Journal of Inequalities and Applications
a SpringerOpen Journal

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Generalized extragradient iterative method for systems of variational inequalities

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Abstract

The purpose of this article is to investigate the problem of finding a common element of the solution sets of two different systems of variational inequalities and the set of fixed points a strict pseudocontraction mapping defined in the setting of a real Hilbert space. Based on the well-known extragradient method, viscosity approximation method and Mann iterative method, we propose and analyze a generalized extra-gradient iterative method for computing a common element. Under very mild assumptions, we obtain a strong convergence theorem for three sequences generated by the proposed method. Our proposed method is quite general and flexible and includes the iterative methods considered in the earlier and recent literature as special cases. Our result represents the modification, supplement, extension and improvement of some corresponding results in the references.

Mathematics Subject Classification (2000): Primary 49J40; Secondary 65K05; 47H09.

Keywords: systems of variational inequalities, generalized extragradient iterative method, strict pseudo-contraction mappings, inverse-strongly monotone mappings, strong convergence

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C. We denote by Fix(S) the set of fixed points of S and by P_C the metric projection of H onto C. Moreover, we also denote by $\mathbf{R}$ the set of all real numbers. For a given nonlinear mapping $A : C \to H$, consider the following classical variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by $VI(A, C)$. It is now well known that the variational inequalities are equivalent to the fixed-point problems, the origin of which can be traced back to Lions and Stampacchia [1]. This alternative formulation has been used to suggest and analyze Picard successive iterative method for solving variational inequalities under the conditions that the involved operator must be strongly monotone and Lipschitz continuous. Related to the variational inequalities, we have the problem of finding fixed points of nonexpansive mappings or strict pseudo-contractions, which is the current interest in functional analysis. Several authors considered some approaches to solve fixed point problems, optimization problems, variational inequality problems and equilibrium problems; see, for example, [2-32] and the references therein.

For finding an element of Fix(S) ∩ $VI(A, C)$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [20] introduced the following iterative algorithm:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. It was proven in [20] that if $\text{Fix}(S) \cap VI(A, C) \neq \emptyset$ then the sequence $\{x_n\}$ converges weakly to some $z \in$ Fix(S) ∩ $VI(A, C)$. Recently, Nadezhkina and Takahashi [19] and Zeng and Yao [32] proposed some so-called extragra-dient method motivated by the idea of Korpelevich [33] for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality. Further, these iterative methods were extended in [27] to develop a general iterative method for finding a element of Fix(S) ∩ $VI(A, C)$.

Let $A_1, A_2 : C \to H$ be two mappings. In this article, we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 A_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_2 A_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.2}$$

which is called a general system of variational inequalities, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants. It was introduced and considered by Ceng et al. [7]. In particular, if $A_1 = A_2 = A$, then problem (1.2) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda_1 A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_2 A x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

which was defined by Verma [22] (see also [21]) and it is called a new system of variational inequalities. Further, if $x^* = y^*$ additionally, then problem (1.3) reduces to the classical variational inequality problem (1.1). We remark that in [34], Ceng et al. proposed a hybrid extragradient method for finding a common element of the solution set of a variational inequality problem, the solution set of problem (1.2) and the fixed-point set of a strictly pseudocontractive mapping in a real Hilbert space. Recently, Ceng et al. [7] transformed problem (1.2) into a fixed point problem in the following way:

Lemma 1.1.[7]. *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem* (1.2) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\left[P_C(x - \lambda_2 A_2 x) - \lambda_1 A_1 P_C(x - \lambda_2 A_2 x)\right], \quad \forall x \in C, \tag{1.4}$$

where $\bar{y} = P_C(\bar{x} - \lambda_2 A_2 \bar{x})$

In particular, if the mapping $A_i : C \to H$ is $\widehat{\alpha}_i$-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\lambda_i \in (0, 2\widehat{\alpha}_i)$ fori $= 1, 2$.

Utilizing Lemma 1.1, they proposed and analyzed a relaxed extragradient method for solving problem (1.2). Throughout this article, the set of fixed points of the mapping G is denoted by Γ. Based on the extragradient method [33] and viscosity approximation method [23], Yao et al. [26] introduced and studied a relaxed extragradient iterative algorithm for finding a common solution of problem (1.2) and the fixed point problem of a strictly pseudocontraction in a real Hilbert space H.

Theorem 1.1. [[26], Theorem 3.2]. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mapping $A_i : C \to H$ be $\widehat{\alpha}_i$-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strict pseudocontraction mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction mapping with $\rho \in \left[0, \frac{1}{2}\right)$ For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C(x_n - \lambda_2 A_2 x_n), \\ y_n = \alpha_n Q x_n + (1-\alpha_n) P_C(z_n - \lambda_1 A_1 z_n), \\ x_{x+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_1 A_1 z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where $\lambda_i \in (0, 2\widehat{\alpha}_i)$ for $i = 1, 2$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)k \leq \gamma_n <(1 - 2\rho)\delta_n$ for all $n \geq 0$;

(ii) $\lim_{x\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$

Then the sequence $\{x_n\}$ generated by (1.5) *converges strongly to $x^* = P_\Omega Q x^*$ and (x^*, y^*) is a solution of the general system of variational inequalities* (1.2), *where $y^* = P_C(x^* - \lambda_2 A_2 x^*)$.*

Let $B_1, B_2 : C \to H$ be two mappings. In this article, we also consider another general system of variational inequalities, that is, finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.6}$$

where $\mu_1 >0$ and $\mu_2 >0$ are two constants.

Utilizing Lemma 1.1, we know that for given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem (1.6) if and only if $\bar{x}$ is a fixed point of the mapping $F : C \to C$ defined by

$$F(x) = P_C\left[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\right], \quad \forall x \in C, \tag{1.7}$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$ In particular, if the mapping $B_i : C \to H$ is $\widehat{\beta}_i$-inverse strongly monotone for $i = 1, 2$, then the mapping F is nonexpansive provided $\mu_i \in (0, 2\widehat{\alpha}_i)$ for $i = 1, 2$. Throughout this article, the set of fixed points of the mapping F is denoted by Γ_0.

Assume that $A_i : C \to H$ is $\widehat{\alpha}_i$-inverse strongly monotone and and $B_i : C \to H$ is $\widehat{\beta}_i$-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strict pseudocontraction mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap \Gamma_0 \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction mapping with $\rho \in \left[0, \frac{1}{2}\right)$. Motivated and inspired by the research work going on in this area, we propose and analyze the following iterative scheme for computing a common element of the solution set Γ of one general system of variational inequalities (1.2), the solution set Γ_0 of another general system of variational inequalities (1.6), and the fixed point set Fix(S) of the mapping S:

$$\begin{cases} z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ y_n = \alpha_n Q x_n + (1-\alpha_n) P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right] + \delta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\mu_i \in (0, 2\widehat{\beta}_i)$ for i = 1, 2, and $\{\alpha_n\}$,$\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\} \subset [0, 1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Furthermore, it is proven that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (1.8) converge strongly to the same point $x^* = P_\Omega Q x^*$ under very mild conditions, and (x^*, y^*) and $(x^*, \bar{y}^*)$ are a solution of general system of variational inequalities (1.2) and a solution of general system of variational inequalities (1.6), respectively, where $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ and $\bar{y}^* = P_C(x^* - \mu_2 B_2 x^*)$

Our result represents the modification, supplement, extension and improvement of the above Theorem 1.1 in the following aspects.

(a) our problem of finding an element of Fix(S) ∩ Γ ∩ Γ_0 is more general and more complex than the problem of finding an element of Fix(S) ∩ Γ in the above Theorem 1.1.

(b) Algorithm (1.8) for finding an element of Fix(S)∩Γ∩Γ_0 is also more general and more flexible than algorithm (1.5) for finding an element of Fix(S) ∩ Γ in the above Theorem 1.1. Indeed, whenever $B_1 = B_2 = 0$, we have

$$z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right] = x_n, \quad \forall n \geq 0.$$

In this case, algorithm (1.8) reduces essentially to algorithm (1.5).

(c) Algorithm (1.8) is very different from algorithm (1.5) in the above Theorem YLK because algorithm (1.8) is closely related to the viscosity approximation method with the ρ-contraction $Q : C \to C$ and involves the Picard successive iteration for the general system of variational inequalities (1.6).

(d) The techniques of proving strong convergence in our result are very different from those in the above Theorem 1.1 because our techniques depend on the norm inequality in Lemma 2.2 and the inverse-strong monotonicity of mappings A_i, $B_i : C \to H$ for i = 1, 2, the demiclosed-ness principle for strict pseudocontractions, and the transformation of two general systems of variational inequalities (1.2) and (1.6) into the fixed-point problems of the nonexpansive self-mappings G: $C \to C$ and F: $C \to C$ (see the above Lemma 1.1, respectively.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) := \left\{x : x_{n_i} \rightharpoonup x \text{ for some subsequence} \{x_{n_i}\} \text{ of } \{x_n\}\right\}.$$

Recall that a mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any α-inverse strongly monotone mapping is Lipschitz continuous. A mapping $S : C \to C$ is called a strict pseudocontraction [35] if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x + y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.1}$$

In this case, we also say that S is a k-strict pseudocontraction. Meantime, observe that (2.1) is equivalent to the following

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.2}$$

It is easy to see that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $\frac{1-k}{2}$-inverse strongly monotone and hence $\frac{2}{1-k}$-Lipschitz continuous; for further detail, we refer to [30] and the references therein. It is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall x \in C.$$

The mapping P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is, there holds the following relation

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, \ y \in C. \tag{2.4}$$

See [36] for more details.

In order to prove our main result in the next section, we need the following lemmas. The following lemma is an immediate consequence of an inner product.

Lemma 2.1. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \tag{2.5}$$

Recall that $S : C \to C$ is called a quasi-strict pseudocontraction if the fixed point set of S, Fix(S), is nonempty and if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2 \quad \text{for all } x \in C \text{ and } p \in \text{Fix}(S). \tag{2.6}$$

We also say that S is a k-quasi-strict pseudocontraction if condition (2.6) holds.

The following lemma was proved by Suzuki [37].

Lemma 2.2.[37]*Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* [0, 1] *with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.3. [[17], Proposition 2.1] *Assume C is a nonempty closed convex subset of a real Hilbert space H and let S: C → C be a self-mapping on C.*

(a) If S is a k-strict pseudocontraction, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \le \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \tag{2.7}$$

(b) if S is a k-strict pseudocontraction, then the mapping I - S is demiclosed (*at* 0). *That is, if* $\{x_n\}$ *is a sequence in C such that* $x_n \rightharpoonup \tilde{x}$ *and* $(I - S)x_n \to 0$, *then* $(I - S)\tilde{x} = 0$, *i.e.,* $\tilde{x} \in \mathrm{Fix}(S)$

(c) if S is a k-quasi-strict pseudocontraction, then the fixed point set Fix(*S*) *of S is closed and convex so that the projection* $P_{\mathrm{Fix}(S)}$ *is well defined.*

Lemma 2.4.[24]*Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \le (1 - \delta_n)a_n + \delta_n\sigma_n, \quad \forall n \ge 0,$$

where $\{\delta_n\}$, $\{\sigma_n\}$ *are sequences of real numbers such that*

(i) $\{\delta_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \delta_n) := \lim_{n\to\infty}\prod_{j=0}^{n}(1 - \delta_j) = 0;$$

(ii) $\limsup_{n\to\infty} \sigma_n \le 0$ *or*

(ii) $\sum_{n=0}^{\infty} \delta_n\sigma_n$ *is convergent.*

Then $\lim_{n\to\infty} a_n = 0$.

3. Strong convergence theorems

We are now in a position to state and prove our main result.

Lemma 3.1. [[26], Lemma 3.1] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S: C → C be a k-strict pseudocontraction mapping. Let γ and δ be two nonnegative real numbers. Assume* $(\gamma + \delta)k \le \gamma$. *Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C. \tag{3.1}$$

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Assume that for i* = 1, 2, *the mappings* A_i, $B_i : C \to H$ *are* $\widehat{\alpha}_i$*-inverse strongly monotone and* $\widehat{\beta}_i$*-inverse strongly monotone, respectively. Let* $S : C \to C$ *be a k-strict pseudocontraction mapping such that* $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap \Gamma_0 \ne \emptyset$. *Let* $Q : C \to C$ *be a ρ-contraction mapping with* $\rho \in \left[0, \frac{1}{2}\right)$ *For given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{cases} z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ y_n = \alpha_n Q x_n + (1-\alpha_n) P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right] + \delta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{3.2}$$

where $\lambda_i \in (0, 2\widehat{\alpha}_i)$ *and* $\mu_i \in (0, 2\widehat{\beta}_i)$ *for* $i = 1, 2$, *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *are four sequences in* [0, 1] *such that:*

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \geq 0$;

(ii) $\lim_{x\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$

Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *generated by* (3.2) *converge strongly to the same point* $x^* = P_\Omega Q x^*$, *and* (x^*, y^*) *and* $(x^*, \bar{y}^*)$ *are a solution of general system of variational inequalities* (1.2) *and a solution of general system of variational inequalities* (1.6), *respectively, where* $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ *and* $\bar{y}^* = P_C(x^* - \mu_2 B_2 x^*)$.

Proof. Let us show that the mappings $I - \lambda_i A_i$ and $I - \mu_i Bi$ are nonexpansive for $i = 1, 2$. Indeed, since for $i = 1, 2$, A_i, B_i are $\widehat{\alpha}_i$-inverse strongly monotone and $\widehat{\beta}_i$-inverse strongly monotone, respectively, we have for all $x, y \in C$

$$\begin{aligned} \left\|(I - \lambda_i A_i)x - (I - \lambda_i A_i)y\right\|^2 &= \left\|(x-y) - \lambda_i(A_i x - A_i y)\right\|^2 \\ &= \|x-y\|^2 - 2\lambda_i\langle A_i x - A_i y, x - y\rangle + \lambda_i^2\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2 - 2\lambda_i\widehat{\alpha}_i\|A_i x - A_i y\|^2 + \lambda_i^2\|A_i x - A_i y\|^2 \\ &= \|x-y\|^2 - \lambda_i(2\widehat{\alpha}_i - \lambda_i)\|A_i x - A_i y\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

and

$$\begin{aligned} \left\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\right\|^2 &= \left\|(x-y) - \mu_i(B_i x - B_i y)\right\|^2 \\ &\leq \|x-y\|^2 - \mu_i(2\widehat{\beta}_i - \mu_i)\|B_i x - B_i y\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

This shows that both $I - \lambda_i A_i$ and $I - \mu_i Bi$ are nonexpansive for $i = 1, 2$.

We divide the rest of the proof into several steps.

Step 1. $\lim_{x\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, first, we can write (3.2) as $x_{n+1} = \beta_n x_n + (1 - \beta_n)u_n$, $\forall n \geq 0$, where $u_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$. Set $\tilde{z}_n = P_C(z_n - \lambda_2 A_2 z_n), \forall n \geq 0$. It follows that

$$\begin{aligned} u_{n+1} - u_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1}P_C(\tilde{z}_{n+1} - \lambda_1 A_1 \tilde{z}_{n+1}) + \delta_{n+1}Sy_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) + \delta_n S y_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1}\left[P_C(\tilde{z}_{n+1} - \lambda_1 A_1 \tilde{z}_{n+1}) - P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n)\right] + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1-\beta_{n+1}} \\ &\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\right)Sy_n. \end{aligned} \tag{3.3}$$

From Lemma 3.1 and (3.2), we get

$$\begin{aligned}&\left\|\gamma_{n+1}\left[P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right]+\delta_{n+1}(Sy_{n+1}-Sy_n)\right\|\\&\le\left\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\right\|+\gamma_{n+1}\left\|\left[P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})-y_{n+1}\right]\right.\\&\quad\left.+\left[y_n-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right]\right\|\\&\le(\gamma_{n+1}+\delta_{n+1})\left\|y_{n+1}-y_n\right\|+\gamma_{n+1}\alpha_{n+1}\left\|Qx_{n+1}-P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})\right\|\\&\quad+\gamma_{n+1}\alpha_n\left\|Qx_n-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right\|.\end{aligned}\tag{3.4}$$

Note that

$$\begin{aligned}\left\|P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right\|&\le\left\|(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})-(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right\|\\&\le\|\tilde{z}_{n+1}-\tilde{z}_n\|\\&=\left\|P_C(z_{n+1}-\lambda_2A_2z_{n+1})-P_C(z_n-\lambda_2A_2z_n)\right\|\\&\le\left\|(z_{n+1}-\lambda_2A_2z_{n+1})-(z_n-\lambda_2A_2z_n)\right\|\\&\le\|z_{n+1}-z_n\|,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\|z_{n+1}-z_n\|&=\left\|P_C\left[P_C(x_{n+1}-\mu_2B_2x_{n+1})-\mu_1B_1P_C(x_{n+1}-\mu_2B_2x_{n+1})\right]\right.\\&\quad\left.-P_C\left[P_C(x_n-\mu_2B_2x_n)-\mu_1B_1P_C(x_n-\mu_2B_2x_n)\right]\right\|\\&\le\left\|\left[P_C(x_{n+1}-\mu_2B_2x_{n+1})-\mu_1B_1P_C(x_{n+1}-\mu_2B_2x_{n+1})\right]\right.\\&\quad\left.-\left[P_C(x_n-\mu_2B_2x_n)-\mu_1B_1P_C(x_n-\mu_2B_2x_n)\right]\right\|\\&\le\left\|P_C(x_{n+1}-\mu_2B_2x_{n+1})-P_C(x_n-\mu_2B_2x_n)\right\|\\&\le\left\|(x_{n+1}-\mu_2B_2x_{n+1})-(x_n-\mu_2B_2x_n)\right\|\\&\le\|x_{n+1}-x_n\|.\end{aligned}\tag{3.6}$$

Then it follows from (3.5) and (3.6) that

$$\begin{aligned}&\left\|y_{n+1}-y_n\right\|\\&\le\|P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z})\|+\alpha_{n+1}\left\|Qx_{n+1}P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})\right\|\\&\quad+\alpha_n\left\|Qx_n-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\right\|\\&\le\|x_{n+1}-x_n\|+\alpha_n\|Qx_n-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\|+\alpha_{n+1}\|Qx_{n+1}-P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})\|.\end{aligned}\tag{3.7}$$

Therefore, from (3.3), (3.4) and (3.7), we have

$$\begin{aligned}\|u_{n+1}-u_n\|\le\|x_{n+1}-x_n\|&+\left(1+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right)\alpha_n\|Qx_n-P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)\|\\&+\left(1+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\right)\alpha_{n+1}\|Qx_{n+1}-P_C(\tilde{z}_{n+1}-\lambda_1A_1\tilde{z}_{n+1})\|\\&+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\left(\left\|P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n\right\|+\left\|Sy_n\right\|\right).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|)\le0.$$

Hence by Lemma 2.2 we get $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|u_n-x_n\|=0.\tag{3.8}$$

Step 2. $\lim_{n\to\infty}\left\|A_1\tilde{z}_n-A_1y^*\right\|=\lim_{n\to\infty}\left\|A_2z_n-A_2x^*\right\|=\lim_{n\to\infty}\left\|B_1\tilde{x}_n-B_1\tilde{y}^*\right\|=\lim_{n\to\infty}\left\|B_2x_n-B_2x^*\right\|=0.$

Indeed, let $x^* \in \Omega$. Utilizing Lemma 1.1 we have $x^* = Sx^*$, $x^* = P_C[P_C(x^* - \lambda_2A_2x^*) - \lambda_1A_1P_C(x^* - \lambda_2A_2x^*)]$ and

$$x^* = P_C\left[P_C\left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right].$$

Put $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ and $\overline{y}^* = P_C\left(x^* - \mu_2 B_2 x^*\right)$. Then $x^* = P_C(y^* - \lambda_1 A_1 y^*)$ and $x^* = P_C\left(\overline{y}^* - \mu_1 B_1 \overline{y}^*\right)$. Thus it follows that

$$\begin{aligned}
&\left\|P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - P_C(y^* - \lambda_1 A_1 y^*)\right\|^2 \\
&\le \left\|(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - \left(y^* - \lambda_1 A_1 y^*\right)\right\|^2 \\
&\le \left\|\tilde{z}_n - y^*\right\|^2 - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2 \\
&= \left\|P_C(z_n - \lambda_2 A_2 z_n) - P_C(x^* - \lambda_2 A2 x^*)\right\|^2 - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2 \\
&\le \left\|(z_n - \lambda_2 A_2 z_n) - \left(x^* - \lambda_2 A_2 x^*\right)\right\|^2 - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2 \\
&\le \left\|z_n - x^*\right\|^2 - \lambda_2\left(\widehat{\alpha}_2 - \lambda_2\right)\left\|A_2 z_n - A_2 x^*\right\|^2 - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\left\|z_n - x^*\right\|^2 &= \left\|P_C\left(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n\right) - P_C\left(\overline{y}^* - \mu_1 B_1 y^*\right)\right\|^2 \\
&\le \left\|\left(\tilde{x}_n - \mu_1 B_1 \tilde{x}_n\right) - \left(\overline{y}^* - \mu_1 B_1 \overline{y}^*\right)\right\|^2 \\
&\le \left\|\tilde{x}_n - y^*\right\|^2 - \mu_1\left(2\widehat{\beta}_1 - \mu_1\right)\left\|B_1 \tilde{x}_n - B_1 \overline{y}^*\right\|^2 \\
&= \left\|P_C\left(x_n - \mu_2 B_2 x_n\right) - P_C\left(x^* - \mu_2 B_2 x^*\right)\right\|^2 - \mu_1\left(2\widehat{\beta}_1 - \mu_1\right)\left\|B_1 \tilde{x}_n - B_1 \overline{y}^*\right\|^2 \\
&\le \left\|x_n - x^*\right\|^2 - \mu_2\left(\widehat{\beta}_2 - \mu_2\right)\|B_2 x_n - B_2 x\|^2 - \mu_1\left(2\widehat{\beta}_1 - \mu_1\right)\left\|B_1 \tilde{x}_n - B_1 \overline{y}^*\right\|^2
\end{aligned} \tag{3.10}$$

It follows from (3.2), (3.9) and (3.10) that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\le \alpha_n\left\|Q x_n - x^*\right\|^2 + (1 - \alpha_n)\left\|P_C\left(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n\right) - P_C\left(y^* - \lambda_1 A_1 y^*\right)\right\|^2 \\
&\le \alpha_n\left\|Q x_n - x^*\right\|^2 + \left\|P_C\left(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n\right) - P_C(y^* - \lambda_1 A_1 y^*)\right\|^2 \\
&\le \alpha_n\left\|Q x_n - x^*\right\|^2 + \left\|z_n - x^*\right\|^2 - \lambda_2\left(\widehat{\alpha}_2 - \lambda_2\right)\left\|A_2 z_n - A_2 x^*\right\|^2 \\
&\quad - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2 \\
&\le \alpha_n\left\|Q x_n - x^*\right\|^2 + \left\|x_n - x^*\right\|^2 - \mu_2\left(\widehat{\beta}_2 - \mu_2\right)\left\|B_2 x_n - B_2 x^*\right\|^2 \\
&\quad - \mu_1\left(2\widehat{\beta}_1 - \mu\right)\left\|B_1 \tilde{x}_n - B_1 \overline{y}^*\right\|^2 - \lambda_2\left(\widehat{\alpha}_2 - \lambda_2\right)\left\|A_2 z_n - A_2 x^*\right\|^2 \\
&\quad - \lambda_1\left(2\widehat{\alpha}_1 - \lambda_1\right)\left\|A_1 \tilde{z}_n - A_1 y^*\right\|^2.
\end{aligned} \tag{3.11}$$

Utilizing the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &= \left\|\beta_n\left(x_n - x^*\right) + (1 - \beta_n)\frac{1}{1 - \beta_n}\left[P_C\left(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n\right) - x^*\right) + \delta_n\left(S y_n - x^*\right)\right]\right\|^2 \\
&\le \beta_n\left\|x_n - x^*\right\|^2 + (1 - \beta_n)\left\|\frac{\gamma_n}{1 - \beta_n}\left(P_C\left(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n\right) - x^*\right) + \frac{\delta_n}{1 - \beta_n}\left(S y_n - x^*\right)\right\|^2 \\
&= \beta_n\left\|x_n - x^*\right\|^2 + (1 - \beta_n)\left\|\frac{\gamma_n\left(y_n - x^*\right) + \delta_n\left(S y_n - x^*\right)}{1 - \beta_n} + \frac{\alpha_n \gamma_n}{1 - \beta_n}\left(P_C\left(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n\right) - Q x_n\right)\right\|^2 \\
&\le \beta_n\left\|x_n - x^*\right\|^2 + (1 - \beta_n)\left\|\frac{\gamma_n\left(y_n - x^*\right) + \delta_n\left(S y_n - x^*\right)}{1 - \beta^*}\right\|^2 + M\alpha_n \\
&\le \beta_n\left\|x_n - x^*\right\|^2 + (1 - \beta_n)\left\|y_n - x^*\right\|^2 + M\alpha_n,
\end{aligned} \tag{3.12}$$

where $M > 0$ is some appropriate constant. So, from (3.11) and (3.12) we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le{}& \|x_n-x^*\|^2 - \mu_2\left(\widehat{\beta}_2-\mu_2\right)(1-\beta_n)\|B_2x_n-B_2x^*\|^2\\ &-\mu_1\left(2\widehat{\beta}_1-\mu_1\right)(1-\beta_n)\|B_1\tilde{x}_n-B_1\overline{y}^*\|^2-\lambda_2\left(\widehat{\alpha}_2-\lambda_2\right)(1-\beta_n)\|A_2z_n-A_2x^*\|^2\\ &-\lambda_1\left(2\widehat{\alpha}_1-\lambda_1\right)(1-\beta_n)\|A_1\tilde{z}_n-A_1y^*\|^2+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lambda_1\left(2\widehat{\alpha}_1-\lambda_1\right)(1-\beta_n)\|A_1\tilde{z}_n-A_1y^*\|^2+\lambda_2\left(\widehat{\alpha}_2-\lambda_2\right)(1-\beta_n)\|A_2z_n-A_2x^*\|^2\\ &+\mu_1\left(2\widehat{\beta}_1-\mu_1\right)(1-\beta_n)\|B_1\tilde{x}_n-B_1\overline{y}^*\|^2+\mu_2\left(\widehat{\beta}_2-\mu_2\right)(1-\beta_n)\|B_2x_n-B_2x^*\|^2\\ &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n\\ &\le \left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\|x_n-x_{n+1}\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.\end{aligned}$$

Since $\liminf_{n\to\infty}\lambda_1(2\widehat{\alpha}_1-\lambda_1)(1-\beta_n)>0, \liminf_{n\to\infty}\lambda_2(2\widehat{\alpha}_2-\lambda_2)(1-\beta_n)>0, \liminf_{n\to\infty}\mu_1\left(2\widehat{\beta}_1-\mu_1\right)(1-\beta_n)>0, \liminf_{n\to\infty}\mu_2\left(\widehat{\beta}_2-\mu_2\right)(1-\beta_n)>0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we have

$$\lim_{n\to\infty}\|A_1\tilde{z}_n-A_1y^*\|=\lim_{n\to\infty}\|A_2z_n-A_2x^*\|=\lim_{n\to\infty}\|B_1\tilde{x}_n-B_1\overline{y}^*\|=\lim_{n\to\infty}\|B_2x_n-B_2x^*\|=0.$$

Step 3. $\lim_{n\to\infty}\|z_n - y_n\| = \lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|Sy_n - y_n\| = 0$.
Indeed, set $v_n = P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)$. Noting that P_C is firmly nonexpansive, we have

$$\begin{aligned}\|\tilde{z}_n-y^*\|^2 &= \|P_C(z_n-\lambda_2A_2z_n)-P_C\left(x^*-\lambda_2A_2x^*\right)\|^2\\ &\le \langle (z_n-\lambda_2A_2z_n)-\left(x^*-\lambda_2A_2x^*\right),\tilde{z}_n-y^*\rangle\\ &= \tfrac{1}{2}\left(\|z_n-x^*-\lambda_2\left(A_2z_n-A_2x^*\right)\|^2+\|\tilde{z}_n-y^*\|^2\right.\\ &\quad\left.-\|(z_n-x^*)-\lambda_2\left(A_2z_n-A_2x^*\right)-(\tilde{z}_n-y^*)\|^2\right)\\ &\le \tfrac{1}{2}\left(\|z_n-x^*\|^2+\|\tilde{z}_n-y^*\|^2-\|(z_n-\tilde{z}_n)-\lambda_2\left(A_2z_n-A_2x^*\right)-\left(x^*-y^*\right)\|^2\right)\\ &= \tfrac{1}{2}\left(\|z_n-x^*\|^2+\|\tilde{z}_n-y^*\|^2-\|z_n-\tilde{z}-\left(x^*-y^*\right)\|^2\right)\\ &\quad+2\lambda_2\langle z_n-\tilde{z}_n-\left(x^*-y^*\right),A_2z_n-A_2x^*\rangle-\lambda_2^2\|A_2z_n-A_2x^*\|^2\Big),\end{aligned}$$

and

$$\begin{aligned}\|v_n-x^*\|^2 &= \|P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)-P_C\left(y^*-\lambda_1A_1y^*\right)\|^2\\ &\le \left\langle \tilde{z}_n-\lambda_1A_1\tilde{z}_n-\left(y^*-\lambda_1A_1y^*\right),v_n-x^*\right\rangle\\ &= \tfrac{1}{2}\left(\|\tilde{z}_n-\lambda_1A_1\tilde{z}_n-\left(y^*-\lambda_1A_1y^*\right)\|^2+\|v_n-x^*\|^2\right.\\ &\quad\left.-\|\tilde{z}_n-\lambda_1A_1\tilde{z}_n-\left(y^*-\lambda_1A_1y^*\right)-\left(v_n-x^*\right)\|^2\right.\\ &\le \tfrac{1}{2}\left(\|\tilde{z}_n-y^*\|^2+\|v_n-x^*\|^2-\|\tilde{z}_n-v_n+\left(x^*-y^*\right)\|^2\right.\\ &\quad\left.+2\lambda_1\left\langle A_1\tilde{z}_n-A_1y^*,\tilde{z}_n-v_n+\left(x^*-y^*\right)\right\rangle-\lambda_1^2\|A_1\tilde{z}_n-A_1y^*\|^2\right)\\ &\le \tfrac{1}{2}\left(\|z_n-x^*\|^2+\|v_n-x^*\|^2-\|\tilde{z}_n-v_n+\left(x^*-y^*\right)\|^2\right.\\ &\quad\left.+2\lambda_1\left\langle A_1\tilde{z}_n-A_1y^*,\tilde{z}_n-v_n+\left(x^*-y^*\right)\right\rangle\right)\end{aligned}$$

due to (3.9). Thus, we have

$$\|\tilde{z}_n-y^*\|^2\le\|z_n-z^*\|^2-\|z_n-\tilde{z}_n-\left(x^*-y^*\right)\|^2+2\lambda_2\langle z_n-\tilde{z}_n-\left(x^*-y^*\right),A_2z_n-A_2x^*\rangle-\lambda_2^2\|A_2z_n-A_2x^*\|^2, \quad (3.13)$$

and

$$\|v_n - x^*\|^2 \leq \|z_n - x^*\|^2 - \|\tilde{z}_n - v_n + (x^* - y^*)\|^2 + 2\lambda_1 \|A_1\tilde{z}_n - A_1y^*\| \|\tilde{z}_n - v_n + (x^* - y^*)\|.$$

It follows that

$$\begin{aligned}\|y_n - x^*\| &\leq \alpha_n \|Qx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\leq \alpha_n \|Qx_n - x^*\|^2 + \|v_n - x^*\|^2 \\ &\leq \alpha_n \|Qx_n - x^*\|^2 + \|z_n - x^*\|^2 - \|\tilde{z}_n - v_n + (x^* - y^*)\|^2 \\ &\quad + 2\lambda_1 \|A_1\tilde{z}_n - A_1y^*\| \|\tilde{z}_n - v_n + (x^* - y^*)\|.\end{aligned} \tag{3.14}$$

Utilizing (3.2), (3.10), (3.12) and (3.13), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 \\ &\quad + (1-\alpha_n)\|P_C(\tilde{z}_n - \lambda_1A_1\tilde{z}_n) - P_C(y^* - \lambda_1A_1y^*)\|^2\Big] + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 + (1-\alpha_n)\|(\tilde{z}_n - \lambda_1A_1\tilde{z}_n) - (y^* - \lambda_1A_1y^*)\|^2\Big]^2 + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 + (1-\alpha_n)\|\tilde{z}_n - y^*\|^2\Big] + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 + (1-\beta_n)\|(\tilde{z}_n - y^*)\|^2\Big] + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\alpha_n \|Qx_n - x^*\|^2 + (1-\beta_n)\Big[\|\tilde{z}_n - x^*\|^2 - \|z_n - \tilde{z}_n - (x^* - y^*)\|^2 \\ &\quad + 2\lambda_2 \langle z_n - \tilde{z}_n - (x^* - y^*), A_2z_n - A_2x^*\rangle\Big] + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\alpha_n \|Qx_n - x^*\|^2 + (1-\beta_n)\Big[\|x_n - x^*\|^2 - \|z_n - \tilde{z}_n - (x^* - y^*)\|^2 \\ &\quad + 2\lambda_2 \|z_n - \tilde{z}_n - (x^* - y^*)\| \|A_2z_n - A_2x^*\|\Big] + M\alpha_n \\ &= \|x_n - x^*\|^2 - (1-\beta_n)\|z_n - \tilde{z}_n - (x^* - y^*)\|^2 \\ &\quad + 2(1-\beta_n)\lambda_2 \|z_n - \tilde{z}_n - (x^* - y^*)\| \|A_2z_n - A_2x^*\| + \Big[M + (1-\beta_n)\|Qx_n - x^*\|^2\Big]\alpha_n.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n)\|z_n - \tilde{z}_n - (x^* - y^*)\|^2 &\leq (\|x_n - x^*\| \ \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + \Big(M + \|Qx_n - x^*\|^2\Big)\alpha_n \\ &\quad + 2(1-\beta_n)\lambda_2 \|z_n - \tilde{z}_n - (x^* - y^*)\| \|A_2z_n - A_2x^*\|.\end{aligned}$$

Note that $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\|A_2z_n - A_2x^*\| \to 0$. Then we immediately deduce that

$$\lim_{n\to\infty} \|z_n - \tilde{z}_n - (x^* - y^*)\| = 0. \tag{3.15}$$

In the meantime, utilizing (3.10), (3.12) and (3.14) we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 + \|z_n - x^*\|^2 \\ &\quad - \|\tilde{z}_n - v_n + (x^* - y^*)\|^2 + 2\lambda_1 \|A_1\tilde{z}_n - A_1y^*\| \|\tilde{z}_n - v_n + (x^* - y^*)\|\Big] + M\alpha_n \\ &\leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Big[\alpha_n \|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad - \|\tilde{z}_n - v_n + (x^* - y^*)\|^2 + 2\lambda_1 \|A_1\tilde{z}_n - A_1y^*\| \|\tilde{z}_n - v_n + (x^* - y^*)\|\Big] + M\alpha_n \\ &\leq \|x_n - x^*\|^2 - (1-\beta_n)\|\tilde{z}_n - v_n + (x^* - y^*)\|^2 \\ &\quad + 2\lambda_1(1-\beta_n)\|A_1\tilde{z}_n - A_1y^*\| \|\tilde{z}_n - v_n + (x^* - y^*)\| + (M + \|Qx_n - x^*\|^2)\alpha_n.\end{aligned}$$

So, we obtain

$$\begin{aligned}(1-\beta_n)\|\tilde{z}_n - v_n + (x^* - y^*)\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\lambda_1(1-\beta_n)\|A_1\tilde{z}_n - A_1y^*\| \\ &\quad \times \|\tilde{z}_n - v_n + (x^* - y^*)\| + \Big(M + \|Qx_n - x^*\|^2\Big)\alpha_n.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\left\|\tilde{z}_n - v_n + (x^* - y^*)\right\| = 0.$$

This together with $\|y_n - v_n\| \le \alpha_n\|Qx_n - v_n\| \to 0$, implies that

$$\lim_{n\to\infty}\left\|\tilde{z}_n - y_n + (x^* - y^*)\right\| = 0. \quad (3.16)$$

Thus, from (3.15) and (3.16) we conclude that

$$\lim_{n\to\infty}\left\|z_n - y_n\right\| = 0.$$

On the other hand, by firm nonexpansiveness of P_C, we have

$$\begin{aligned}
&\|\tilde{x}_n - \overline{y}^*\|^2 = \|P_C(x_n - \mu_2 B_2 x_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\le \langle (x_n - \mu_2 B_2 x_n) - (x^* - \mu_2 B_2 x^*), \tilde{x}_n - \overline{y}^*\rangle \\
&= \frac{1}{2}\left(\|x_n - x^* - \mu_2(B_2x_n - B_2x^*)\|^2 + \|\tilde{x}_n - \overline{y}^*\|^2 - \|(x_n - x^*) - \mu_2(B_2x_n - B_2x^*) - (\tilde{x}_n - \overline{y}^*)\|^2\right) \\
&\le \frac{1}{2}\left(\|x_n - x^*\|^2 + \|\tilde{x}_n - \overline{y}^*\|^2 - \|(x_n - \tilde{x}_n) - \mu_2(B_2x_n - B_2x^*) - (x^* - \overline{y}^*)\|^2\right) \\
&= \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|\tilde{x}_n - \overline{y}^*\|^2 - \|x_n - \tilde{x}_n - (x^* - \overline{y}^*)\|^2 \\
&\quad +2\mu_2\langle x_n - \tilde{x}_n - (x^* - \overline{y}^*), B_2x_n - B_2x^*\rangle - \mu_2^2\|B_2x_n - B_2x^*\|^2\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\|z_n - x^*\|^2 = \|P_C(\tilde{x}_n - \mu_1 B_1\tilde{x}_n) - P_C(\overline{y}^* - \mu_1 B_1\overline{y}^*)\|^2 \\
&\le \langle \tilde{x}_n - \mu_1 B_1\tilde{x}_n - (\overline{y}^* - \mu_1 B_1\overline{y}^*), z_n - x^*\rangle \\
&= \frac{1}{2}\Big(\|\tilde{x}_n - \mu_1 B_1\tilde{x}_n - (\overline{y}^* - \mu_1 B_1\overline{y}^*)\|^2 + \|z_n - x^*\|^2 \\
&\quad -\|\tilde{x}_n - \mu_1 B_1\tilde{x}_n - (\overline{y}^* - \mu_1 B_1\overline{y}^*) - (z_n - x^*)\|^2\Big) \\
&\le \frac{1}{2}\Big(\|\tilde{x}_n - \overline{y}^*\|^2 + \|z_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|^2 \\
&\quad + 2\mu_1\langle B_1\tilde{x}_n - B_1\overline{y}^*, \tilde{x}_n - z_n + (x^* - \overline{y}^*)\rangle - \mu_1^2\|B_1\tilde{x}_n - B_1\overline{y}^*\|^2 \\
&\le \frac{1}{2}\left(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|^2 + 2\mu_1\langle B_1\tilde{x}_n - B_1\overline{y}^*, \tilde{x}_n - z_n + (x^* - \overline{y}^*)\rangle\right).
\end{aligned}$$

Thus, we have

$$\|\tilde{x}_n - \overline{y}^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - \tilde{x}_n - (x^* - \overline{y}^*)\|^2 + 2\mu_2\langle x_n - \tilde{x}_n - (x^* - \overline{y}^*), B_2x_n - B_2x^*\rangle - \mu_2^2\|B_2x_n - B_2x^*\|^2, \quad (3.17)$$

and

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|^2 + 2\mu_1\|B_1\tilde{x}_n - B_1\overline{y}^*\|\,\|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|. \quad (3.18)$$

Consequently, from (3.10), (3.11), (3.12) and (3.17) we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \le \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|y_n - x^*\|^2 + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\left(\alpha_n\|Qx_n - x^*\|^2 + \|z_n - x^*\|^2\right) + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\left(\alpha_n\|Qx_n - x^*\|^2 + \|\tilde{x}_n - \overline{y}^*\|^2\right) + M\alpha_n \\
&\le \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\Big[\alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - \tilde{x}_n - (x^* - \overline{y}^*)\|^2 \\
&\quad + 2\mu_2\langle x_n - \tilde{x}_n - (x^* - \overline{y}^*), B_2x_n - B_2x^*\rangle\Big] + M\alpha_n \\
&\le \|x_n - x^*\|^2 - (1 - \beta_n)\|x_n - \tilde{x}_n - (x^* - \overline{y}^*)\|^2 \\
&\quad + 2(1 - \beta_n)\mu_n\|x_n - \tilde{x}_n - (x^* - \overline{y}^*)\|\,\|B_2x_n - B_2x^*\| + \left(M + \|Qx_n - x^*\|^2\right)\alpha_n.
\end{aligned}$$

It follows that

$$(1-\beta_n)\left\|x_n-\tilde{x}_n-\left(x^*-\overline{y}^*\right)\right\|^2 \le \left(\|x_n-x^*\|+\|x_{n+1}-x^*\|\right)\|x_{n+1}-x_n\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n + 2(1-\beta_n)\mu_2\left\|x_n-\tilde{x}_n-\left(x^*-\overline{y}^*\right)\right\|\|B_2x_n-B_2x^*\|.$$

Note that $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\|B_2x_n - B_2x^*\| \to 0$. Then we immediately deduce that

$$\lim_{n\to\infty}\left\|x_n-\tilde{x}_n-\left(x^*-\overline{y}^*\right)\right\|=0. \tag{3.19}$$

Furthermore, utilizing (3.11), (3.12) and (3.18) we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+M\alpha_n\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left(\alpha_n\|Qx_n-x^*\|^2+\|z_n-x^*\|^2\right)+M\alpha_n\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\Big[\alpha_n\|Qx_n-x^*\|^2+\|x_n-x^*\|^2\\
&\quad-\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|^2+2\mu_1\|B_1\tilde{x}_n-B_1\overline{y}^*\|\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|\Big]+M\alpha_n\\
&\le \|x_n-x^*\|^2-(1-\beta_n)\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|^2\\
&\quad+2\mu_1(1-\beta_n)\|B_1\tilde{x}_n-B_1\overline{y}^*\|\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.
\end{aligned}$$

So, we get

$$(1-\beta_n)\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|^2 \le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\mu_1(1-\beta_n)\|B_1\tilde{x}_n-B_1\overline{y}^*\| \times\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.$$

Hence,

$$\lim_{n\to\infty}\left\|\tilde{x}_n-z_n+\left(x^*-\overline{y}^*\right)\right\|=0. \tag{3.20}$$

Thus, from (3.19) and (3.20) we conclude that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Since

$$\begin{aligned}
\left\|\delta_n\left(Sy_n-x_n\right)\right\| &\le \|x_{n+1}-x_n\|+\gamma_n\|P_C(\tilde{z}_n-\lambda_1A_1\tilde{z}_n)-x_n\|\\
&\le \|x_{n+1}-x_n\|+\gamma_n\|y_n-x_n\|+\gamma_n\alpha_n\|Qx_n-P_C(\tilde{z}_n-\lambda_1A_n\tilde{z}_n)\|,
\end{aligned}$$

so we obtain that

$$\lim_{n\to\infty}\|Sy_n-x_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|Sy_n-y_n\|=0.$$

Step 4. $\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle \le 0$ where $x^* = P_\Omega Qx^*$.

Indeed, as H is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ and

$$\limsup_{n\to\infty}\langle Qx^*-x^*,x_n-x^*\rangle=\limsup_{i\to\infty}\langle Qx^*-x^*,x_{n_i}-x\rangle.$$

From Step 3 it is known that $\|x_n - y_n\| \to 0$ as $n \to \infty$. This together with $x_{n_i} \rightharpoonup v$, implies that $y_{n_i} \rightharpoonup v$. Again from Step 3 it is known that $\|y_n - Sy_n\| \to 0$ as $n \to \infty$. Thus it is clear from

Lemma 2.3 (ii) that $\upsilon \in \mathrm{Fix}(S)$. Next, we prove that $\upsilon \in \Gamma \cap \Gamma_0$. As a matter of fact, observe that

$$\begin{aligned}
&\|y_n - G(y_n)\| \\
&\le \alpha_n \|Qx_n - G(y_n)\| + (1-\alpha_n)\|P_C[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n) - G(y_n)\| \\
&= \alpha_n \|Qx_n - G(y_n)\| + (1-\alpha_n)\|G(y_n)\| \\
&\le \alpha_n \|Qx_n - G(y_n)\| + (1-\alpha_n)\|z_n - y_n\| \\
&\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\|z_n - F(z_n)\| \\
&\le \alpha_n \|Qx_n - F(z_n)\| + (1-\alpha_n)\|P_C[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n) - F(z_n)\| \\
&= \alpha_n \|Qx_n - F(z_n)\| + (1-\alpha_n)\|F(x_n) - F(z_n)\| \\
&\le \alpha_n \|Qx_n - F(z_n)\| + (1-\alpha_n)\|x_n - z_n\| \\
&\to 0,
\end{aligned}$$

where G and F are given in (1.4) and (1.7), respectively. According to Lemma 2.3 (ii) we obtain $\upsilon \in \Gamma \cap \Gamma_0$. Therefore, $\upsilon \in \Omega$. Hence, it follows from (2.3) that

$$\begin{aligned}
\limsup_{n\to\infty} \langle Qx^* - x^*, x_n - x^*\rangle &= \lim_{i\to\infty} \langle Qx^* - x^*, x_{n_i} - x^*\rangle \\
&= \langle Qx^* - x^*, \upsilon - x^*\rangle \\
&\le 0.
\end{aligned}$$

Step 5. $\lim_{n\to\infty} x_n = x^*$.

Indeed, from (3.2) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*) + \gamma_n\alpha_n(P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - Qx_n) = \|^2 \\
&\le \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\|^2 + 2\gamma_n\alpha_n\langle P_C(\tilde{z}_n - |\lambda_1 A_1 \tilde{z}_n) - Qx_n, x_{n+1} - x^*\rangle \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\tfrac{1}{1-\beta_n}[\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)]\|^2 \\
&\quad + 2\gamma_n\alpha_n\langle P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - x^*, x_{n+1} - x^*\rangle + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.21}$$

By Lemma 3.1 and (3.21), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\gamma_n\alpha_n\|P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - x^*\|\,\|x_{n+1} - x^*\| \\
&\quad + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - x^*\|^2 + 2\alpha_n\langle Qx_n - x^*, y_n - x^*\rangle\right] \\
&\quad + 2\gamma_n\alpha_n\|P_C(\tilde{z}_n - \lambda_1 A_1 \tilde{z}_n) - x^*\|\,\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|G(z_n) - G(x^*)\|^2 + 2\alpha_n\langle Qx_n - x^*, y_n - x^*\rangle\right] \\
&\quad + 2\gamma_n\alpha_n\|G(z_n) - G(x^*)\|\,\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle Qx_n - x^*, y_n - x^*\rangle\right] \\
&\quad + 2\gamma_n\alpha_n\|z_n - x^*\|\,\|x_{n+1} - x*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle.
\end{aligned}$$

From (3.10), we note that $\|z_n - x^*\| \le \|x_n - x^*\|$. Hence, according to $1 - \beta_n = \gamma_n + \delta_n$ we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n(1-\beta_n)\langle Qx_n-x^*,y_n-x^*\rangle\\
&\quad+2\gamma_n\alpha_n\|x_n-x^*\|\,\|x_{n+1}-x^*\|+2\gamma_n\alpha_n\langle x^*-Qx_n,x_{n+1}-x^*\rangle\\
&=[1-(1-\beta_n)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\gamma_n\langle Qx_n-x^*,y_n-x_{n+1}\rangle\\
&\quad+2\alpha_n\delta_n\langle Qx_n-x^*,y_n-x^*\rangle+2\alpha_n\gamma_n\|x_n-x^*\|\,\|x_{n+1}-x^*\|\\
&\le[1-(1-\beta_n)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\gamma_n\|Qx_n-x^*\|\,\|y_n-x_{n+1}\|\\
&\quad+2\alpha_n\delta_n\langle Qx_n-x^*,x_n-x^*\rangle+2\alpha_n\delta_n\langle Qx_n-x^*,y_n-x_n\rangle+2\alpha_n\gamma_n\|x_n-x^*\|\,\|x_{n+1}-x^*\|\\
&\le[1-(1-\beta_n)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\gamma_n\|Qx_n-x^*\|\,\|y_n-x_{n+1}\|\\
&\quad+2\alpha_n\delta_n\rho\|x_n-x^*\|^2+2\alpha_n\delta_n\langle Qx^*-x^*,x_n-x^*\rangle\\
&\quad+2\alpha_n\delta_n\|Qx_n-x^*\|\,\|y_n-x_n\|+2\alpha_n\gamma_n\|x_n-x^*\|\,\|x_{n+1}-x^*\|\\
&\le[1-(1-\beta_n)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\gamma_n\|Qx_n-x^*\|\,\|y_n-x_{n+1}\|\\
&\quad+2\alpha_n\delta_n\rho\|x_n-x^*\|^2+2\alpha_n\delta_n\langle Qx^*-x^*,x_n-x^*\rangle\\
&\quad+2\alpha_n\delta_n\|Qx_n-x^*\|\,\|y_n-x_n\|+\alpha_n\gamma_n\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right),
\end{aligned}
$$

that is,

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\le&\left[1-\frac{(1-2\rho)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n\right]\|x_n-x^*\|^2+\frac{[(1-2\rho)\delta_n-\gamma_n]\alpha_n}{1-\alpha_n\gamma_n}\\
&\times\left\{\frac{2\gamma_n}{(1-2\rho)\delta_n-\gamma_n}\|Qx_n-x^*\|\,\|y_n-x_{n+1}\|\right.\\
&\left.+\frac{2\delta_n}{(1-2\rho)\delta_n-\gamma_n}\|Qx_n-x^*\|\,\|y_n-x_n\|+\frac{2\delta_n}{(1-2\rho)\delta_n-\gamma_n}\langle Qx^*-x^*,x_n-x^*\rangle\right\}.
\end{aligned}
$$

Note that $\liminf_{n\to\infty}\frac{(1-2\rho)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}>0$. It follows that $\sum_{n=0}^{\infty}\frac{(1-2\rho)\delta_n-\gamma_n}{1-\alpha_n\gamma_n}\alpha_n=\infty$. It is obvious that

$$
\begin{aligned}
&\limsup_{x\to\infty}\left\{\frac{2\gamma_n}{(1-2\rho)\delta_n-\gamma_n}\|Qx_n-x^*\|\,\|y_n-x_{n+1}\|+\frac{2\delta_n}{(1-2\rho)\delta_n-\gamma_n}\|Qx_n-x^*\|\,\|y_n-x_n\|\right.\\
&\left.+\frac{2\delta_n}{(1-2\rho)\delta_n-\gamma_n}\langle Qx^*-x^*,x_n-x^*\rangle\right\}\le 0.
\end{aligned}
$$

Therefore, all conditions of Lemma 2.4 are satisfied. Consequently, in terms of Lemma 2.4 we immediately deduce that $x_n\to x^*$. This completes the proof. □

Next we present some applications of Theorem 3.1 in several special cases.

Corollary 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mapping $A_i: C\to H$ be $\widehat{\alpha}_i$-inverse strongly monotone for $i = 1, 2$. Let $S: C\to C$ be a k-strict pseudocontraction such that $\Omega := \mathrm{Fix}(S)\cap\Gamma\neq\emptyset$. Let $Q: C\to C$ be a ρ-contraction with $\rho\in\left[0,\frac{1}{2}\right)$. For given $x_0\in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ be generated iteratively by*

$$
\begin{cases}
y_n=\alpha_nQx_n+(1-\alpha_n)P_C\left[P_C(x_n-\lambda_2A_2x_n)-\lambda_1A_1P_C(x_n-\lambda_2A_2x_n)\right],\\
x_{n+1}=\beta_nx_n+\gamma_nP_C\left[P_C(x_n-\lambda_2A_2x_n)-\lambda_1A_1P_C(x_n-\lambda_2A_2x_n)\right]+\delta_nSy_n,\quad\forall n\ge 0,
\end{cases}
$$

where $\lambda_i\in(0,2\widehat{\alpha}_i)$ for $i = 1, 2$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n+\gamma_n+\delta_n=1$ and $(\gamma_n+\delta_n)k\le\gamma_n<(1-2\rho)\delta_n$ for all $n\ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *converge strongly to the same point* $x^* = P_\Omega Q x^*$, *and* (x^*, y^*) *is a solution of general system* (1.2) *of variational inequalities, where* $y^* = P_C(x^* - \lambda_2 A_2 x^*)$.

Proof. It is easy to see that if $B_i = 0$ for $i = 1, 2$, then for any given $\widehat{\beta_i} \in (0, \infty)$, B_i is $\widehat{\beta_i}$-inverse strongly monotone. In Theorem 3.1, putting $B_i = 0$ and taking $\mu_i \in (0, 2\widehat{\beta_i})$ for $i = 1, 2$ we have $\Omega := \mathrm{Fix}(S) \cap \Gamma$ n $\Gamma_0 = \mathrm{Fix}(S) \cap \Gamma$ and

$$z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right] = x_n, \quad \forall n \ge 0.$$

In this case, algorithm (3.2) reduces to the following algorithm

$$\begin{cases} y_n = \alpha_n Q x_n + (1-\alpha_n) P_C\left[P_C(x_n - \lambda_2 A_2 x_n) - \lambda_1 A_1 P_C(x_n - \lambda_2 A_2 x_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(x_n - \lambda_2 A_2 x_n) - \lambda_1 A_1 P_C(x_n - \lambda_2 A_2 x_n)\right] + \delta_n S y_n, \quad \forall n \ge 0, \end{cases}$$

Therefore, in terms of Theorem 3.1 we immediately obtain the desired result. □

Remark 3.1. Compared with Theorem YLK (i.e., [[26], Theorem 3.2]), Corollary 3.1 coincides essentially with Theorem YLK. Therefore, Theorem 3.1 includes Theorem YLK as a special case.

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Assume that for i = 1, 2, the mappings* A_i, $B_i : C \to H$ *are* $\widehat{\alpha}_i$ *-inverse strongly monotone and* $\widehat{\beta}_i$ *-inverse strongly monotone, respectively. Let* $S : C \to C$ *be a k-strict pseudocontraction such that* $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap \Gamma_0 \ne \emptyset$. *For fixed* $u \in C$ *and given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{cases} z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ y_n = \alpha_n u + (1-\alpha_n) P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right] + \delta_n S y_n, \quad \forall n \ge 0, \end{cases}$$

where $\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|^2 + 2\mu_1 \|B_1 \tilde{x}_n - B_1 \overline{y}^*\| \|\tilde{x}_n - z_n + (x^* - \overline{y}^*)\|$.
and $\mu_i \in (0, 2\widehat{\beta}_i)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *are four sequences in* [0, 1] *such that:*

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n < \delta_n$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ *converge strongly to the same point* $x^* = P_\Omega u$, *and* (x^*, y^*) *and* $(x^*, \overline{y}^*)$ *are a solution of general system* (1.2) *of variational inequalities and a solution of general system* (1.6) *of variational inequalities, respectively, where* $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ *and* $\overline{y}^* = P_C(x^* - \mu_2 B_2 x^*)$.

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Assume that for $i = 1, 2$, the mappings A_i, $B_i : C \to H$ are $\widehat{\alpha}_i$ -inverse strongly monotone and $\widehat{\beta}_i$ -inverse strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap \Gamma_0 \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in \left[0, \frac{1}{2}\right)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right] + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\mu_i \in (0, 2\widehat{\beta}_i)$ for $i = 1, 2$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right) = 0.$

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to the same point $x^* = P_\Omega Q x^*$, and (x^*, y^*) and $(x^*, \bar{y}^*)$ are a solution of general system (1.2) *of variational inequalities and a solution of general system* (1.6) *of variational inequalities, respectively, where $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ and $\bar{y}^* = P_C(x^* - \mu_2 B_2 x^*)$.*

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Assume that for $i = 1, 2$, the mappings A_i, $B_i : C \to H$ are $\widehat{\alpha}_i$ -inverse strongly monotone and $\widehat{\beta}_i$ -inverse strongly monotone, respectively. Let $S : C \to C$ be a nonexpansive mapping such that $\Omega := \mathrm{Fix}(S) \cap \Gamma \cap \Gamma_0 \neq \emptyset$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{cases} z_n = P_C\left[P_C(x_n - \mu_2 B_2 x_n) - \mu_1 B_1 P_C(x_n - \mu_2 B_2 x_n)\right], \\ y_n = \alpha_n u + (1 - \alpha_n) P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n P_C\left[P_C(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 P_C(z_n - \lambda_2 A_2 z_n)\right] + \delta_n S y_n, \quad \forall n \geq 0, \end{cases}$$

where $\lambda_i \in (0, 2\widehat{\alpha}_i)$ and $\mu_i \in (0, 2\widehat{\beta}_i)$ for $i = 1, 2$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that:

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < \delta_n$ *for all* $n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \gamma_n > 0$;
(iv) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\right) = 0.$

Then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to the same point $x^* = P_\Omega u$, and (x^*, y^*) and $(x^*, \bar{y}^*)$ are a solution of general system (1.2) *of variational inequalities and* *a solution of general system* (1.6) *of variational inequalities, respectively, where* $y^* = P_C(x^* - \lambda_2 A_2 x^*)$ *and* $\overline{y}^* = P_C(x^* - \mu_2 B_2 x^*)$.

Acknowledgements
In this research, the first author was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), Leading Academic Discipline Project of Shanghai Normal University (DZL707), Ph.D. Program Foundation of Ministry of Education of China (20070270004), Science and Technology Commission of Shanghai Municipality Grant (075105118), and Shanghai Leading Academic Discipline Project (S30405). The second author was partially supported by the NSC100-2115-M-033-001. The third author gratefully acknowledge the financial support from the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 16 August 2011 Accepted: 16 April 2012 Published: 16 April 2012

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doi:10.1186/1029-242X-2012-88
Cite this article as: Ceng *et al.*: **Generalized extragradient iterative method for systems of variational inequalities.** *Journal of Inequalities and Applications* 2012 **2012**:88.