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Invariant mean and a Korovkin-type approximation theorem

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Abstract

In this paper we apply this form of convergence to prove some Korovkin-type approximation theorem by using the test functions 1, e^{-x}, e^{-2x}, which generalizes the results of Boyanov and Veselinov (Bull. Math. Soc. Sci. Math. Roum. 14(62):9-13, 1970).

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1 Introduction and preliminaries

Let c and ℓ_∞ denote the spaces of all convergent and bounded sequences, respectively, and note that $c \subset \ell_\infty$. In the theory of sequence spaces, an application of the well-known Hahn-Banach extension theorem gave rise to the concept of the Banach limit. That is, the lim functional defined on c can be extended to the whole of ℓ_∞ and this extended functional is known as the Banach limit. In 1948, Lorentz [1] used this notion of a generalized limit to define a new type of convergence, known as almost convergence. Later on, Raimi [2] gave a slight generalization of almost convergence and named it σ-convergence. Before proceeding further, we recall some notations and basic definitions used in this paper.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional φ defined on the space ℓ_∞ of all bounded sequences is called an invariant mean (or a σ-mean; *cf.* [2]) if it is non-negative, normal and $\varphi(x) = \varphi((x_{\sigma(n)}))$.

A sequence $x = (x_k)$ is said to be *σ-convergent* to the number L if and only if all of its σ-means coincide with L, *i.e.*, $\varphi(x) = L$ for all φ. A bounded sequence $x = (x_k)$ is σ-convergent (*cf.* [3]) to the number L if and only if $\lim_{p\to\infty} t_{pm} = L$ uniformly in m, where

$$t_{pm} = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p+1}.$$

We denote the set of all σ-convergent sequences by V_σ and in this case we write $x_k \to L(V_\sigma)$ and L is called the σ-limit of x. Note that a σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits (*cf.* [4]) and $c \subset V_\sigma \subset \ell_\infty$.

If σ is a translation then the σ-mean is called a *Banach limit* and σ-convergence is reduced to the concept of almost convergence introduced by Lorentz [1].

In [5], the idea of statistical σ-convergence is defined which is further applied to prove some approximation theorems in [6] and [7].

If $m = 1$, then we get $(C,1)$ convergence, and in this case we write $x_k \to \ell(C,1)$, where $\ell = (C,1)\text{-}\lim x$.

Remark 1.1 Note that
(a) a convergent sequence is also σ-convergent;
(b) a σ-convergent sequence implies $(C,1)$ convergence.

Example 1.1 Let $\sigma(n) = n + 1$. Define the sequence $z = (z_n)$ by

$$z_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Then x is σ-convergent to 1/2 but not convergent.

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with the norm $\|f\|_\infty := \sup_{a\le x\le b} |f(x)|$, $f \in C[a,b]$. Suppose that $T_n : C[a,b] \to C[a,b]$. We write $T_n(f,x)$ for $T_n(f(t),x)$ and we say that T is a positive operator if $T(f,x) \ge 0$ for all $f(x) \ge 0$.

The classical Korovkin approximation theorem states the following [7]: Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Quite recently, such type of approximation theorem has been studied in [8, 9] and [10] by using λ-statistical convergence, while in [11] lacunary statistical convergence has been used. Boyanov and Veselinov [12] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions 1, e^{-x}, e^{-2x}. In this paper, we generalize the result of Boyanov and Veselinov by using the notion of σ-convergence. Our results also generalize the results of Mohiuddine [13], in which the author has used almost convergence and the test functions 1, x, x^2.

2 Korovkin-type approximation theorem

We prove the following σ-version of the classical Korovkin approximation theorem.

Theorem 2.1 *Let (T_k) be a sequence of positive linear operators from $C(I)$ into $C(I)$. Then, for all $f \in C(I)$,*

$$\sigma\text{-}\lim_{k\to\infty} \left\| T_k(f;x) - f(x) \right\|_\infty = 0 \tag{2.1}$$

if and only if

$$\sigma\text{-}\lim_{k\to\infty} \left\| T_k(1;x) - 1 \right\|_\infty = 0, \tag{2.2}$$

$$\sigma\text{-}\lim_{k\to\infty} \left\| T_k\left(e^{-s};x\right) - e^{-x} \right\|_\infty = 0, \tag{2.3}$$

$$\sigma\text{-}\lim_{k\to\infty} \left\| T_k\left(e^{-2s};x\right) - e^{-2x} \right\|_\infty = 0. \tag{2.4}$$

Proof Since each 1, e^{-x}, e^{-2x} belongs to $C(I)$, conditions (2.2)-(2.4) follow immediately from (2.1). Let $f \in C(I)$. Then there exists a constant $M > 0$ such that $|f(x)| \leq M$ for $x \in I$. Therefore,

$$\left|f(s) - f(x)\right| \leq 2M, \quad -\infty < s, x < \infty. \tag{2.5}$$

It is easy to prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|f(s) - f(x)\right| < \varepsilon, \tag{2.6}$$

whenever $|e^{-s} - e^{-x}| < \delta$ for all $x \in I$.

Using (2.5), (2.6), putting $\psi_1 = \psi_1(s,x) = (e^{-s} - e^{-x})^2$, we get

$$\left|f(s) - f(x)\right| < \varepsilon + \frac{2M}{\delta^2}(\psi_1), \quad \forall |s - x| < \delta.$$

This is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1) < f(s) - f(x) < \varepsilon + \frac{2M}{\delta^2}(\psi_1).$$

Now, we operate $T_{\sigma^{k}(n)}(1,x)$ for all n to this inequality since $T_{\sigma^{k}(n)}(f,x)$ is monotone and linear. We obtain

$$\begin{aligned} T_{\sigma^{k}(n)}(1;x)\left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1)\right) &< T_{\sigma^{k}(n)}(1;x)\bigl(f(s) - f(x)\bigr) \\ &< T_{\sigma^{k}(n)}(1;x)\left(\varepsilon + \frac{2M}{\delta^2}(\psi_1)\right). \end{aligned}$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore

$$\begin{aligned} -\varepsilon T_{\sigma^{k}(n)}(1;x) - \frac{2M}{\delta^2} T_{\sigma^{k}(n)}(\psi_1;x) &< T_{\sigma^{k}(n)}(f;x) - f(x) T_{\sigma^{k}(n)}(1;x) \\ &< \varepsilon T_{\sigma^{k}(n)}(1;x) + \frac{2M}{\delta^2} T_{\sigma^{k}(n)}(\psi_1;x). \end{aligned} \tag{2.7}$$

But

$$\begin{aligned} &T_{\sigma^{k}(n)}(f;x) - f(x) \\ &\quad = T_{\sigma^{k}(n)}(f;x) - f(x) T_{\sigma^{k}(n)}(1;x) + f(x) T_{\sigma^{k}(n)}(1;x) - f(x) \\ &\quad = \bigl[T_{\sigma^{k}(n)}(f;x) - f(x) T_{\sigma^{k}(n)}(1;x)\bigr] + f(x)\bigl[T_{\sigma^{k}(n)}(1;x) - 1\bigr]. \end{aligned} \tag{2.8}$$

Using (2.7) and (2.8), we have

$$\begin{aligned} T_{\sigma^{k}(n)}(f;x) - f(x) &< \varepsilon T_{\sigma^{k}(n)}(1;x) + \frac{2M}{\delta^2} T_{\sigma^{k}(n)}(\psi_1;x) \\ &\quad + f(x)\bigl(T_{\sigma^{k}(n)}(1;x) - 1\bigr). \end{aligned} \tag{2.9}$$

Now

$$\begin{aligned}
T_{\sigma^k(n)}(\psi_1;x) &= T_{\sigma^k(n)}\bigl(\bigl(e^{-s}-e^{-x}\bigr)^2;x\bigr) = T_{\sigma^k(n)}\bigl(e^{-2s}-2e^{-s}e^{-x}+e^{-2x};x\bigr) \\
&= T_{\sigma^k(n)}\bigl(e^{-2s};x\bigr) - 2e^{-x}T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) + \bigl(e^{-2x}\bigr)T_{\sigma^k(n)}(1;x) \\
&= \bigl[T_{\sigma^k(n)}\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr] - 2e^{-x}\bigl[T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr] \\
&\quad + e^{-2x}\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr].
\end{aligned}$$

Using (2.9), we obtain

$$\begin{aligned}
T_{\sigma^k(n)}(f;x) - f(x) &< \varepsilon T_{\sigma^k(n)}(1;x) + \frac{2M}{\delta^2}\bigl\{\bigl[T_{\sigma^k(n)}\bigl(\bigl(e^{-2s}\bigr);x\bigr) - e^{-2x}\bigr] \\
&\quad - 2e^{-x}\bigl[T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr] + e^{-2x}\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr]\bigr\} \\
&\quad + f(x)\bigl(T_{\sigma^k(n)}(1;x) - 1\bigr) \\
&= \varepsilon[T_{\sigma^k(n)}(1;x) - 1] + \varepsilon + \frac{2M}{\delta^2}\bigl\{\bigl[T_{\sigma^k(n)}\bigl(\bigl(e^{-2s}\bigr);x\bigr) - e^{-2x}\bigr] \\
&\quad - 2e^{-x}\bigl[T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr] + e^{-2x}\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr]\bigr\} \\
&\quad + f(x)\bigl(T_{\sigma^k(n)}(1;x) - 1\bigr).
\end{aligned}$$

Since ε is arbitrary, we can write

$$\begin{aligned}
T_{\sigma^k(n)}(f;x) - f(x) &\le \varepsilon\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr] + \frac{2M}{\delta^2}\bigl\{\bigl[T_{\sigma^k(n)}\bigl(\bigl(e^{-2s}\bigr);x\bigr) - e^{-2x}\bigr] \\
&\quad - 2e^{-x}\bigl[T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr] + e^{-2x}\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr]\bigr\} \\
&\quad + f(x)\bigl[T_{\sigma^k(n)}(1;x) - 1\bigr].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\bigl|T_{\sigma^k(n)}(f;x) - f(x)\bigr| \\
&\quad \le \varepsilon + (\varepsilon + M)\bigl|T_{\sigma^k(n)}(1;x) - 1\bigr| + \frac{2M}{\delta^2}\bigl|e^{-2x}\bigr|\bigl|T_{\sigma^k(n)}(1;x,y) - 1\bigr| \\
&\qquad + \frac{2M}{\delta^2}\bigl|T_{\sigma^k(n)}\bigl(e^{-2s};x\bigr)\bigr|\bigl|-e^{-2x}\bigr| + \frac{4M}{\delta^2}\bigl|e^{-x}\bigr|\bigl|T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr| \\
&\quad \le \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2}\right)\bigl|T_{\sigma^k(n)}(1;x) - 1\bigr| + \frac{2M}{\delta^2}\bigl|e^{-2x}\bigr|\bigl|T_{\sigma^k(n)}(1;x) - 1\bigr| \\
&\qquad + \frac{2M}{\delta^2}\bigl|T_{\sigma^k(n)}\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr| + \frac{4M}{\delta^2}\bigl|T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr|
\end{aligned}$$

since $|e^{-x}| \le 1$ for all $x \in I$. Now, taking $\sup_{x\in I}$

$$\begin{aligned}
\bigl\|T_{\sigma^k(n)}(f;x) - f(x)\bigr\|_\infty &\le \varepsilon + K\bigl(\bigl\|T_{\sigma^k(n)}(1;x) - 1\bigr\|_\infty + \bigl\|T_{\sigma^k(n)}\bigl(e^{-s};x\bigr) - e^{-x}\bigr\|_\infty \\
&\quad + \bigl\|T_{\sigma^k(n)}\bigl(e^{-2s};x\bigr) - e^{-2x}\bigr\|_\infty\bigr),
\end{aligned}$$

where $K = \max\{\varepsilon + M + \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\}$. Now writing

$$D_{n,p}(f,x) = \frac{1}{p}\sum_{k=0}^{p-1} T_{\sigma^k(n)}(f,x),$$

we get

$$\begin{aligned}\|D_{n,p}(f,x) - f(x)\|_\infty \le{}& \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right)\|D_{n,p}(1,x) - 1\|_\infty \\ &+ \frac{4Mb}{\delta^2}\|D_{n,p}(t,x) - e^{-x}\|_\infty + \frac{2M}{\delta^2}\|D_{n,p}(t^2,x) - e^{-2x}\|_\infty.\end{aligned}$$

Letting $p \to \infty$ and using (2.2), (2.3), (2.4), we get

$$\lim_{p\to\infty}\|D_{n,p}(f,x) - f(x)\|_\infty = 0, \quad \text{uniformly in } n. \qquad \square$$

In the following example we construct a sequence of positive linear operators satisfying the conditions of Theorem 2.1 but not satisfying the conditions of the Korovkin theorem of Boyanov and Veselinov [12].

Example 2.1 Consider the sequence of classical Baskakov operators [14]

$$V_n(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n-1+k}{k} x^k (1+x)^{-n-k},$$

where $0 \le x, y < \infty$.

Let the sequence (L_n) be defined by $L_n : C(I) \to C(I)$ with $L_n(f;x) = (1+z_n)V_n(f;x)$, where z_n is defined as above. Since

$$\begin{aligned}&L_n(1;x) = 1, \\ &L_n(e^{-s};x) = \left(1 + x - xe^{-\frac{1}{n}}\right)^{-n}, \\ &L_n(e^{-2s};x) = \left(1 + x^2 - x^2 e^{-\frac{1}{n}}\right)^{-n},\end{aligned}$$

and the sequence (P_n) satisfies the conditions (2.1), (2.2) and (2.3). Hence we have

$$\sigma\text{-}\lim\|L_n(f,x) - f(x)\|_\infty = 0.$$

On the other hand, we get $L_n(f,0) = (1+z_n)f(0)$ since $L_n(f,0) = f(0)$, and hence

$$\|L_n(f,x) - f(x)\|_\infty \ge |L_n(f,0) - f(0)| = z_n|f(0)|.$$

We see that (L_n) does not satisfy the classical Korovkin theorem since $\limsup_{n\to\infty} z_n$ does not exist. Hence our Theorem 2.1 is stronger than that of Boyanov and Veselinov [12].

3 A consequence

Now we present a slight general result.

Theorem 3.1 *Let (T_n) be a sequence of positive linear operators on $C(I)$ such that*

$$\lim_n \sup_m \frac{1}{n} \sum_{k=0}^{n-1} \| T_n - T_{\sigma^k(m)} \| = 0.$$

If

$$\sigma\text{-}\lim_n \left\| T_n\left(e^{-\nu s}, x\right) - e^{-\nu x} \right\|_\infty = 0 \quad (\nu = 0, 1, 2), \tag{3.1}$$

then, for any function $f \in C(I)$ bounded on the real line, we have

$$\lim_n \left\| T_n(f, x) - f(x) \right\|_\infty = 0. \tag{3.2}$$

Proof From Theorem 2.1, we have that if (3.1) holds, then

$$\sigma\text{-}\lim_n \left\| T_n(f, x) - f(x) \right\|_\infty = 0,$$

which is equivalent to

$$\lim_n \left\| \sup_m D_{m,n}(f, x) - f(x) \right\|_\infty = 0.$$

Now

$$\begin{aligned} T_n - D_{m,n} &= T_n - \frac{1}{n} \sum_{k=0}^{n-1} T_{\sigma^k(m)} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} (T_n - T_{\sigma^k(m)}). \end{aligned}$$

Therefore

$$T_n - \sup_m D_{m,n} = \sup_m \frac{1}{n} \sum_{k=0}^{n-1} (T_n - T_{\sigma^k(m)}).$$

Hence, using the hypothesis, we get

$$\lim_n \left\| T_n(f, x) - f(x) \right\|_\infty = \lim_n \left\| \sup_m D_{m,n}(f, x) - f(x) \right\|_\infty = 0,$$

that is, (3.2) holds. □

Competing interests
The author declares that they have no competing interests.

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