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Some properties of meromorphically multivalent functions

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Abstract

By using the method of differential subordinations, we derive certain properties of meromorphically multivalent functions.

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1 Introduction

Let $\Sigma(p)$ denotes the class of meromorphically multivalent functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad \left(p \in N = \{1, 2, 3, \ldots\}\right), \tag{1.1}$$

which are analytic in the punctured unit disk

$$U^* = \left\{z : z \in C \text{ and } 0 < |z| < 1\right\} = U \setminus \{0\}.$$

Let $f(z)$ and $g(z)$ be analytic in U. Then, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U, such that $|w(z)| \le |z|$ and $f(z) = g(w(z))$ $(z \in U)$. If $g(z)$ is univalent in U, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p(z) = 1 + p_1 z + \ldots$ be analytic in U. Then for $-1 \le B < A \le 1$, it is clear that

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in U) \tag{1.2}$$

if and only if

$$\left|p(z) - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2} \quad (-1 < B < A \le 1; z \in U) \tag{1.3}$$

and

$$\text{Re}\, p(z) > \frac{1 - A}{2} \quad (B = -1; z \in U). \tag{1.4}$$

Recently, several authors (see, e.g., [1-7]) considered some interesting properties of meromorphically multivalent functions. In the present article, we aim at proving some subordination properties for the class $\Sigma(p)$.

To derive our results, we need the following lemmas.

Lemma 1 (see [8]. Let $h(z)$ be analytic and starlike univalent in U with $h(0) = 0$. If $g(z)$ is analytic in U and $zg'(z) \prec h(z)$, then

$$g(z) \prec g(0) + \int_0^z \frac{h(t)}{t} dt.$$

Lemma 2 (see [9]. Let $p(z)$ be analytic and nonconstant in U with $p(0) = 1$. If $0 < |z_0| < 1$ and $\mathrm{Re}\, p(z_0) = \min_{|z| \leq |z_0|} \mathrm{Re}\, p(z)$, then

$$z_0 p'(z_0) \leq -\frac{|1 - p(z_0)|^2}{2\left(1 - \mathrm{Re}\, p(z_0)\right)}.$$

2 Main results

Our first result is contained in the following.

Theorem 1. Let $\alpha \in (0, \frac{1}{2}]$ and $\beta \in (0,1)$. If $f(z) \in \Sigma(p)$ satisfies $f(z) \neq 0$ $(z \in U^*)$ and

$$\left| \frac{z^{-p}}{f(z)} \left(\frac{zf'(z)}{f(z)} + p \right) \right| < \delta \quad (z \in U), \tag{2.1}$$

where δ is the minimum positive root of the equation

$$\alpha \sin\left(\frac{\pi\beta}{2}\right) x^2 - x + (1 - \alpha) \sin\left(\frac{\pi\beta}{2}\right) = 0, \tag{2.2}$$

then

$$\left| \arg\left(\frac{f(z)}{z^{-p}} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in U). \tag{2.3}$$

The bound β is the best possible for each $\alpha \in (0, \frac{1}{2}]$.

Proof. Let

$$g(x) = \alpha \sin\left(\frac{\pi\beta}{2}\right) x^2 - x + (1 - \alpha) \sin\left(\frac{\pi\beta}{2}\right). \tag{2.4}$$

We can see that the Equation (2.2) has two positive roots. Since $g(0) > 0$ and $g(1) < 0$, we have

$$0 < \frac{\alpha}{1 - \alpha}\delta \leq \delta < 1. \tag{2.5}$$

Put

$$\frac{f(z)}{z^{-p}} = \alpha + (1 - \alpha) p(z). \tag{2.6}$$

Then from the assumption of the theorem, we see that $p(z)$ is analytic in U with $p(0) = 1$ and $\alpha + (1 - \alpha)p(z) \neq 0$ for all $z \in U$. Taking the logarithmic differentiations in both sides of (2.6), we get

$$\frac{zf'(z)}{f(z)} + p = \frac{(1 - \alpha) zp'(z)}{\alpha + (1 - \alpha) p(z)} \tag{2.7}$$

and

$$\frac{z^{-p}}{f(z)}\left(\frac{zf'(z)}{f(z)}+p\right)=\frac{(1-\alpha)\,zp'(z)}{\left(\alpha+(1-\alpha)\,p(z)\right)^2} \tag{2.8}$$

for all $z \in U$. Thus the inequality (2.1) is equivalent to

$$\frac{(1-\alpha)\,zp'(z)}{\left(\alpha+(1-\alpha)\,p(z)\right)^2} \prec \delta z. \tag{2.9}$$

By using Lemma 1, (2.9) leads to

$$\int_0^z \frac{(1-\alpha)\,p'(t)}{\left(\alpha+(1-\alpha)\,p(t)\right)^2}dt \prec \delta z$$

or to

$$1-\frac{1}{\alpha+(1-a)\,p(z)} \prec \delta z. \tag{2.10}$$

In view of (2.5), (2.10) can be written as

$$p(z) \prec \frac{1+\frac{\alpha}{1-\alpha}\delta z}{1-\delta z}. \tag{2.11}$$

Now by taking $A = \frac{\alpha}{1-\alpha}\delta$ and $B = -\delta$ in (1.2) and (1.3), we have

$$\begin{aligned}\left|\arg\left(\frac{f(z)}{z^{-p}}-\alpha\right)\right| &= \left|\arg p(z)\right| \\ &< \arcsin\left(\frac{\delta}{1-\alpha+\alpha\delta^2}\right) \\ &= \frac{\pi}{2}\beta\end{aligned}$$

for all $z \in U$ because of $g(\delta) = 0$. This proves (2.3).

Next, we consider the function $f(z)$ defined by

$$f(z)=\frac{z^{-p}}{1-\delta_z}\quad\left(z\in U^*\right).$$

It is easy to see that

$$\left|\frac{z^{-p}}{f(z)}\left(\frac{zf'(z)}{f(z)}+p\right)\right| = |\delta z| < \delta \quad (z \in U).$$

Since

$$\frac{f(z)}{z^{-p}}-\alpha=(1-\alpha)\frac{1+\frac{\alpha}{1-\alpha}\delta z}{1-\delta z},$$

it follows from (1.3) that

$$\sup_{z\in U}\left|\arg\left(\frac{f(z)}{z^{-p}}-\alpha\right)\right| = \arcsin\left(\frac{\delta}{1-\alpha+\alpha\delta^2}\right)=\frac{\pi}{2}\beta.$$

Hence, we conclude that the bound β is the best possible for each $\alpha \in (0, \frac{1}{2}]$.

Next, we derive the following.

Theorem 2. If $f(z) \in \Sigma(p)$ satisfies $f(z) \neq 0$ ($z \in U^*$) and

$$\operatorname{Re}\left\{\frac{z^{-p}}{f(z)}\left(\frac{zf'(z)}{f(z)}+p\right)\right\}<\gamma \quad (z \in U), \tag{2.12}$$

where

$$0<\gamma<\frac{1}{2\log 2}, \tag{2.13}$$

then

$$\operatorname{Re}\frac{z^{-p}}{f(z)}>1-2\gamma\log 2 \quad (z \in U). \tag{2.14}$$

The bound in (2.14) is sharp.

Proof. Let

$$p(z)=\frac{f(z)}{z^{-p}}. \tag{2.15}$$

Then $p(z)$ is analytic in U with $p(0) = 1$ and $p(z) \neq 0$ for $z \in U$. In view of (2.15) and (2.12), we have

$$1-\frac{zp'(z)}{\gamma p^2(z)} \prec \frac{1+z}{1-z},$$

i.e.,

$$z\left(\frac{1}{p(z)}\right)' \prec \frac{2\gamma z}{1-z}.$$

Now by using Lemma 1, we obtain

$$\frac{1}{p(z)} \prec 1-2\gamma\log(1-z). \tag{2.16}$$

Since the function $1 - 2\gamma\log(1 - z)$ is convex univalent in U and

$$\operatorname{Re}\left(1-2\gamma\log(1-z)\right)>1-2\gamma\log 2 \quad (z \in U),$$

from (2.16), we get the inequality (2.14).

To show that the bound in (2.14) cannot be increased, we consider

$$f(z)=\frac{z^{-p}}{1-2\gamma\log(1-z)} \quad \left(z \in U^*\right).$$

It is easy to verify that the function $f(z)$ satisfies the inequality (2.12). On the other hand, we have

$$\operatorname{Re}\frac{z^{-p}}{f(z)} \to 1-2\gamma\log 2$$

as $z \to -1$. Now the proof of the theorem is complete.

Finally, we discuss the following theorem.

Theorem 3. Let $f(z) \in \Sigma(p)$ with $f(z) \neq 0$ $(z \in U^*)$. If

$$\left|\mathrm{Im}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z^{-p}}-\lambda\right)\right\}\right| < \sqrt{\lambda\left(\lambda+2p\right)} \quad (z \in U) \tag{2.17}$$

for some $\lambda(\lambda > 0)$, then

$$\mathrm{Re}\,\frac{f(z)}{z^{-p}} > 0 \quad (z \in U). \tag{2.18}$$

Proof. Let us define the analytic function $p(z)$ in U by

$$\frac{f(z)}{z^{-p}} = p(z).$$

Then $p(0) = 1$, $p(z) \neq 0$ $(z \in U)$ and

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z^{-p}}-\lambda\right) = \left(p(z)-\lambda\right)\left(\frac{zp'(z)}{p(z)}-p\right) \quad (z \in U). \tag{2.19}$$

Suppose that there exists a point $z_0(0 < | z_0 | < 1)$ such that

$$\mathrm{Re}\,p(z) > 0 \ \ (|z| < |z_0|) \quad \text{and} \quad p(z_0) = i\beta, \tag{2.20}$$

where β is real and $\beta \neq 0$. Then, applying Lemma 2, we get

$$z_0p'(z_0) \leq -\frac{1+\beta^2}{2}. \tag{2.21}$$

Thus it follows from (2.19), (2.20), and (2.21) that

$$I_0 = \mathrm{Im}\left\{\frac{z_0f'(z_0)}{f(z_0)}\left(\frac{f(z_0)}{z_0^{-p}}-\lambda\right)\right\} = -p\beta + \frac{\lambda}{\beta}z_0p'(z_0). \tag{2.22}$$

In view of $\lambda > 0$, from (2.21) and (2.22) we obtain

$$I_0 \geq -\frac{\lambda+\left(\lambda+2p\right)\beta^2}{2\beta} \geq \sqrt{\lambda\left(\lambda+2p\right)} \quad (\beta < 0) \tag{2.23}$$

and

$$I_0 \leq -\frac{\lambda+\left(\lambda+2p\right)\beta^2}{2\beta} \leq -\sqrt{\lambda\left(\lambda+2p\right)} \ \ (\beta > 0). \tag{2.24}$$

But both (2.23) and (2.24) contradict the assumption (2.17). Therefore, we have Re$p(z) > 0$ for all $z \in U$. This shows that (2.18) holds true.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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