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An interior proximal cutting hyperplane method for equilibrium problems

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Abstract

We propose a new method for solving equilibrium problems on polyhedra, where the underlying function is continuous and satisfies a pseudomonotone assumption which is called an interior proximal cutting hyperplane method. The method is based on the special interior proximal function which replaces the usual quadratic function. This leads to an interior proximal algorithm. The algorithm can be viewed as combining the cutting hyperplane method and the special interior proximal function. Finally some preliminary computational results are given.

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1 Introduction

Equilibrium problems appear frequently in many practical problems arising, for instance, physics, engineering, game theory, transportation, economics and network (see [1,2]). They become an attractive field for many researchers both theory and applications (see [3-8]). These problems are models whose formulation includes optimization, variational inequalities, (vector) optimization problems, fixed point problems, saddle point problems, Nash equilibria and complementarity problems as particular cases (see [1,5,9]). In this article, we consider the equilibrium problems (shortly EP (f, C)):

Find $x^* \in C$ such that $f(x^*, y) \geq 0$ for all $y \in C$,

where C is a polyhedral set on \mathbb{R}^n defined by

$$C := \{x \in \mathbb{R}^n | Ax \leq b\}, \quad (1.1)$$

A is a $p \times n$ matrix, $b \in \mathbb{R}^p$, $f: C \times C \rightarrow \mathbb{R}$ is a *bifunction* such that $f(x, x) = 0$ for every $x \in C$.

Throughout this article, we assume that:

(A.1) $\text{int}C = \{x | Ax < b\}$ is nonempty.

(A.2) $f(x, \cdot)$ is convex on C for all $x \in C$.

(A.3) f is continuous on $C \times C$.

(A.4) The solution set S of EP (f, C) is nonempty.

Theory of equilibrium problems has been studied extensively and intensively on the existence of solutions and generalization to many abstract ways. However, methods for solving $EP(f, C)$ still restrict and have not satisfy the need of applications. There are popular approaches for solving $EP(f, C)$ to our knowledge. The first approach is based on the gap function (see [10]), the second way is to use the proximal point method [7] and the third one is the auxiliary subproblem principle [8].

In [3,4], Anh proposed interior proximal methods for solving monotone equilibrium problems when C is a polyhedral convex set. The method is based on a special interior proximal function which replaces the usual quadratic function. The method has also been studied to variational inequalities by many authors (see [11,12]). This leads to an interior proximal-type algorithm, which can be viewed as combining an Armijo-type line search technique and the special interior proximal function. The only assumption required is that f is monotone on C .

In this article, we propose an algorithm for solving $EP(f, C)$, by making no assumptions on the problem other than continuity and pseudomonotonicity of the bifunction f . Recently, Anh and Kuno [13] introduced a new method for solving multivalued variational inequalities on a closed convex set, where the underlying function is upper semicontinuous and generalized monotone. We extend the cutting hyperplane method to $EP(f, C)$. First, we construct an appropriate hyperplane which separates the current iterative point from the solution set. Next we combine this technique with Armijo-type line search technique to obtain a convergent algorithm for pseudomonotone equilibrium problems. Then the next iteration is obtained as the projection of the current iteration onto the intersection of the feasible set with the halfspace containing the solution set.

The article is organized as follows. In Section 2, we give formal definitions of our target $EP(f, C)$ and the pseudomonotonicity of f . We then combined an idea often used for multivalued variational inequalities to $EP(f, C)$ and interior proximal technique to develop an iterative algorithm. Section 3 is devoted to the proof of its global convergence to a solution of $EP(f, C)$. In the last section, we apply the algorithm for the Nash-Cournot oligopolistic market equilibrium model. The numerical results are implemented to verify our development.

2 The interior proximal cutting hyperplane algorithm

We list some well known definitions and the projection under the Euclidean norm which will be required in our following analysis.

Definition 2.1 Let C be a closed convex subset of \mathbb{R}^n , we denote the projection on C by $Pr_C(\cdot)$, i.e,

$$Pr_C(x) = \arg \min\{\|y - x\| \mid y \in C\} \quad \forall x \in \mathbb{R}^n.$$

Then the bifunction $f: C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be

(i) monotone on C if for each $x, y \in C$,

$$f(x, y) + f(y, x) \leq 0;$$

(ii) pseudomonotone on C if for each $x, y \in C$,

$$f(x, y) \geq 0 \quad \text{implies} \quad f(y, x) \leq 0.$$

It is observe that $(i) \Rightarrow (ii)$. But the converse is not true. There are some examples in [14].

Classical variational inequality problems (shortly VIP) are to find a vector $x^* \in C$ such that

$$\langle F(x^*), \gamma - x^* \rangle \geq 0 \quad \forall \gamma \in C,$$

where $C \subseteq \mathbb{R}^n$ is a nonempty closed convex subset of \mathbb{R}^n and F is a continuous mapping from C into \mathbb{R}^n . Then they can be alternatively formulated as finding the zero point of the operator $T(x) = F(x) + N_C(x)$ where

$$N_C(x) = \begin{cases} \{\gamma \in C \mid \langle \gamma, z - x \rangle \leq 0, \forall z \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

A well known method to solve this problem is the proximal point algorithm [2], which starting with any point $x^0 \in C$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k T(x) + \nabla_x h(x, x^k), \tag{2.1}$$

where

$$h(x, x^k) = \frac{1}{2} \|x - x^k\|^2.$$

Motivation for studying the algorithm of problem (2.1) could be found in [11,15,16].

Auslender et al. [12] have proposed an interior proximal-type method for solving (VIP) on $C := \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x \geq 0\}$ through replacing function $h(x, x^k)$ by $d_\phi(x, x^k)$ which is defined as

$$d_\phi(x, \gamma) = \sum_{i=1}^n \gamma_i^2 \phi(\gamma_i^{-1} x_i),$$

where

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

with $\nu > \mu > 0$. The fundamental difference here is that the term d_ϕ is used to force the iterations $\{x^{k+1}\}$ to stay in the interior of \mathbb{R}_+^n . This technique is extended by many authors to variational inequalities and equilibrium problems (see [1,3]).

Applying this idea to the equilibrium problem $EP(f, C)$, we consider another interior proximal function defined by

$$D(x, \gamma) = \begin{cases} \frac{1}{2} \|x - \gamma\|^2 + \mu \sum_{i=1}^p l_i^2(\gamma) \left(\frac{l_i(x)}{l_i(\gamma)} \log \frac{l_i(x)}{l_i(\gamma)} - \frac{l_i(x)}{l_i(\gamma)} + 1 \right) & \text{if } x \in \text{int } C, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\mu \in (0, 1)$, a_i ($i = 1, \dots, p$) are the rows of the matrix A , and

$$l_i(x) = b_i - \langle a_i, x \rangle, \\ l(x) = (l_1(x), l_2(x), \dots, l_p(x))^T.$$

We denote by $\nabla_1 D(x, y)$ the gradient of $D(\cdot, y)$ at x for every $y \in C$. It is easy to see that

$$\nabla_1 D(x, y) = x - y - \mu A^T X_y \log \frac{l(x)}{l(y)},$$

where

$$X_y := \text{diag} (l_1(y), \dots, l_p(y)) \text{ and } \log \frac{l(x)}{l(y)} := \left(\log \frac{l_1(x)}{l_1(y)}, \dots, \log \frac{l_p(x)}{l_p(y)} \right).$$

Then we consider the following regularized auxiliary problem (shortly RAP):

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) + \frac{1}{c} D(y, x^*) \geq 0 \text{ for all } y \in C,$$

where $c > 0$ is a regularization parameter.

The equivalence between $\text{EP}(f, C)$ and (RAP) is due to the following lemma (see [1]).

Lemma 2.2 *Let $f: C \times C \rightarrow \mathbb{R} \cup \{+\infty\}$ be a bifunction and $x^* \in C$. Then x^* is a solution to $\text{EP}(f, C)$ if and only if x^* is a solution to (RAP).*

Lemma 2.2 shows that the solution of the equilibrium problem $\text{EP}(f, C)$ can be approximated by an iterative procedure $x^{k+1} = h(x^k)$, $k = 0, 1, \dots$, where $c > 0$, x^0 is any starting point in C and $h(x^k)$ is the unique solution of the strongly convex program:

$$\min \left\{ f(x^k, y) + \frac{1}{c} D(y, x^k) \mid y \in C \right\}.$$

However, generally, the sequence $\{x^k\}$ does not converge to a solution of the equilibrium problems (see [2]).

Let f be a mapping defined by

$$f(x, y) := \sup \{ \langle w, y - x \rangle \mid w \in F(x) \},$$

where $F: C \rightarrow 2^{\mathbb{R}^n}$ is a multivalued mapping such that $F(x) \neq \emptyset$ for all $x \in C$. Then $\text{EP}(f, C)$ can be formulated as the multivalued variational inequality problems (shortly MVIP):

Find $x^* \in C$, $w^* \in F(x^*)$ such that

$$\langle w^*, x - x^* \rangle \geq 0 \quad \forall x \in C.$$

In this case, it is known that solutions coincide with zeros of the following projected residual function

$$T(x) := x - \text{Pr}_C(x - w^*).$$

In other words, with $x^0 \in C$, $w^0 \in F(x^0)$, the point (x^0, w^0) is a solution of (MVIP) if and only if $T(x^0) = 0$, where $T(x^0) = x^0 - \text{Pr}_C(x^0 - w^0)$ (see [16]). Applying this idea and interior proximal function technique $D(\cdot, \cdot)$ to the equilibrium problem $\text{EP}(f, C)$, we obtain the solution scheme: Let x^k be a current approximation to the solution of $\text{EP}(f, C)$. First, we compute $y^k = \arg \min \{ f(x^k, y) + \beta D(y, x^k) \mid y \in C \}$ for some positive constant β . Next, we search the line segment between x^k and $r(x^k) = x^k - y^k$ for a point (\bar{w}^k, z^k) such that the hyperplane $\partial H_k = \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k \rangle = 0\}$ strictly separates x^k

from the solution set S of $EP(f, C)$. To find such (\bar{w}^k, z^k) , we may use a computationally inexpensive Armijo-type procedure. Then we compute the next iterate x^{k+1} by projecting x^k onto the intersection of the feasible set C with the halfspace $H_k = \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k \rangle \leq 0\}$.

Then, the algorithm is described as follows.

Algorithm 2.3

Step 0. Choose $\sigma > 0, x^0 \in C, 0 < \sigma < \frac{\beta}{2}$, and $\gamma \in (0, 1)$.

Step 1. Compute

$$y^k := \arg \min \{f(x^k, \gamma) + \beta D(y, x^k) \mid y \in C\}, r(x^k) := x^k - y^k. \tag{2.3}$$

Find the smallest nonnegative number m_k of m such that

$$f(x^k - \gamma^{m_k} r(x^k), y^k) + \sigma \|r(x^k)\|^2 \leq 0. \tag{2.4}$$

Step 2. (Cutting hyperplane) Choose $\bar{w}^k \in \partial_2 f(z^k, z^k)$, where $z^k := x^k - \gamma^{m_k} r(x^k)$.

Set

$$H_k := \{x \in \mathbb{R}^n \mid \langle \bar{w}^k, x - z^k \rangle \leq 0\}.$$

Find $x^{k+1} := \text{Pr}_{C \cap H_k}(x^k)$.

Step 3. Set $k = k + 1$, and go to Step 1.

3 Convergence of the algorithm

In the next lemma, we justify the stopping criterion.

Lemma 3.1 *If $r(x^k) = 0$, then x^k is a solution to equilibrium problem $EP(f, C)$.*

Proof. Since y^k is the solution to problem (2.3) and an optimization result in convex programming (see [1]), we have

$$0 \in \partial_2 f(x^k, y^k) + \beta \nabla_1 D(y^k, x^k) + N_C(y^k),$$

where N_C denotes the normal cone. From $y^k \in \text{int}C$, it follows that $N_C(y^k) = \{0\}$. Hence

$$\xi^k + \beta \nabla_1 D(y^k, x^k) = 0,$$

where $\xi^k \in \partial_2 f(x^k, y^k)$. Replacing $y^k = x^k$ in this equality, we get

$$\xi^k + \beta \nabla_1 D(x^k, x^k) = 0.$$

Since

$$\nabla_1 D(x, y) = x - y - \mu A^T X_y \log \frac{l(x)}{l(y)} \quad \forall x, y \in C, \tag{3.1}$$

we have

$$\nabla_1 D(x^k, x^k) = 0.$$

Thus $\zeta^k = 0$. Combining this with $f(x^k, x^k) = 0$, we obtain

$$f(x^k, y) \geq \langle \xi^k, y - \xi^k \rangle = 0 \quad \forall y \in C.$$

which means that x^k is a solution to $EP(f, C)$.

In Algorithm 2.3, we need to show the existence of the nonnegative integer m_k .

Lemma 3.2 For $\gamma \in (0, 1)$, $0 < \sigma < \frac{\beta}{2}$, if $r(x^k) > 0$ then there exists the smallest nonnegative integer m_k such that the inequality (2.4) holds.

Proof. Assume on the contrary, the inequality (2.4) is not satisfied for any nonnegative integer i , i.e.,

$$f(x^k - \gamma^i r(x^k), \gamma^k) + \sigma \|r(x^k)\|^2 > 0.$$

Letting $i \rightarrow \infty$, from the continuity of f we have

$$f(x^k, \gamma^k) + \sigma \|r(x^k)\|^2 \geq 0. \tag{3.2}$$

Otherwise, for each $t > 0$ we have $1 - \frac{1}{t} \leq \log t$. We obtain after multiplication by $\frac{l_i(\gamma^k)}{l_i(x^k)} > 0$ for each $i = 1, \dots, p$,

$$\frac{l_i(\gamma^k)}{l_i(x^k)} - 1 \leq \frac{l_i(\gamma^k)}{l_i(x^k)} \log \frac{l_i(\gamma^k)}{l_i(x^k)}.$$

Then,

$$\begin{aligned} D(\gamma^k, x^k) &= \frac{1}{2} \|x^k - \gamma^k\|^2 + \mu \sum_{i=1}^n l_i^2(x^k) \left(\frac{l_i(\gamma^k)}{l_i(x^k)} \log \frac{l_i(\gamma^k)}{l_i(x^k)} - \frac{l_i(\gamma^k)}{l_i(x^k)} + 1 \right) \\ &\geq \frac{1}{2} \|r(x^k)\|^2. \end{aligned} \tag{3.3}$$

Since γ^k is the solution to the strongly convex program (2.4), we have

$$f(x^k, y) + \beta D(y, x^k) \geq f(x^k, \gamma^k) + \beta D(\gamma^k, x^k) \quad \forall y \in C.$$

Substituting $y = x^k \in C$ and using assumptions $f(x^k, x^k) = 0$, $D(x^k, x^k) = 0$, we get

$$f(x^k, \gamma^k) + \beta D(\gamma^k, x^k) \leq 0. \tag{3.4}$$

Combining (3.3) with (3.4), we obtain

$$f(x^k, \gamma^k) + \frac{\beta}{2} \|r(x^k)\|^2 \leq 0. \tag{3.5}$$

Then, inequalities (3.2) and (3.5) imply that

$$-\sigma \|r(x^k)\|^2 \leq f(x^k, \gamma^k) \leq -\frac{\beta}{2} \|r(x^k)\|^2.$$

Hence it must be either $r(x^k) = 0$ or $\sigma \geq \frac{\beta}{2}$. The first case contradicts to $r(x^k) \neq 0$, while the second one contradicts to the fact $\sigma < \frac{\beta}{2}$.

The following results perform some property of the cutting hyperplane H_k .

Lemma 3.3 Let $\{x^k\}$ be the sequence generated by Algorithm 2.3. Then the following hold:

- (i) $x^k \notin H_k, S \subseteq C \cap H_k$.
- (ii) $x^{k+1} = \text{Pr}_{C \cap H_k}(\bar{y}^k)$, where $\bar{y}^k = \text{Pr}_{H_k}(x^k)$.

Proof. (i) Since $\bar{w}^k \in \partial_2 f(z^k, z^k), \gamma^k \in C, f(z^k, z^k) = 0$, and $z^k = x^k - \gamma^{m_k} r(x^k)$, we have

$$\begin{aligned} f(z^k, \gamma^k) &\geq \langle \bar{w}^k, \gamma^k - z^k \rangle \\ &= -(1 + \gamma^{m_k}) \langle \bar{w}^k, r(x^k) \rangle. \end{aligned}$$

Combining this with (2.4), we obtain that

$$(1 + \gamma^{m_k}) \langle \bar{w}^k, r(x^k) \rangle \geq \sigma \|r(x^k)\|^2. \tag{3.6}$$

Hence

$$\begin{aligned} \langle \bar{w}^k, x^k - z^k \rangle &= \gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle \\ &\geq \frac{\sigma \gamma^{m_k}}{1 + \gamma^{m_k}} \|r(x^k)\|^2 \\ &> 0. \end{aligned}$$

This implies $x^k \notin H_k$.

Since f is assumed to be pseudomonotone on $C, z^k \in C$ and $x^* \in S$,

$$f(x^*, z^k) \geq 0 \Rightarrow f(z^k, x^*) \leq 0.$$

Combining this with $\bar{w}^k \in \partial_2 f(z^k, z^k)$, we get

$$\begin{aligned} \langle \bar{w}^k, x^* - z^k \rangle &\leq f(z^k, x^*) - f(z^k, z^k) \\ &\leq 0. \end{aligned}$$

Thus, $x^* \in H_k$.

(ii) We know that

$$H = \{x \in \mathbb{R}^n \mid \langle w, x - x^0 \rangle \leq 0\}, \text{Pr}_H(\gamma) = \gamma - \frac{\langle w, \gamma - x^0 \rangle}{\|w\|^2} w.$$

Hence,

$$\begin{aligned} \bar{y}^k &= \text{Pr}_{H_k}(x^k) \\ &= x^k - \frac{\langle \bar{w}^k, x^k - z^k \rangle}{\|\bar{w}^k\|^2} \bar{w}^k \\ &= x^k - \frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \bar{w}^k. \end{aligned}$$

Otherwise, for every $y \in C \cap H_k$ there exists $\lambda \in (0, 1)$ such that

$$\hat{x} = \lambda x^k + (1 - \lambda)y \in C \cap \partial H_k,$$

where $\partial H_k = \{x \in \mathbb{R}^n \mid \langle \bar{u}^k, x - z^k \rangle = 0\}$, because $x^k \in C$ but $x^k \notin H_k$.

$$\begin{aligned}
 \|\gamma - \bar{\gamma}^k\|^2 &\geq (1 - \lambda)^2 \|\gamma - \bar{\gamma}^k\|^2 \\
 &= \|\hat{x} - \lambda x^k - (1 - \lambda)\bar{\gamma}^k\|^2 \\
 &= \|(\hat{x} - \bar{\gamma}^k) - \lambda(x^k - \bar{\gamma}^k)\|^2 \\
 &= \|\hat{x} - \bar{\gamma}^k\|^2 + \lambda^2 \|x^k - \bar{\gamma}^k\|^2 - 2\lambda \langle \hat{x} - \bar{\gamma}^k, x^k - \bar{\gamma}^k \rangle \\
 &= \|\hat{x} - \bar{\gamma}^k\|^2 + \lambda^2 \|x^k - \bar{\gamma}^k\|^2 \\
 &\geq \|\hat{x} - \bar{\gamma}^k\|^2,
 \end{aligned} \tag{3.7}$$

because $\bar{\gamma}^k = \text{Pr}_{H_k}(x^k)$. Also we have

$$\begin{aligned}
 \|\hat{x} - x^k\|^2 &= \|\hat{x} - \bar{\gamma}^k + \bar{\gamma}^k - x^k\|^2 \\
 &= \|\hat{x} - \bar{\gamma}^k\|^2 - 2\langle \hat{x} - \bar{\gamma}^k, x^k - \bar{\gamma}^k \rangle + \|\bar{\gamma}^k - x^k\|^2 \\
 &= \|\hat{x} - \bar{\gamma}^k\|^2 + \|\bar{\gamma}^k - x^k\|^2.
 \end{aligned}$$

Since $x^{k+1} = \text{Pr}_{C \cap H_k}(x^k)$, using the Pythagorean theorem we can reduce that

$$\begin{aligned}
 \|\hat{x} - \bar{\gamma}^k\|^2 &= \|\hat{x} - x^k\|^2 - \|\bar{\gamma}^k - x^k\|^2 \\
 &\geq \|x^{k+1} - x^k\|^2 - \|\bar{\gamma}^k - x^k\|^2 \\
 &= \|x^{k+1} - \bar{\gamma}^k\|^2.
 \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\|x^{k+1} - \bar{\gamma}^k\| \leq \|\gamma - \bar{\gamma}^k\| \quad \forall \gamma \in C \cap H_k,$$

which implies

$$x^{k+1} = \text{Pr}_{C \cap H_k}(\bar{\gamma}^k).$$

In order to prove the convergence of Algorithm 2.3, we give the following key property of the sequence $\{x^k\}$ generated by the algorithm.

Lemma 3.4 *The sequence $\{x^k\}$ generated by Algorithm 2.3 satisfies the following inequality.*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - \gamma^k\|^2 - \left(\frac{\gamma^{m_k} \sigma}{\|\bar{u}^k\| (1 + \gamma^{m_k})} \right)^2 \|r(x^k)\|^4. \tag{3.9}$$

Proof. Since $x^{k+1} = \text{Pr}_{C \cap H_k}(y^k)$, we have

$$\langle y^k - x^{k+1}, z - x^{k+1} \rangle \leq 0 \quad \forall z \in C \cap H_k.$$

Substituting $z = x^* \in C \cap H_k$, then we have

$$\langle \gamma^k - x^{k+1}, x^* - x^{k+1} \rangle \leq 0 \Leftrightarrow \langle \gamma^k - x^{k+1}, x^* - \gamma^k + \gamma^k - x^{k+1} \rangle \leq 0,$$

which implies

$$\|x^{k+1} - \gamma^k\|^2 \leq \langle x^{k+1} - \gamma^k, x^* - \gamma^k \rangle.$$

Hence,

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^{k+1} - \gamma^k + \gamma^k - x^*\|^2 \\ &= \|x^{k+1} - \gamma^k\|^2 + \|\gamma^k - x^*\|^2 + 2\langle x^{k+1} - \gamma^k, \gamma^k - x^* \rangle \\ &\leq \langle x^* - \gamma^k, x^{k+1} - \gamma^k \rangle + \|\gamma^k - x^*\|^2 + 2\langle x^{k+1} - \gamma^k, \gamma^k - x^* \rangle \\ &= \|\gamma^k - x^*\|^2 + \langle x^{k+1} - \gamma^k, \gamma^k - x^* \rangle \\ &= \|\gamma^k - x^*\|^2 - \|x^{k+1} - \gamma^k\|^2. \end{aligned} \tag{3.10}$$

Since $z^k = x^k - \gamma^{m_k} r(x^k)$ and

$$\gamma^k - \text{Pr}_{H_k}(x^k) = x^k - \frac{\langle \bar{w}^k, x^k - z^k \rangle}{\|\bar{w}^k\|^2} \bar{w}^k,$$

we have

$$\begin{aligned} &\|\gamma^k - x^*\|^2 \\ &= \|x^k - x^*\|^2 + \frac{\langle \bar{w}^k, x^k - z^k \rangle^2}{\|\bar{w}^k\|^4} \|\bar{w}^k\|^2 - \frac{2\langle \bar{w}^k, x^k - z^k \rangle}{\|\bar{w}^k\|^2} \langle \bar{w}^k, x^k - x^* \rangle \\ &= \|x^k - x^*\|^2 + \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 - \frac{2\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \langle \bar{w}^k, x^k - x^* \rangle \\ &= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 \\ &\quad - 2 \left[\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \langle \bar{w}^k, x^k - x^* \rangle - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 \right] \\ &= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 \\ &\quad - \frac{2\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \left[\langle \bar{w}^k, x^k - x^* \rangle - \gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle \right] \\ &= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 \\ &\quad - \frac{2\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \langle \bar{w}^k, x^k - x^* - \gamma^{m_k} r(x^k) \rangle \\ &= \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 - \frac{2\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|^2} \langle \bar{w}^k, z^k - x^* \rangle. \end{aligned} \tag{3.11}$$

From (3.6), it follows that

$$\langle \bar{w}^k, r(x^k) \rangle \geq \frac{\sigma}{1 + \gamma^{m_k}} \|r(x^k)\|^2.$$

Thus, (3.11) reduces to

$$\begin{aligned} \|y^k - x^*\|^2 &\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \langle \bar{w}^k, r(x^k) \rangle}{\|\bar{w}^k\|} \right)^2 \\ &\leq \|x^k - x^*\|^2 - \left(\frac{\gamma^{m_k} \sigma}{\|\bar{w}^k\| (1 + \gamma^{m_k})} \right)^2 \|r(x^k)\|^4. \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we obtain the inequality (3.9)

Theorem 3.5 *Suppose that Assumptions A.1-A.4 hold, the mapping $\partial_2 f(\cdot, z^k)$ is uniformly bounded by $M > 0$, and f is pseudomonotone on C . Then the sequence $\{x^k\}$ generated by Algorithm 2.3 converges to a solution of $EP(f, C)$.*

Proof. The inequality (3.9) implies that the sequence $\{\|x^k - x^*\|\}$ is nonincreasing and hence convergent. Consequently, the sequence $\{x^k\}$ is bounded.

Since the mapping $\partial_2 f(\cdot, z^k)$ is uniformly bounded by $M > 0$, i.e.,

$$\|w^k\| \leq M \quad \forall k = 1, \dots$$

This, together with (3.9), implies

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - y^k\|^2 - \left(\frac{\gamma^{m_k} \sigma}{M(1 + \gamma^{m_k})} \right)^2 \|r(x^k)\|^4. \tag{3.13}$$

Since $\{\|x^k - x^*\|\}$ converges to zero, it is easy to see that

$$\lim_{k \rightarrow \infty} \gamma^{m_k} \|r(x^k)\| = 0.$$

The cases remaining to consider are the following.

Case 1. $\limsup_{k \rightarrow \infty} \gamma^{m_k} > 0$. This case must follow that $\liminf_{k \rightarrow \infty} \|r(x^k)\| = 0$. Since $\{x^k\}$ is bounded, there exists \bar{x} which is an accumulation point of $\{x^k\}$. In other words, a subsequence $\{x^{k_i}\}$ converges to some \bar{x} such that $r(\bar{x}) = 0$, as $i \rightarrow \infty$. Then we see from Lemma 3.3 that $\bar{x} \in S$, and besides we can take $x^* = \bar{x}$, in particular in (3.13). Thus $\{\|x^k - \bar{x}\|\}$ is a convergent sequence. Since \bar{x} is an accumulation point of $\{x^k\}$, the sequence $\{\|x^k - x^*\|\}$ converges to zero, i.e., $\{x^k\}$ converges to $\bar{x} \in S$.

Case 2. $\lim_{k \rightarrow \infty} \gamma^{m_k} = 0$. Since m_k is the smallest nonnegative integer, $m_k - 1$ does not satisfy (2.4). Hence, we have

$$f(x^k - \gamma^{m_k-1} r(x^k), y^k) > -\sigma \|r(x^k)\|^2,$$

and besides

$$f(x^{k_i} - \gamma^{m_{k_i}-1} r(x^{k_i}), y^{k_i}) > -\sigma \|r(x^{k_i})\|^2. \tag{3.14}$$

Passing onto the limit in (3.14), as $i \rightarrow \infty$, and using the continuity of f , we have

$$f(\bar{x}, \bar{y}) \geq -\sigma \|\bar{x} - \bar{y}\|^2. \tag{3.15}$$

From (3.5) we have

$$f(x^{k_i}, y^{k_i}) \leq -\frac{\beta}{2} \|r(x^{k_i})\|^2.$$

Since f is continuous, passing onto the limit, as $i \rightarrow \infty$, we obtain

$$f(\bar{x}, \bar{y}) \leq -\frac{\beta}{2} \|\bar{x} - \bar{y}\|^2.$$

Combining this with (3.15), we have

$$s\sigma \|\bar{x} - \bar{y}\|^2 \geq \frac{\beta}{2} \|\bar{x} - \bar{y}\|^2,$$

which implies $r(\bar{x}) := \|\bar{x} - \bar{y}\| = 0$ or $\sigma \geq \frac{\beta}{2}$. The second case contradicts to the fact $0 < \sigma < \frac{\beta}{2}$ and hence $r(\bar{x}) = 0$, $\bar{x} \in S$. Letting $x^* = \bar{x}$ and repeating the previous arguments, we conclude that the whole sequence $\{x^k\}$ converges to $\bar{x} \in S$.

4 Numerical results

We applied the algorithm to solve a problem of production competition under the Nash-Cournot oligopolistic market equilibrium model (see [1,2,17]). In this model, it is assumed that there are n -firms producing a common homogenous commodity and that the price p_i of firm i depends on the total quantity $\sigma_x = \sum_{i=1}^n x_i$ of the commodity.

Let $h_i(x_i)$ denote the cost of the firm i when its production level is x_i . Suppose that the profit of firm i is given by

$$f_i(x_1, \dots, x_n) := x_i p_i(\sigma_x) - h_i(x_i) \quad i = 1, \dots, n, \tag{4.1}$$

where h_i is the cost function of firm i that is assumed to be dependent only on its production level.

Let $C \subset \mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x \geq 0\}$ be closed convex which denotes the strategy set of firms. Each firm seeks to maximize its own profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric input. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables. Thus under this equilibrium concept, each firm determines its best response given other firms' actions. Mathematically, a point $x^* = (x_1^*, \dots, x_n^*) \in C$ is said to be a Nash equilibrium point if

$$f_i(x_1^*, \dots, x_{i-1}^*, y_i, x_{i+1}^*, \dots, x_n^*) \leq f_i(x_1^*, \dots, x_n^*) \quad \forall y \in C. \tag{4.2}$$

When h_i is affine, this market problem can be formulated as a special Nash equilibrium problem in the n -person nonco-operative game theory.

Set

$$\phi(x, y) := - \sum_{i=1}^n f_i(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \tag{4.3}$$

and

$$\begin{aligned}
 f(x, \gamma) &:= \phi(x, \gamma) - \phi(x, x) \\
 &= \sum_{i=1}^n \left(h_i(\gamma_i) - h_i(x_i) - \gamma_i p \left(\gamma_i + \sum_{j \neq i} x_j \right) + x_i p \left(\sum_{i=1}^n x_i \right) \right). \tag{4.4}
 \end{aligned}$$

Then it has been proved in [17] that the problem of finding an equilibrium point of this model can be formulated as EP(f, C):

Find $x \in C$ such that $f(x^*, \gamma) \geq 0$, for all $\gamma \in C$.

Proposition 4.1 [2] *A point x^* is an equilibrium point for the oligopolistic market problem if and only if it is a solution to EP(f, C), where*

$$\begin{aligned}
 f(x, \gamma) &:= \langle H(x) - p(\sigma_x)e - p'(\sigma_x)x, \gamma - x \rangle \\
 H(x) &= \left(h'_1(x_1), \dots, h'_n(x_n) \right)^T, e = (1, \dots, 1)^T, \sigma_x = \langle x, e \rangle.
 \end{aligned}$$

The following proposition gives some properties of the bifunction f .

Proposition 4.2 [2] *Let $p : C \rightarrow \mathbb{R}_+$ be convex, twice continuously differentiable, and nonincreasing and let the function $\mu_\tau : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by $\mu_\tau(\sigma_x) = \sigma_x p(\sigma_x + \tau)$ be concave for every $\tau \geq 0$. Also, let the function $h_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, $i = 1, \dots, n$, be convex and twice continuously differentiable. Then, the cost bifunction*

$$f(x, \gamma) := \langle H(x) - p(\sigma_x)e - p'(\sigma_x)x, \gamma - x \rangle$$

is monotone on C .

We now applied the algorithm to the example with seven firms ($n = 7$) provided in [9,17], where the cost and inverse demand functions have the form

$$\begin{aligned}
 H(x) &:= (2x_1 + 1, 3x_2 + 4, 4x_3 + 2, 1.5x_4 + 3, 4x_5 + 1, x_6 - 2, 3x_7 + 1)^T, \\
 p(t) &:= \frac{2}{3t} t \in (0, +\infty).
 \end{aligned}$$

Then Propositions 4.1 and 4.2 show that the bifunction defined by (4.4) is monotone on $C \times C$ and therefore assumptions of our algorithm are satisfied.

In this example, we choose

$$\begin{aligned}
 n &= 7, \\
 \eta &= 0.1, \\
 \gamma &= 2, \\
 \beta &= 5, \\
 \mu &= 0.5, \\
 x^0 &= (3, 3, 3, 3, 3, 3, 3)^T, \\
 C &= \left\{ x \in \mathbb{R}^n \mid 13 \leq \sum_{i=1}^n x_i \leq 25, 1 \leq x_i \leq 5 (i = 1, \dots, n) \right\}.
 \end{aligned}$$

Note that in this case, at iteration k , we have

$$\partial_2 f(z^k, z^k) = \{ H(z^k) - p(\sigma_{z^k})e - p'(\sigma_{z^k})z^k \},$$

Table 1 Approximate solutions for k

Iter (k)	x_1^k	x_2^k	x_3^k	x_4^k	x_5^k	x_6^k	x_7^k
0	3	3	3	3	3	3	3
1	2.0919	1.0015	1.0036	1.4664	1.0493	5.0019	1.3946
2	2.0943	1.0020	0.9991	1.4648	1.0509	5.0019	1.3979
3	2.0933	0.9993	0.9982	1.4608	1.0472	4.9992	1.3955
4	2.0938	1.0000	1.0011	1.4617	1.0483	5.0003	1.3964
5	2.0939	1.0002	1.0009	1.4618	1.0487	5.0007	1.3970
6	2.0939	0.9998	0.9995	1.4607	1.0480	4.9998	1.3965
7	2.0940	1.0000	1.0003	1.4610	1.0482	5.0001	1.3968

where $p'(\sigma_{z^k}) = -\frac{2}{3\sigma_{z^k}^2}$. Lemma 3.1 shows that if $r(x^k) = 0$, then x^k is a solution to EP (f, C). So we can say that x^k is an ϵ -solution to EP(f, C) if we have $\|r(x^k)\| \leq \epsilon$. The tolerance is taken by $\epsilon = 10^{-6}$, we obtained the following Table 1.

The approximate solution obtained after seven iterations is

$$x^7 = (2.0940, 1.0000, 1.0003, 1.4610, 1.0482, 5.0001, 1.3968)^T.$$

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Authors' contributions

JKK conceived the study and participated in its design and coordination. JKK suggested many good ideas that are useful for achievement this paper and made the revision. PNA and JKK prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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