satisfies $(I_\alpha^* I_\alpha f)^\wedge(\xi) = |\xi|^{-2\alpha}\hat{f}(\xi)$ for $f \in D$ and $0 < \alpha < \frac{1}{2}$.

Proof As proved in Examples 2.1, 2.2, it is sufficient to show that for $f \in D$,

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi). \tag{2.2}$$

One proves (2.2) firstly for $f \in C_0^\infty(\mathbb{R}^2)$. Take $\mu(r) \in C^\infty([0,\infty))$ with $0 \le \mu(r) \le 1$ and

$$\mu(r) = \begin{cases} 1, & r \ge 2; \\ 0, & 0 \le r \le 1. \end{cases}$$

Define $\psi_N(\xi) := \mu(N|\xi|)\hat{f}(\xi)$. Then $\psi_N(\xi) \in \Psi$ and $f_N(x) := \check{\psi}_N(x) = \hat{\psi}_N(-x) \in \Phi$. By Lemma 2.1,

$$(I_\alpha f_N)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}_N(\xi). \tag{2.3}$$

Let $k(x)$ be the inverse Fourier transform of the function $1 - \mu(|x|)$ and $k_N(x) := \frac{1}{N^2}k(\frac{x}{N})$. Then $\int_{\mathbb{R}^2} k(x)\,dx = 1$ and $f_N(x) = f(x) - k_N * f(x)$. Moreover, the classical approximation theorem [11] tells

$$\lim_{N\to\infty} \|f_N - f\|_p = 0$$

for $p > 1$. On the other hand, $\|I_\alpha f_N - I_\alpha f\|_2 = \|I_\alpha(f_N - f)\|_2 \le C\|f_N - f\|_{\frac{2}{1+\alpha}}$ due to Theorem 16 [12, p.69]. Hence, $\lim_{N\to\infty} \|(I_\alpha f_N)^\wedge(\xi) - (I_\alpha f)^\wedge(\xi)\|_2 = \lim_{N\to\infty} \|I_\alpha f_N - I_\alpha f\|_2 = 0$. That is,

$$\lim_{N\to\infty} (I_\alpha f_N)^\wedge(\xi) = (I_\alpha f)^\wedge(\xi) \tag{2.4}$$

in $L^2(\mathbb{R}^2)$ sense. Note that $\||\xi|^{-\alpha}\hat{f}_N(\xi) - |\xi|^{-\alpha}\hat{f}(\xi)\|_2^2 = \int_{\mathbb{R}^2} |\xi|^{-2\alpha}|\hat{f}(\xi)|^2[1 - \mu(N|\xi|)]^2\,d\xi$; $|\xi|^{-2\alpha}|\hat{f}(\xi)|^2 \in L^1(\mathbb{R}^2)$ with $0 < \alpha < \frac{1}{2}$ and $\lim_{N\to\infty}[1 - \mu(N|\xi|)] = 0$. Then

$$\lim_{N\to\infty} \bigl\| |\xi|^{-\alpha}\hat{f}_N(\xi) - |\xi|^{-\alpha}\hat{f}(\xi)\bigr\|_2 = 0 \tag{2.5}$$

thanks to the Lebesgue dominated convergence theorem, which means $\lim_{N\to\infty} |\xi|^{-\alpha} \times \hat{f}_N(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$ in $L^2(\mathbb{R}^2)$ sense. This with (2.3), (2.4) shows (2.2) for $f \in C_0^\infty(\mathbb{R}^2)$.

In order to show (2.2) for $f \in D$, one can find $g \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} g(x)\,dx = 1$ and $\lim_{N\to\infty} \|f * g_N - f\|_p = 0$ $(p \ge 1)$ by Theorem 4.2.1 in [13], where $g_N(\cdot) = N^2 g(N\cdot)$. Since

$f * g_N \in C_0^\infty(\mathbb{R}^2)$, the above proved fact says

$$\big(I_\alpha(f * g_N)\big)^\wedge(\xi) = |\xi|^{-\alpha}(f * g_N)^\wedge(\xi). \tag{2.6}$$

The same arguments as (2.4) and (2.5) show that $\lim_{N\to\infty}(I_\alpha(f * g_N))^\wedge(\xi) = (I_\alpha f)^\wedge(\xi)$ and $\lim_{N\to\infty}|\xi|^{-\alpha}(f * g_N)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$. Hence,

$$(I_\alpha f)^\wedge(\xi) = |\xi|^{-\alpha}\hat{f}(\xi).$$

This completes the proof of (2.2) for $f \in D$. □

Next, we prove a lemma which will be used in the next section. For convenience, here and in what follows, we define $\mathcal{M} = N \cup M$ with $N = \mathbb{Z}^2$, $M := \{\mu = (j,l,k,d) : j \geq j_0, -2^j \leq l \leq 2^j - 1, k \in \mathbb{Z}^2, d = 0,1\}$. Then the shearlet system (introduced in Section 1) can be represented as $\{s_\mu : \mu \in \mathcal{M}\}$, where $s_\mu = \psi_\mu = \psi_{j,l,k}^{(d)}$ if $\mu \in M$, and $s_\mu = \varphi_\mu = \varphi_{j_0,k}$ if $\mu \in N$.

Lemma 2.2 *Let K satisfy $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$ and $\{s_\mu, \mu \in \mathcal{M}\}$ be shearlets introduced in the first section. Define $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$ and $U_\mu := 2^{-2\alpha j}K\sigma_\mu$. Then $\|U_\mu\| \leq C$ and for $\mu \in \mathcal{M}$,*

$$\langle f, s_\mu\rangle = 2^{2\alpha j}\langle Kf, U_\mu\rangle.$$

Proof By the Plancherel formula and the assumption $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$, one knows that $\langle f, s_\mu\rangle = \langle \hat{f}, \hat{s}_\mu\rangle = \langle \hat{f}(\xi), (b + |\xi|^2)^{-\alpha}\hat{\sigma}_\mu(\xi)\rangle$. Moreover,

$$\langle f, s_\mu\rangle = \big\langle \hat{f}(\xi), \big(K^*K\sigma_\mu\big)^\wedge(\xi)\big\rangle = \big\langle f, K^*K\sigma_\mu\big\rangle = \langle Kf, K\sigma_\mu\rangle = 2^{2\alpha j}\langle Kf, U_\mu\rangle$$

due to $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$ and $U_\mu := 2^{-2\alpha j}K\sigma_\mu$.

Next, one shows $\|U_\mu\| \leq C$. Note that $\|K\sigma_\mu\|^2 = \langle K\sigma_\mu, K\sigma_\mu\rangle = \langle K^*K\sigma_\mu, \sigma_\mu\rangle = \langle (K^*K\sigma_\mu)^\wedge, \hat{\sigma}_\mu\rangle$, $(K^*K\sigma_\mu)^\wedge = (b + |\xi|^2)^{-\alpha}\hat{\sigma}_\mu(\xi)$ and $\hat{\sigma}_\mu(\xi) = (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)$. Then $\|K\sigma_\mu\|^2 = \langle \hat{s}_\mu(\xi), (b + |\xi|^2)^\alpha \hat{s}_\mu(\xi)\rangle$ and

$$\|U_\mu\|^2 = 2^{-4\alpha j}\|K\sigma_\mu\|^2 = 2^{-4\alpha j}\int_{\mathbb{R}^2}\big(b + |\xi|^2\big)^\alpha \big|\hat{s}_\mu(\xi)\big|^2\,d\xi.$$

Because $\operatorname{supp}\hat{s}_\mu \subseteq C_j := [-2^{2j-1}, 2^{2j-1}]^2 \setminus [-2^{2j-4}, 2^{2j-4}]^2$, one receives $\|U_\mu\|^2 = 2^{-4\alpha j}\int_{C_j}(b + |\xi|^2)^\alpha |\hat{s}_\mu(\xi)|^2\,d\xi \leq C$. This completes the proof of Lemma 2.2. □

At the end of this section, we introduce two theorems which are important for our discussions. As in [8], we use $STAR^2(A)$ to denote all sets $B \subseteq [0,1]^2$ with C^2 boundary ∂B given by

$$\beta(\theta) = \begin{pmatrix}\rho(\theta)\cos\theta \\ \rho(\theta)\sin\theta\end{pmatrix}$$

in a polar coordinate system. Here, $\rho(\theta) \leq \rho_0 < 1$ and $|\rho''(\theta)| \leq A$. Define $\varepsilon^2(A) := \{f = f_0 + f_1 X_B, B \in STAR^2(A)\}$, where $f_0, f_1 \in C_0^2([0,1]^2)$ are compactly supported on $[0,1]^2$. Let

$c_\mu := \langle f, s_\mu \rangle$, $M_j := \{(j,l,k,d), |k| \le 2^{2j+1}, -2^j \le l \le 2^j - 1, d = 0,1\}$ and

$$R(j,\varepsilon) = \{\mu \in M_j : |c_\mu| > \varepsilon\}.$$

Then with $\sharp R(j,\varepsilon)$ standing for the cardinality of $R(j,\varepsilon)$, the following conclusion holds [8].

Theorem 2.3 *For* $f \in \varepsilon^2(A)$, $\sharp R(j,\varepsilon) \le C\varepsilon^{-\frac{2}{3}}$ *and*

$$\sum_{\mu \in M_j} |c_\mu|^2 \le C 2^{-2j}.$$

Theorem 2.4 [14] *Let* $X \sim N(u,1)$ *and* $t = \sqrt{2\log(\eta^{-1})}$ *with* $0 < \eta \le \frac{1}{2}$. *Then*

$$E\left|T_s(X,t) - u\right|^2 = \left[2\log\left(\eta^{-1}\right) + 1\right]\left(\eta + \min\left\{u^2, 1\right\}\right),$$

where $N(u,1)$ *denotes the normal distribution with mean u and variance* 1, *while* $T_s(y,t) := \operatorname{sgn}(y)(|y| - t)_+$ *is the soft thresholding function.*

3 Main theorem

In this section, we give an approximation result, which extends the result [8, Theorem 4.2] from the Radon transform to a family of linear operators. To do that, we introduce a set $\mathcal{N}(\varepsilon)$ of significant shearlet coefficients as follows. Let

$$s_1 = \frac{1}{\frac{9}{2} + 6\alpha}\log_2\left(\varepsilon^{-1}\right), \qquad s_2 = \frac{1}{\frac{3}{2} + 2\alpha}\log_2\left(\varepsilon^{-1}\right),$$

and $j_0 = \lceil s_1 \rceil$, $j_1 = \lceil s_2 \rceil$. Define $\mathcal{N}(\varepsilon) := M(\varepsilon) \cup N(\varepsilon) \subseteq \mathcal{M}$, where

$$N(\varepsilon) = \left\{\mu = k \in \mathbb{Z}^2 : |k| \le 2^{2j_0+1}\right\};$$
$$M(\varepsilon) = \left\{\mu = (j,l,k,d) : j_0 \le j \le j_1, -2^j \le l \le 2^j - 1, |k| \le 2^{2j+1}, d = 0,1\right\}.$$

Consider the model $Y = Kf + \varepsilon W$ with $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$. Lemma 2.2 tells that $y_\mu := 2^{2\alpha j}\langle Y, U_\mu \rangle = \langle f, s_\mu \rangle + \varepsilon 2^{2\alpha j} n_\mu$, where n_μ is Gaussian noise with zero mean and bounded variance $\sigma_\mu^2 = \|U_\mu\|^2 \le C$ [15]. Let $c_\mu = \langle f, s_\mu \rangle$ and $\tilde{f} = \sum_{\mu \in \mathcal{N}(\varepsilon)} \tilde{c}_\mu s_\mu$ with

$$\tilde{c}_\mu = \begin{cases} T_s(y_\mu, \varepsilon\sqrt{2\log(\sharp\mathcal{N}(\varepsilon))}2^{2\alpha j}\sigma_\mu), & \mu \in \mathcal{N}(\varepsilon); \\ 0, & \text{otherwise,} \end{cases}$$

where $T_s(y,t)$ is the soft thresholding function. Then the following result holds.

Theorem 3.1 *Let* $f \in \varepsilon^2(A)$ *be the solution to* $Y = Kf + \varepsilon W$ *with* $(K^*Kf)^\wedge(\xi) = (b + |\xi|^2)^{-\alpha}\hat{f}(\xi)$ *and* $\tilde{f}$ *be defined as above. Then*

$$\sup_{f \in \varepsilon^2(A)} E\|\tilde{f} - f\|^2 \le C\log\left(\varepsilon^{-1}\right)\varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}} \quad (\varepsilon \to 0).$$

Here and in what follows, E stands for the expectation operator.

Proof Since $\{s_\mu, \mu \in \mathcal{M}\}$ is a Parseval frame, $f = \sum_{\mu\in\mathcal{M}} c_\mu s_\mu$ and $\tilde{f} = \sum_{\mu\in\mathcal{N}(\varepsilon)} \tilde{c}_\mu s_\mu$, $\tilde{f} - f = \sum_{\mu\in\mathcal{M}} (\tilde{c}_\mu - c_\mu) s_\mu$. Moreover, $\|\tilde{f} - f\|^2 = \sum_{\mu\in\mathcal{M}} |\tilde{c}_\mu - c_\mu|^2$ and

$$E\|\tilde{f} - f\|^2 = \sum_{\mu\in\mathcal{N}(\varepsilon)} E|\tilde{c_\mu} - c_\mu|^2 + \sum_{\mu\in\mathcal{N}(\varepsilon)^C} |c_\mu|^2. \tag{3.1}$$

In order to estimate $\sum_{\mu\in N(\varepsilon)^C} |c_\mu|^2$, one observes $\sum_{\mu\in M_j} |c_\mu|^2 \le C2^{-2j}$ due to Theorem 2.3. Then $\sum_{j>j_1}\sum_{\mu\in M_j} |c_\mu|^2 \le C\sum_{j>j_1} 2^{-2j} \le C2^{-2j_1}$. By $2^{j_1} \lesssim \varepsilon^{-\frac{1}{\frac{3}{2}+2\alpha}}$,

$$\sum_{j>j_1}\sum_{\mu\in M_j} |c_\mu|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}. \tag{3.2}$$

(Here and in what follows, $A \lesssim B$ denotes $A \le CB$ for some constant $C > 0$).

Next, one considers c_μ for $j_0 \le j \le j_1$ and $|k| \ge 2^{2j+1}$. Note that $|\psi^{(d)}(x)| \le C_m(1 + |x|)^{-m}$ ($d = 0, 1$, $m = 1, 2, \ldots$). Then $|\psi_{j,l,k}^{(d)}(x)| \le C_m 2^{\frac{3}{2}j}(1 + |B_d^l A_d^j x - k|)^{-m}$. Since $f \in \varepsilon^2(A)$, supp$f \subset Q_0 := [0,1]^2$ and

$$\bigl|\langle f, \psi_{j,l,k}^{(d)}\rangle\bigr| \le C_m 2^{\frac{3}{2}j}\|f\|_\infty \int_{Q_0} \bigl(1 + \bigl|B_d^l A_d^j x - k\bigr|\bigr)^{-m}\, dx.$$

On the other hand, $|B_d^l A_d^j x| \le \|B_d^l A_d^j\||x| \le 2^{2j}|x| \le \sqrt{2}2^{2j}$ for $x \in Q_0$. Hence, $(1 + |B_d^l A_d^j x - k|)^{-m} \le (1 + |k| - |B_d^l A_d^j x|)^{-m} \le (|k| - \sqrt{2}2^{2j})^{-m}$ for $|k| \ge 2^{2j+1}$. Moreover, $\sum_{|k|\ge 2^{2j+1}} |c_\mu|^2 \le 2^{3j}\sum_{|k|\ge 2^{2j+1}}(|k| - \sqrt{2}2^{2j})^{-2m} = 2^{3j}\sum_{n=1}^\infty \sum_{2^{2j+n}\le|k|\le 2^{2j+n+1}} 2^{-4mj}(2^n - \sqrt{2})^{-2m} \lesssim 2^{3j}2^{-4mj} \times \sum_{n=1}^\infty 2^{2(2j+n+1)}(2^n - \sqrt{2})^{-2m} \lesssim 2^{3j}2^{-2j(2m-2)}$, since m can be chosen big enough. Therefore,

$$\sum_{j=j_0}^{j_1}\sum_{l=-2^j}^{2^j-1}\sum_{|k|\ge 2^{2j+1}} |c_\mu|^2 \le C_m \sum_{j=j_0}^{\infty} 2^{8j}2^{-4mj} \lesssim 2^{-8j_0} \le 2^{-6j_0} \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}} \tag{3.3}$$

due to the choice of j_0. The similar (even simpler) arguments show $\sum_{|k|\ge 2^{2j_0+1}} |\langle f, \varphi_{j_0,k}\rangle|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}$ with $\varphi_{j_0,k}(x) = 2^{j_0}\varphi(2^{j_0}x - k)$. This with (3.2) and (3.3) leads to

$$\sum_{\mu\in\mathcal{N}(\varepsilon)^C} |c_\mu|^2 \lesssim \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}. \tag{3.4}$$

Finally, one estimates $\sum_{\mu\in\mathcal{N}(\varepsilon)} E|\tilde{c_\mu} - c_\mu|^2$. By the definition of y_μ, $\varepsilon^{-1}2^{-2\alpha j}\sigma_\mu^{-1}y_\mu \sim N(\varepsilon^{-1}2^{-2\alpha j}\sigma_\mu^{-1}c_\mu, 1)$. Applying Theorem 2.4 with $\eta^{-1} = \sharp\mathcal{N}(\varepsilon)$, one obtains that

$$\begin{aligned} &E\bigl|T_s\bigl[\varepsilon^{-1}2^{-2\alpha j}\sigma_\mu^{-1}y_\mu, \sqrt{2\log\bigl(\sharp\mathcal{N}(\varepsilon)\bigr)}\bigr] - \varepsilon^{-1}2^{-2\alpha j}\sigma_\mu^{-1}c_\mu\bigr|^2 \\ &\quad = \bigl[2\log\bigl(\sharp\mathcal{N}(\varepsilon)\bigr) + 1\bigr]\left[\frac{1}{\sharp\mathcal{N}(\varepsilon)} + \min\bigl\{\varepsilon^{-2}2^{-4\alpha j}\sigma_\mu^{-2}c_\mu^2, 1\bigr\}\right]. \end{aligned}$$

Hence, $E|T_s[y_\mu, \varepsilon 2^{2\alpha j}\sigma_\mu\sqrt{2\log(\sharp\mathcal{N}(\varepsilon))}] - c_\mu|^2 \lesssim [2\log(\sharp\mathcal{N}(\varepsilon)) + 1][\varepsilon^2\frac{2^{4\alpha j}\sigma_\mu^2}{\sharp\mathcal{N}(\varepsilon)} + \min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\sigma_\mu^2\}]$. By $\tilde{c}_\mu = T_s[y_\mu, \varepsilon 2^{2\alpha j}\sigma_\mu\sqrt{2\log(\sharp\mathcal{N}(\varepsilon))}]$ for $\mu \in \mathcal{N}(\varepsilon)$, one knows that

$$E\sum_{\mu\in\mathcal{N}(\varepsilon)}|\tilde{c_\mu} - c_\mu|^2 \lesssim \bigl[2\log\bigl(\sharp\mathcal{N}(\varepsilon)\bigr) + 1\bigr] \times\left[\varepsilon^2\sum_{\mu\in\mathcal{N}(\varepsilon)}\frac{2^{4\alpha j}\sigma_\mu^2}{\sharp\mathcal{N}(\varepsilon)} + \sum_{\mu\in\mathcal{N}(\varepsilon)}\min\bigl\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\sigma_\mu^2\bigr\}\right]. \tag{3.5}$$

Note that $\mathcal{N}(\varepsilon)\cap M_j \subset \{(j,l,k,d) : |k| \le 2^{2j+1}, |l| \le 2^j\}$. Then $\sharp\mathcal{N}(\varepsilon) \le C\sum_{j\le j_1}2^{5j} \lesssim 2^{5j_1} \lesssim \varepsilon^{-\frac{5}{\frac{3}{2}+2\alpha}}$, and $\log(\sharp\mathcal{N}(\varepsilon)) \lesssim \frac{10}{\frac{3}{2}+2\alpha}\log(\varepsilon^{-1}) \lesssim \log(\varepsilon^{-1})$. Since $\{\sigma_\mu : \mu \in \mathcal{M}\}$ is uniformly bounded, $[2\log(\sharp\mathcal{N}(\varepsilon)) + 1]\varepsilon^2\sum_{\mu\in\mathcal{N}(\varepsilon)}\frac{2^{4\alpha j}\sigma_\mu^2}{\sharp\mathcal{N}(\varepsilon)} \lesssim \log(\varepsilon^{-1})\varepsilon^2 2^{4\alpha j_1}$. This with the choice of 2^{j_1} shows that

$$\bigl[2\log\bigl(\sharp\mathcal{N}(\varepsilon)\bigr) + 1\bigr]\varepsilon^2\sum_{\mu\in\mathcal{N}(\varepsilon)}\frac{2^{4\alpha j}\sigma_\mu^2}{\sharp\mathcal{N}(\varepsilon)} \lesssim \log\bigl(\varepsilon^{-1}\bigr)\varepsilon^{\frac{3}{\frac{3}{2}+2\alpha}} \lesssim \log\bigl(\varepsilon^{-1}\bigr)\varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}. \tag{3.6}$$

It remains to estimate $[2\log(\sharp\mathcal{N}(\varepsilon)) + 1]\sum_{\mu\in\mathcal{N}(\varepsilon)}\min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\sigma_\mu^2\}$. Clearly,

$$\sum_{\mu\in\mathcal{N}(\varepsilon)}\min\bigl\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\bigr\} = \sum_{\{\mu\in\mathcal{N}(\varepsilon):|c_\mu|\ge 2^{2\alpha j}\varepsilon\}}2^{4\alpha j}\varepsilon^2 + \sum_{\{\mu\in\mathcal{N}(\varepsilon):|c_\mu|<2^{2\alpha j}\varepsilon\}}|c_\mu|^2. \tag{3.7}$$

By Theorem 2.3, $\sum_{\{\mu\in M_j:|c_\mu|\ge 2^{2\alpha j}\varepsilon\}}2^{4\alpha j}\varepsilon^2 \lesssim (2^{2\alpha j}\varepsilon)^{-\frac{2}{3}}2^{4\alpha j}\varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j}\varepsilon^{\frac{4}{3}}$. Hence,

$$\sum_{\{\mu\in\mathcal{N}(\varepsilon):|c_\mu|\ge 2^{2\alpha j}\varepsilon\}}2^{4\alpha j}\varepsilon^2 = \sum_{j=j_0}^{j_1}\sum_{\{\mu\in M_j:|c_\mu|\ge 2^{2\alpha j}\varepsilon\}}2^{4\alpha j}\varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j_1}\varepsilon^{\frac{4}{3}}. \tag{3.8}$$

On the other hand, $\sum_{\{\mu\in\mathcal{N}(\varepsilon):|c_\mu|<2^{2\alpha j}\varepsilon\}}|c_\mu|^2 = \sum_{j=j_0}^{j_1}\sum_{n=0}^{\infty}\sum_{\{2^{2\alpha j-n-1}\varepsilon<|c_\mu|\le 2^{2\alpha j-n}\varepsilon\}}|c_\mu|^2$. According to Theorem 2.3, $\sharp R(j, 2^{2\alpha j-n-1}\varepsilon) \lesssim 2^{-\frac{2}{3}(2\alpha j-n-1)}\varepsilon^{-\frac{2}{3}}$ and

$$\sum_{\{2^{2\alpha j-n-1}\varepsilon<|c_\mu|\le 2^{2\alpha j-n}\varepsilon\}}|c_\mu|^2 \lesssim 2^{-\frac{2}{3}(2\alpha j-n-1)}\varepsilon^{-\frac{2}{3}}2^{2(2\alpha j-n)}\varepsilon^2 \lesssim 2^{\frac{8}{3}\alpha j}2^{-\frac{4}{3}n}2^{\frac{2}{3}}\varepsilon^{\frac{4}{3}}.$$

Therefore,

$$\sum_{\{\mu\in\mathcal{N}(\varepsilon):|c_\mu|<2^{2\alpha j}\varepsilon\}}|c_\mu|^2 \le \sum_{j=j_0}^{j_1}\sum_{n=0}^{\infty}2^{\frac{8}{3}\alpha j}2^{-\frac{4}{3}n}\varepsilon^{\frac{4}{3}} \le 2^{\frac{8}{3}\alpha j_1}\varepsilon^{\frac{4}{3}}.$$

Combining this with (3.7) and (3.8), one knows that $\sum_{\mu\in\mathcal{N}(\varepsilon)}\min\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\} \lesssim 2^{\frac{8}{3}\alpha j_1}\varepsilon^{\frac{4}{3}}$. Furthermore,

$$\sum_{\mu\in\mathcal{N}(\varepsilon)}\min\bigl\{c_\mu^2, \varepsilon^2 2^{4\alpha j}\bigr\} \lesssim \varepsilon^{\frac{2}{2\alpha+\frac{3}{2}}} \tag{3.9}$$

thanks to $2^{j_1} \lesssim \varepsilon^{-\frac{1}{2\alpha+\frac{3}{2}}}$. Now, it follows from (3.5), (3.6) and (3.9) that

$$\sum_{\mu\in\mathcal{N}(\varepsilon)} E|\tilde{c_\mu} - c_\mu|^2 \lesssim \log(\varepsilon^{-1})\varepsilon^{\frac{2}{2\alpha+\frac{3}{2}}}.$$

This with (3.1) and (3.4) leads to the desired conclusion $\sup_{f\in\varepsilon^2(A)} E\|\tilde{f} - f\|^2 \le C\log(\varepsilon^{-1}) \times \varepsilon^{\frac{2}{\frac{3}{2}+2\alpha}}$. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LH and YL finished this work together. Two authors read and approved the final manuscript.

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