Homoclinic classes with shadowing

Jiweon Ahn[1], Keonhee Lee[1] and Manseob Lee[2*]

* Correspondence:
lmsds@mokwon.ac.kr
[2]Department of Mathematics,
Mokwon University, Daejeon, 302-729, Korea
Full list of author information is available at the end of the article

Abstract

We show that for C^1 generic diffeomorphisms, an isolated homoclinic class is shadow-able if and only if it is a hyperbolic basic set.

Mathematics Subject Classification 2000: 37C20; 37C05; 37C29; 37D05.

Keywords: shadowing property, basic set, hyperbolic, generically, homoclinic class

1 Introduction

Let M be a closed C^∞ manifold, and denote by d the distance on M induced from the Riemannian metric $\| \cdot \|$ on the tangent bundle TM. Denote by $\mathrm{Diff}(M)$ the space of diffeo-morphisms of M endowed with the C^1-topology. Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b \ (-\infty \leq a < b \leq \infty)$ in M is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$. A closed f-invariant set $\Lambda \subset M$ is said to be *chain transitive* if for any points $x, y \leadsto \Lambda$ and $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta} \subset \Lambda \ (a_\delta) < b_\delta$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = \gamma$. For given $x, y \in M$, we write $x \leadsto y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a}^b \ (a < b)$ of f such that $x_a = x$ and $x_b = y$. Write $x \boxtimes y$ if $x \leadsto y$ and $y \leadsto x$. The set of points $\{x \in M : x \boxtimes x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. If we denote the set of periodic points f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$, where $\Omega(f)$ is the non-wandering set of f. The relation $\boxtimes$ on $\mathcal{R}(f)$ induces an equivalence relation, whose classes are called *chain components* of f. Every chain component of f is a closed f-invariant set.

Denote by $f|_\Lambda$ the restriction of f to the set Λ. We say that $f|_\Lambda$ has the *shadowing property* (or, Λ *is shadowable for* f) if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ of f there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$, for $i \in \mathbb{Z}$.

It is well known that if p is a hyperbolic periodic point f with period k then the sets

$$W^s(p) = \left\{ x \in M : f^{kn}(x) \to p \text{ as } n \to \infty \right\} \text{ and}$$

$$W^u(p) = \left\{ x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty \right\}$$

are C^1-injectively immersed submanifolds of M. Every point in the transversal intersection $(W^s(p) \pitchfork W^u(p))$ of $W^s(p)$ and $W^u(p)$ is called the *homoclinic point of* f *associated to* p. The closure of the homoclinic points of f associated to p is called the *homoclinic class of* f and it is denoted by $H_f(p)$.

Note that the homoclinic class $H_f(p)$ is a subset of the chain component $C_f(p)$ of f containing p. We consider only the hyperbolic periodic orbits of saddle type. We say

that two hyperbolic periodic points p and q are *homoclinically related*, and write $p \sim q$, if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. We know that if $p \sim q$ then index(p) = index(q). Here index(p) denotes the dimension of the stable manifold $W^s(p)$ of p. By Smale's transverse homoclinic theorem, we know that the closure of the set of homoclinically related points with a hyperbolic periodic point p is the homoclinic class $H_f(p)$ of f associated to p.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\| D_x f^n|_{E_x^s} \right\| \leq C\lambda^n \text{ and } \left\| D_x f^{-n}|_{E_x^u} \right\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. It is well-known that if Λ is hyperbolic, then Λ is shadowable.

We say that Λ is *isolated* (or *locally maximal*) if there is a compact neighborhood U of Λ such that $\cap_{n \in \mathbb{Z}} f^n(U) = \Lambda$. We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff (M); in this case, $\mathcal{G}$ is dense in Diff(M). A property "P" is said to be *(C^1)-generic* if "P" holds for all diffeo-morphisms which belong to a residual subset of Diff(M). We use the terminology "for C^1 generic f" to express "there is a residual subset $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$ ...".

In [1], Abdenur and Díaz posed the following conjecture:

Conjecture. For C^1 generic f, f is shadowable if and only if it is hyperbolic.

In this article, we gives a partial answer to the above conjecture. First, we show that C^1-generically, the chain recurrent set is hyperbolic if and only if it has the shadowing property. Next, we prove that C^1-generically, the isolated homoclinic class containing a hyperbolic periodic point is shadowable if and only if it is hyperbolic.

It is explain in [2] that every C^1-generic diffeomorphism come in one of two types: *tame diffeomorphisms*, which have a finite number of homoclinic classes and whose nonwandering sets admits partitions into a finite number of disjoint transitive sets; and *wild diffeomor-phisms*, which have an infinite number of homoclinic classes and whose nonwandering sets admit no such partitions. It is easy to show that if a diffeomorphism has a finite number of chain components, then every chain component is locally maximal, and therefore, every chain component of a tame diffeomorphism is locally maximal. Hence, we can get the following result.

Theorem 1.1 *For C^1 generic f, if f is tame then the following two conditions are equivalent:*

(a) $\mathcal{R}(f)$ *is hyperbolic,*
(b) $\mathcal{R}(f)$ *is shadowable.*

We say that a closed f-invariant set Λ is *basic*, if Λ is isolated, $f|_\Lambda$ is transitive and the periodic orbits are dense in Λ. The main result of this article is the following.

Theorem 1.2 *For C^1 generic f, the isolated homoclinic class $H_f(p)$ of f containing a hy-perbolic periodic point p is shadowable if and only if it is a hyperbolic basic set.*

A similar result for locally maximal chain transitive sets was proved in [3]. More precisely, it is proved that C^1-generically, every locally maximal chain transitive set is hyperbolic if it is shadowable.

2 Proof of Theorem 1.2

Let M and $f \in \text{Diff}(M)$ be as before. In this section, to prove Theorem 1.2, we use the techniques developed by Mañé [4]. Let $\Lambda_j(f)$ be the closure of the set of hyperbolic periodic points of f with index $j(0 \le j \le \dim M)$. If there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, any periodic points of g are hyperbolic, then f satisfies both Axiom A and the no-cycle condition. To prove our result, we first note that if p is homoclinically related to q, then $H_f(p) = H_f(q)$.

Lemma 2.1 *Suppose that f has the shadowing property on $H_f(p)$. Then for any hyperbolic periodic point $q \in H_f(p)$, $W^s(p) \cap W^u(q) \neq \emptyset$, and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof. We will only show that $W^s(p) \cap W^u(q) \neq \emptyset$. Since p and q are hyperbolic saddles, there are $\epsilon(p) > 0$ and $\epsilon(q) > 0$ such that

- both $W^s_{\epsilon(p)}(p)$ and $W^u_{\epsilon(q)}(q)$ are C^1-embedded disks,
- if $d(f^n(x), f^n(p)) < \epsilon(p)$ for $n \ge 0$, then $x \in W^s_{\epsilon(p1)}(p)$,
- if $d(f^n(x), f^n(q)) < \epsilon(q)$ for $n \le 0$ then $x \in W^u_{\epsilon(q)}(q)$.

Set $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$, and let $0 < \delta = \delta(\epsilon) < \epsilon$ be a number which satisfies the shadowing property of $f|_{H_f(p)}$ with respect to ϵ. To simplify, we assume that $f(p) = p$ and $f(q) = q$. Since $H_f(p) = \overline{\{q \in P(f) : q \sim p\}}$, we can choose a hyperbolic periodic point γ such that γ is homoclinically related to p, and $d(q, \gamma) < \delta/2$. For $x \in W^s(p) \pitchfork W^u(\gamma)$, choose $n_1 > 0$ and $n_2 > 0$ such that

$$d\left(f^{n_1}(x), p\right) < \delta/4 \text{ and } d\left(f^{-n_2}(x), \gamma\right) < \delta/4.$$

Thus

$$d\left(f^{-n_2}(x), q\right) < d\left(f^{-n_2}(x), \gamma\right) + d\left(\gamma, q\right) < \delta/2.$$

Therefore, we can construct a δ-pseudo orbit;

$$\xi = \left\{..., q, f^{-n_2+1}(x), ..., f^{-1}(x), x, f(x), ..., f^{n_1-1}(x), p, ...,\right\}.$$

Then we have $\xi \subset H_f(p)$. Since f has the shadowing property on $H_f(p)$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$, for $i \in \mathbb{Z}$. Thus

$$f^{n_1+l}(y) \in W^s_\epsilon(p) \text{ and } f^{-n_2-l}(y) \in W^u_\epsilon(q), \quad \text{for } l > 0.$$

Therefore, $y \in f^{-n_1-l}\left(W^s_\epsilon(p)\right) \cap f^{n_2+l}\left(W^u_\epsilon(q)\right)$. This means $y \in W^s(p) \cap W^u(q)$, and so $W^s(p) \cap W^u(q) \neq \emptyset$.

Lemma 2.2 *There is a residual set $\mathcal{G}_1 \subset \text{Diff}(M)$ such that $f \in \mathcal{G}_1$ satisfies the following properties:*

(a) *Every periodic point of f is hyperbolic and all their invariant manifolds are intersect transversely* (Kupka-Smale).

(b) $C_f(p) = H_f(p)$, *where p is a hyperbolic periodic point* ([5]).

Lemma 2.3 *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that if $f \in \mathcal{G}_2$, and f has the shadowing property on $H_f(p)$, then for any hyperbolic periodic point $q \in H_f(p)$,*

$$W^s(p) \pitchfork W^u(q) \neq \emptyset, \quad and \ W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Proof. Let $\mathcal{G}_2$ be as in Lemma 2.2(a), and let $f \in \mathcal{G}_2$. Take a hyperbolic saddle point $q \in H_f(p) \cap P(f)$. Since f has the shadowing property on $H_f(p)$, we know that $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$ by Lemma 2.1. Since $f \in \mathcal{G}_2$, we know that

$$W^s(p) \pitchfork W^u(q) \neq \emptyset, \quad and \ W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Proposition 2.4 *For C^1 generic f, if the homoclinic class $H_f(p)$ is isolated and shadowable, then there exist constants $m > 0$ and $0 < \lambda < 1$ such that for any periodic point $q \in H_f(p)$, index(q) = $\dim W^s(q)$,*

$$\prod_{i=0}^{\pi(q)-1} \left\| Df^m|_{E^s(f^{im}(q))} \right\| < \lambda^{\pi(q)}, \quad \prod_{i=0}^{\pi(q)-1} \left\| Df^m|_{E^s(f^{im}(q))} \right\| < \lambda^{\pi(q)}$$

and

$$\left\| Df^m|_{E^s(q)} \right\| \cdot \left\| Df^{-m}|_{E^u(f^m(q))} \right\| < \lambda^2,$$

where $\pi(q)$ denotes the period of q.

We introduce the following notion which was introduced in [6]. For $\eta > 0$ and $f \in \mathrm{Diff}(M)$, a C^1 curve γ is called an *η-simply periodic curve* of f if

- γ is diffeomorphic to [0,1] and its two end points are hyperbolic periodic points of f,
- γ is periodic with period $\pi(\gamma)$, i.e., $f^{\pi(\gamma)}(\gamma) = \gamma$, and $l(f^i(\gamma)) < \eta$ for any $0 \leq i \leq \pi(\gamma) - 1$, where $l(\gamma)$ denotes the length of γ.
- γ is normally hyperbolic.

Let p be a periodic point of f. For $\delta \in (0,1)$, we say p has a *δ-weak eigenvalue* if $Df^{\pi(p)}(p)$ has an eigenvalue μ such that $(1 - \delta)^{\pi(p)} < \mu < (1 + \delta)^{\pi(p)}$.

Lemma 2.5 [6]*There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$, any hyper-bolic periodic point p of f, and*

(a) *for any $\eta > 0$, if for any C^1 neighborhood $\mathcal{U}(f)$ of f, some $g \in \mathcal{U}(f)$ has an η-simply periodic curve γ such that two endpoints of γ are homoclinic related with p_g, then f has an 2η-simply periodic curve α such that two endpoints of α are homoclinically related with p;*

(b) *for any $\delta > 0$, if f has a periodic point $q \sim p$ with δ-weak eigenvalue, then f has a periodic point $q' \sim p$ with δ-weak eigenvalue, whose eigenvalues are all real.*

The following lemma shows that the map $f \mapsto C_f(p)$ is upper semi-continuous.

Lemma 2.6 *For any $\epsilon > 0$, there is $\delta > 0$ such that if $d_1(f,g) < \delta$ then $C_g(p_g) \subset B_\epsilon(C_f(p))$, where d_1 denotes the C^1-metric on* Diff(M).

Proof. See [[7], Lemma].

Let $H_f(p)$ be the homoclinic class of f associated to p. It is known that the map $f \mapsto H_f(p)$ is lower semi-continuous. Thus by Lemma 2.3(b), there is a residual set $\mathcal{R}$ in Diff(M) such that for any f in $\mathcal{R}$, the map $f \mapsto H_f(p)(= C_f(p))$ is semi-continuous.

Remark 2.7 *There is a residual set $\mathcal{G}_4 \subset$ Diff(M) such that for any $f \in \mathcal{G}_4$, we have the following property. Let $C_f(p)$ be the isolated chain component of f containing p in an open set U in M. If $C_f(p)$ is semi-continuous, then for any $\epsilon > 0$, there is $\delta > 0$ such that if $d_1(f,g) < \delta$ then $C_g(p_g) \subset B_\epsilon(C_f(p))$ and $C_f(p) \subset B_\epsilon(C_g(p_g))$, where d_1 is the C^1 metric on M.*

Let p be a hyperbolic periodic point $f \in$ Diff(M).

Remark 2.8 [6]*There is a residual set $\mathcal{G}_5 \subset$ Diff(M) such that for any $f \in \mathcal{G}_5$ and any $\delta > 0$, if every periodic point $q \sim p$ has no 2δ-weak eigenvalue, then there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ every periodic point q which is homoclinically related to p_g has no δ-weak eigenvalue, where p_g is the continuation of p.*

Lemma 2.9 *There is a residual set $\mathcal{G}_6 \subset$ Diff(M) such that if $f \in \mathcal{G}_6$, and the isolated homoclinic class $H_f(p)$ is shadowable, then there is a $\delta > 0$ such that every periodic point $q \in H_f(p)$ has no δ-weak eigenvalue.*

Proof. Let $\mathcal{G}_6 = \bigcap_{i=1}^{5} \mathcal{G}_i$ and let $f \in \mathcal{G}_6$. Assume that f has the shadowing property on the homoclinic class $H_f(p)$. Note that $H_f(p) = C_f(p)$. Suppose that for any $\delta > 0$ there is a hyperbolic periodic point $q \in C_f(p)$ such that q has a δ-weak eigenvalue. Since f has the shadowing property on $C_f(p)$, for the point $q \in C_f(p) \cap P(f)$, we see that $q \sim p$ by Lemma 2.3. Let U be an isolated neighborhood of $C_f(p)$, and let $\eta > 0$ be a sufficiently small constant such that $B_{2\eta}(C_f(p)) \subset U$. Then for any C^1-neighborhood $\mathcal{U}(f)$ of f, there is $g \in \mathcal{U}(f)$ such that g has an $\eta/2$-simply periodic curve $\mathcal{I}_q$, whose endpoints are homoclinically related to p_g, and $\eta/2$-simply periodic curve $\mathcal{I}_q \subset C_g(pg) \subset B_{2\eta}(C_f(p))$ (for more details, see Theorem B in [8]). Then we know that for some $l > 0, \mathcal{I}_q$ is a g^l-invariant small curve containing p_g (q is the center of $\mathcal{I}_q$), where $q \sim p_g$. By Lemma 2.5(a), for the above $\eta > 0$ f has a η-simply periodic curve $\mathcal{J}_q$ such that the endpoints of $\mathcal{J}_q$ are homoclinically related to p. Then we see that

$$\mathcal{J}_q \subset B_{2\eta}\left(C_f(p)\right) \subset U.$$

Then for some $l' > 0, \mathcal{J}_p$ is a $f^{l'}$-invariant small curve center at q. To simplify, we denote $f^{l'}$ by f. Since $C_f(p)$ is an isolated in U, we have

$$\mathcal{J}_q \subset \bigcap_{n \in \mathbb{Z}} f^n\left(B_{2\eta}\left(C_f(p)\right)\right) \subseteq \bigcap_{n \in \mathbb{Z}} f^n(U) = C_f(p).$$

Since $\mathcal{J}_q$ is a simple periodic curve of f, $f|_{\mathcal{J}_q}$ is the identity map. Since f has the shadowing property on $C_f(p)$, f must have the shadowing property on $\mathcal{J}_q$. But it is a contradiction. Thus every periodic point $q \in H_f(p)$ has no δ-weak eigenvalue.

Proof of Proposition 2.4. Let $f \in \mathcal{G}_\delta$, and suppose that there is a nonhyperbolic periodic point $q \in H_f(p)$. Then q has a δ-weak eigenvalue. This contradicts Lemma 2.9, and com-pletes the proof of Proposition 2.4 by Mañé [4].

Proof of Theorem 1.2. Let $f \in \mathcal{G}_\delta$, and let $C_f(p)$ be isolated in an open set U. Assume that f has the shadowing property on $C_f(p)$. Then $C_f(p)$ satisfy the assumptions of Propo-sition 2.4. Since f has the shadowing property on $C_f(p)$, $C_f(p)$ is hyperbolic by the main result in [9]. Consequently, we have proved that C^1-generically, $H_f(p)$ is a hyperbolic basic set.

Author details
[1]Department of Mathematics, Chungnam National University, Daejeon, 305-764, Korea [2]Department of Mathematics, Mokwon University, Daejeon, 302-729, Korea

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

References
1. Abdenur, F, Díaz, LJ: Pseudo-orbit shadowing in the C^1 topology. Disc Contin Dynam Syst. **17**, 223–245 (2007)
2. Abdenur, F: Generic robustness of spectral decompositions. Ann Scient Ec Norm Sup. **36**, 213–224 (2003)
3. Lee, K, Wen, X: Shadowable chain transitive sets of C^1-generic diffeomorphisms. Bull Korean Math Soc. **49**, 263–270 (2012)
4. Mañé, R: An ergodic closing lemma. Ann Math. **116**, 503–540 (1982)
5. Bonatti, C, Crovisier, S: Recurrence and genericity. Invent Math. **158**, 33–104 (2004)
6. Yang, D, Gan, S: Expansive homoclinic classes. Nonlinearity. **22**, 729–733 (2009)
7. Sakai, K: A note on C^1-stably shadowable chain components. Far East J Dyn Syst. **11**, 297–300 (2009)
8. Sakai, K: C^1-stably shadowable diffeomorphisms. Ergod Th & Dynam Syst. **28**, 987–1029 (2008)
9. Wen, X, Gan, S, Lan, W: C^1-stably shadowable chain component is hyperbolic. J Diff Equ. **246**, 340–357 (2009)