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Iterated commutators of multilinear fractional operators with rough kernels

Zengyan Si[1,2*] and Yanlong Shi[3]

* Correspondence: sizengyan@yahoo.cn
[1]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China
Full list of author information is available at the end of the article

Abstract

Let $\Omega \in L^s(s^{mn-1})$ for some $s > 1$ be a homogeneous function of degree zero on R^{mn}. We obtain the iterated commutator $I_{\prod \vec{b},\Omega,\alpha}$ of multilinear fractional operator is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p and also is bounded from $L^{p_1}(u_1^{p_1}) \times \cdots \times L^{p_m}(u_m^{p_m})$ to $L^p(v^p)$, when $\vec{b} \in BMO^m$ and $\vec{b} \in BMO^m(v)$, respectively. Similarly results still hold for its corresponding maximal operator $\mathcal{M}_{\prod \vec{b},\Omega,\alpha}$.

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1. Introduction

Let $0 < \alpha < n$, the classical fractional integral operator (or the Riesz potential) T_α is defined by

$$T_\alpha f(x) = \int_{R^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy,$$

which plays important roles in many fields such as PDE and so on. For the classical results for T_α please see [1-3], and see [4,5] for T_α with rough kernels. When $b \lfloor BMO$, Chanillo [6] proved the commutator $T_{\alpha, b}$ is bounded from $L^p(R^n)$ into $L^q(R^n)(p > 1, 1/q = 1/p - \alpha/n > 0)$, where $T_{\alpha, b}f(x) = b(x)T_\alpha f(x) - T_\alpha(bf)(x)$.

The study of multilinear singular integral operator has recently received increasing attention. It is not only motivated by a mere quest to generalize the theory of linear operators but rather by their natural appearance in analysis. In recent years, the study of these operators has made significant advances, many results obtained parallel to the linear theory of classical Calderón-Zygmund operators. As one of the most important operators, the multilinear fractional type operator has also been attracted more attentions. In 1999, Kenig and Stein [7] studied the following multilinear fractional operator I_α, $0 < \alpha < mn$,

$$I_\alpha(\vec{f})(x) = \int_{(R^n)^m} \frac{f_1(y_1)f_2(y_2)\cdots f_m(y_m)}{|(x-y_1, x-y_2, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y},$$

where, and throughout this article, we denote by $\vec{y} = (y_1, \ldots, y_m)$, $d\vec{y} = dy_1, \ldots, dy_m$, and $\vec{f} = (f_1, f_2, \ldots, f_m)$, m, n the nonnegative integers with $m \geq 1$ and $n \geq 2$.

Theorem 1.1. [7]*Let* $m \in N$,

$$\frac{1}{s} = \frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_m} - \frac{\alpha}{n} > 0,$$

with $0 < \alpha < mn$, $1 \le r_i \le \infty$, *then*

(a) If each$r$ $r_i > 1$,

$$\left\| I_\alpha(\vec{f}) \right\|_{L^s(R^n)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{r_i}(R^n)};$$

(b) If $r_i = 1$ *for some* i,

$$\left\| I_\alpha(\vec{f}) \right\|_{L^{s,\infty}(R^n)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{r_i}(R^n)}.$$

Obviously, in the case $m = 1$, I_α is the classical fractional integral operator T_α. The inequality is the multi-version of the well-known Hardy-Littlewood-Sobolev inequality for T_α, i.e., $\|T_\alpha f_1\|_{L^s(R^n)} \le C\|f_1\|_{L^{r_1}(R^n)}$, where $\frac{1}{s} = \frac{1}{r_1} - \frac{\alpha}{n} > 0$ and $r_1 > 1$.

We say a locally integrable nonnegative function w on R^n belongs to $A(p, q)(1 < p, q < \infty)$ if

$$\sup_{Q \subset R^n} \left(\frac{1}{|Q|} \int_Q w(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)^{-p'} dx \right)^{1/p'} < \infty,$$

where Q denotes a cube in R^n with the sides parallel to the coordinate axes and the supremum is taken over all cubes, $p' = \frac{p}{p-1}$ be the conjugate index of p.

Muckenhoupt and Wheede [2] showed that $\|T_\alpha f\|_{L^q(w^q)} \le C\|f\|_{L^p(w^p)}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$ and $w \in A(p, q)$.

García-Cuerva and Martell [8] proved that, when $0 < \alpha < n$, $1 < p < q < \infty$,

$$\left(\int_{R^n} |T_\alpha f(x)|^q u(x) dx \right)^{1/q} \le C \left(\int_{R^n} |f(x)|^p v(x) dx \right)^{1/p},$$

hold for weights (u, v), if there exists $r > 1$ such that for each cube Q in R^n,

$$|Q|^{\frac{1}{q} + \frac{\alpha}{n} - \frac{1}{p}} \left(\frac{1}{|Q|} \int_Q u(x)^r dx \right)^{\frac{1}{rq}} \left(\frac{1}{|Q|} \int_Q v(x)^{r(1-p')} dx \right)^{\frac{1}{rp'}} \le C.$$

Motivated by this observation, Shi and Tao [9] pursued the results bellow parallel to the above two estimates.

Theorm 1.2. [9]*Let* $0 < \alpha < mn$, *suppose that* $f_i \in L^p(w^{p_i})$ *with* $1 < p_i < mn/\alpha (i = 1, 2, \ldots, m)$ *and* $w(x) \in \bigcap_{i=1}^m A(p_i, q_i)$, *where* $1/q_i = 1/p_i - \alpha/mn$. *If let*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n}$$

then there is a constant $C > 0$ *independent of* f_i *such that*

$$\left\| I_\alpha(\vec{f}) \right\|_{L^p(w^p)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w^{p_i})},$$

for $f_i \in \mathscr{S}(R^n)$, $i = 1, \ldots, m$.

Theorem 1.3. [9]*Let* $0 < \alpha < mn$, (u, v) *is a pair of weights. If for every* $i = 1, 2, \ldots, m$, $1 < p_i < mp < \infty$, *there exist* $r_i > 1$ *such that for every cube* Q *in* R^n,

$$|Q|^{\frac{1}{mp} + \frac{\alpha}{mn} - \frac{1}{p_i}} \left(\frac{1}{|Q|} \int_Q u(x)^{r_i} dx \right)^{\frac{1}{r_i mp}} \left(\frac{1}{|Q|} \int_Q v(x)^{r_i(1-p'_i)} dx \right)^{\frac{1}{r_i p'_i}} \leq C,$$

then for every $f_i \in L^{p_i}(v)$, *there is a constant* $C > 0$ *independent of* f_i *such that*

$$\left\| I_\alpha(\vec{f}) \right\|_{L^p(u)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(v)}.$$

Before stating our main results, let's recall some definitions. For $0 < \alpha < n$, suppose Ω is homogeneous of degree zero on R^n and $\Omega \in L^s(S^{n-1})(s > 1)$, where S^{n-1} denotes the unit sphere in R^n. Then the fractional operator $T_{\Omega,\alpha}$ and its corresponding maximal operator $M_{\Omega,\alpha}$ can be defined, respectively, by

$$T_{\Omega,\alpha} f(x) = \int_{R^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f(x-y) dy,$$

$$M_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)| \, |f(x-y)| \, dy.$$

The higher order commutators associated with $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are defined as

$$T^m_{\Omega,\alpha,b} f(x) = \int_{R^n} \frac{\Omega(y)}{|y|^{n-\alpha}} (b(x) - b(x-y))^m f(x-y) dy,$$

$$M^m_{\Omega,\alpha,b} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)| \, |b(x) - b(x-y)|^m \, |f(x-y)| \, dy.$$

For v a nonnegative locally integrable function on R^n, a function b is said to belong to $BMO(v)$, if there is a constant $C > 0$ such that

$$\int_Q |b(x) - b_Q| \, dx \leq C \int_Q v(x) dx,$$

hold for any cube Q in R^n with its sides parallel to the coordinate axes, where $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$.

When $b(x) \in BMO(v)$, Ding and Lu [10] studied the (L^p (u^p), L^q (v^q)) boundedness of the higher order commutators $T^m_{\Omega,\alpha,b}$ and $M^m_{\Omega,\alpha,b}$.

Segovia and Torrea [11] gave the weighted boundedness of higher order commutator for vector-valued integral operators with a pair of weights using the Rubio de Francia extrapolation idea for weighted norm inequalities. As an application of this result, they obtained (L^p (u^p), L^q (v^q)) boundedness for $M^m_{1,\alpha,b}$.

Theorem 1.4. [11]*Suppose that* $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$. *Then for* $b \in BMO(v)$, $u(x)$, $v(x) \in A(p, q)$ *and* $u(x)v(x)^{-1} = v^m$, *there is a constant* $C > 0$, *independent of* f, *such that* $M^m_{1,\alpha,b}$ *satisfies*

$$\left(\int_{R^n} [M^m_{1,\alpha,b}f(x)v(x)]^q dx\right)^{1/q} \le C\left(\int_{R^n} |f(x)u(x)|^p dx\right)^{1/p}.$$

Let $s > 1$, $\Omega \in L^s(S^{mn-1})$ be a homogeneous function of degree zero on R^{mn}. Assume that $\vec{b} = (b_1, \ldots, b_m)$ is a collection of locally integrable functions. In this article, we study the iterated commutator of multilinear fractional integral operator and its corresponding maximal operator defined by

$$I_{\prod\vec{b},\Omega,\alpha}(\vec{f})(x) = \int_{(R^n)^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m (b_i(x) - b_i(x-y_i))f_i(x-y_i)d\vec{y};$$

$$\mathcal{M}_{\prod\vec{b},\Omega,\alpha}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} |\Omega(\vec{y})| \prod_{i=1}^m |b_i(x) - b_i(x-y_i)|\,|f_i(x-y_i)|\,d\vec{y}.$$

Remark 1.5. If $m = 1$, $I_{\prod\vec{b},\Omega,\alpha}$ is the homogeneous fractional commutator $T_{\Omega,\alpha,b}$; If $m = 1$ and $\Omega \equiv 1$, $I_{\prod\vec{b},\Omega,\alpha}$ is the classical fractional commutator for T_α.

Inspired by the above results, one may naturally ask the following questions: Whether the conclusions in [6] can be extended to $I_{\prod\vec{b},\Omega,\alpha}$ and $\mathcal{M}_{\prod\vec{b},\Omega,\alpha}$. Can we obtain similar results as in [10] for the iterated commutators $I_{\prod\vec{b},\Omega,\alpha}$ and $\mathcal{M}_{\prod\vec{b},\Omega,\alpha}$.

The following theorems will give positive answers to the above questions.

Theorem 1.6. *Let* $0 < \alpha < mn, 1 \le s' < p_i < \frac{mn}{\alpha}$ *with* $s' \in \mathbb{N}$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$.

Then for functions $\vec{b} \in BMO^m$, *we have* *(i)* $\mathcal{M}_{\prod\vec{b},\Omega,\alpha}$ *is bounded from* $L^{p_1}(R^n)\times\cdots\times L^{p_m}(R^n)$ *to* $L^p(R^n)$*, that is*

$$\left\|\mathcal{M}_{\prod\vec{b},\Omega,\alpha}(\vec{f})\right\|_{L^p(R_n)}\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(R^n)},$$

(ii) $I_{\prod\vec{b},\Omega,\alpha}$ *is bounded from* $L^{p_1}(R^n)\times\cdots\times L^{p_m}(R^n)$ *to* $L^p(R^n)$*, that is*

$$\left\|I_{\prod\vec{b},\Omega,\alpha}(\vec{f})\right\|_{L^p(R^n)}\le C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(R^n)}.$$

where C is a positive constant independent of f_i *, for* $i = 1,..., m$.

Theorem 1.7. *Let* $0<\alpha<mn, 1\le s'<p_i<\dfrac{mn}{\alpha}$ *with* $s'\in\mathbb{N}$, $v(x)=\prod_{i=1}^{m}v_i(x)$ *and* $v(x)=\prod_{i=1}^{m}v_i(x)$ *with* $u_i(x)^{s'}, v_i(x)^{s'}\in\bigcap_{i=1}^{m}A(p_i/s', q_i/s')$ *and* $u_i(x)^{s'}\left(v_i(x)^{s'}\right)^{-1}=v$. *Then for functions* $\vec{b}\in BMO^m(v)$, *we have*

(i) $\mathcal{M}_{\prod\vec{b},\Omega,\alpha}$ *is bounded from* $L^{p_1}\left(u_1^{p_1}\right)(R^n)\times\cdots\times L^{p_m}\left(u_m^{p_m}\right)(R^n)$ *to* $L^p(v^p)(R^n)$*, that is*

$$\left(\int_{R^n}\left[\mathcal{M}_{\prod\vec{b},\Omega,\alpha}\vec{f}(x)v(x)\right]^p dx\right)^{1/p}\le C\prod_{i=1}^{m}\left(\int_{R^n}\left|f_i(x)u_i(x)\right|^{p_i}dx\right)^{1/p_i},$$

(ii) $I_{\prod\vec{b},\Omega,\alpha}$ *is bounded from* $L^{p_1}\left(u_1^{p_1}\right)(R^n)\times\cdots\times L^{p_m}\left(u_m^{p_m}\right)(R^n)$ *to* $L^p(v^p)(R^n)$*, that is*

$$\left(\int_{R^n}\left[I_{\prod\vec{b},\Omega,\alpha}\vec{f}(x)v(x)\right]^p dx\right)^{1/p}\le C\prod_{i=1}^{m}\left(\int_{R^n}\left|f_i(x)u_i(x)\right|^{p_i}dx\right)^{1/p_i}.$$

where C is a positive constant independent of f_i *, for* $i = 1,..., m$.

Remark 1.8. Theorem 1.6 extend some of the result in [6] significantly. Theorem 1.7 is the multi-version of Theorems 1 and 3 in [10].

Throughout this article, the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of all essential variables.

2. Proof of the main results

To prove Theorems 1.6 and 1.7, we need the following lemmas.

Lemma 2.1. *Let* $0<\alpha<mn, 1\le s'<\dfrac{mn}{\alpha}$, *assume that the function* $f_i\in L^{p_i}(R^n)$ *with* $1\le p_i<\infty (i = 1, 2,..., m)$, *then there exists a constant* $C > 0$ *such that for any* $x\in R^n$,

$$\mathcal{M}_{\prod\vec{b},\Omega,\alpha}(\vec{f})(x)\le C\prod_{i=1}^{m}\left[M^{s'}_{1,\frac{\alpha s'}{m},b_i}f_i^{s'}\right]^{\frac{1}{s'}}(x).$$

Proof. Since $\Omega \in L^s(S^{mn-1})$, by Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|\, |f_i(x-y_i)|\, d\vec{y} \\
&\le \frac{1}{r^{mn-\alpha}} \left(\int_{|\vec{y}|<r} |\Omega(\vec{y})|^s d\vec{y} \right)^{\frac{1}{s}} \left(\int_{|\vec{y}|<r} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le C \sup_{r>0} \frac{1}{r^{mn(1-\frac{1}{s})-\alpha}} \left(\int_{|\vec{y}|<r} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le C \sup_{r>0} \left(\frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}|<r} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le C \left(\sup_{r>0} \frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}|<r} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le C \left(\sup_{r>0} \frac{1}{r^{mn-\alpha s'}} \int_{|y_1|<r} \cdots \int_{|y_m|<r} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} d\vec{y} \right)^{\frac{1}{s'}} \\
&\le C \prod_{i=1}^{m} \left(\sup_{r>0} \frac{1}{r^{n-\alpha s'/m}} \int_{|y_i|<r} |b_i(x) - b_i(x-y_i)|^{s'} |f_i(x-y_i)|^{s'} dy_i \right)^{\frac{1}{s'}} \\
&= C \prod_{i=1}^{m} \left[M^{s'}_{1,\frac{\alpha s'}{m},b_i} f_i^{s'} \right]^{\frac{1}{s'}} (x).
\end{aligned}$$

This completes our proof. □

Lemma 2.2. *Let* $0 < \alpha < mn, f_i \in L^{p_i}(R^n)$ *for* $1 < p_i < \infty (i = 1, 2,..., m)$. *For any* $0 <\in <\min\{\alpha, mn - \alpha\}$, *there exists a constant* $C > 0$ *such that for any* $x \in R^n$,

$$\left| I_{\prod \vec{b},\Omega,\alpha}(\vec{f})(x) \right| \le C \left[\mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x) \right]^{\frac{1}{2}} \left[\mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x) \right]^{\frac{1}{2}}$$

Proof. Fix $x \in R^n$ and $0 < \in <\min\{\alpha, mn - \alpha\}$, for any $\delta > 0$ we have

$$\begin{aligned}
\left| I_{\prod \vec{b},\Omega,\alpha} \vec{f}(x) \right| &\le \int_{(R^n)^m} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|\, |f_i(x-y_i)|\, d\vec{y} \\
&\le \int_{|\vec{y}|\le\delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|\, |f_i(x-y_i)|\, d\vec{y} \\
&\quad + \int_{|\vec{y}|>\delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)|\, |f_i(x-y_i)|\, d\vec{y} \\
&= I + II.
\end{aligned}$$

For I, we have

$$\begin{aligned}
I &= \sum_{j=0}^{\infty} \int_{B(2^{-j}\delta)\backslash B(2^{-j-1}\delta)} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^{-j-1}\delta)^{mn-\alpha}} \int_{B(2^{-j}\delta)\backslash B(2^{-j-1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\delta)^{mn-\alpha}} \int_{B(2^{-j}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{(2^{-j}\delta)^{\varepsilon}}{(2^{-j}\delta)^{mn-\alpha+\varepsilon}} \int_{B(2^{-j}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le C\delta^{\varepsilon} \sum_{j=0}^{\infty} (2^{-j\varepsilon}) \, \mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x) \\
&\le C\delta^{\varepsilon} \mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x).
\end{aligned}$$

For II, we have

$$\begin{aligned}
II &= \sum_{j=0}^{\infty} \int_{B(2^{j+1}\delta)\backslash B(2^{j}\delta)} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^{j}\delta)^{mn-\alpha}} \int_{B(2^{j+1}\delta)\backslash B(2^{j}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^{j}\delta)^{mn-\alpha}} \int_{B(2^{j+1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le \sum_{j=0}^{\infty} \frac{(2^{j}\delta)^{-\varepsilon}}{(2^{j}\delta)^{mn-\alpha-\varepsilon}} \int_{B(2^{j+1}\delta)} |\Omega(\vec{y})| \prod_{i=1}^{m} |b_i(x) - b_i(x-y_i)| \, |f_i(x-y_i)| \, d\vec{y} \\
&\le C\delta^{-\varepsilon} \sum_{j=0}^{\infty} (2^{-j\varepsilon}) \, \mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x) \\
&\le C\delta^{-\varepsilon} \mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x).
\end{aligned}$$

So we get

$$\left| I_{\prod \vec{b},\Omega,\alpha}(\vec{f})(x) \right| \le C\delta^{-\varepsilon} \mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x) + C\delta^{\varepsilon} \mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x).$$

Now, we choose δ, such that

$$\delta^{2\varepsilon} = \frac{\mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x)}{\mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x)}.$$

This implies Lemma 2.2. □

Now let's prove Theorem 1.6.

Proof. We prove conclusion (i) first. Since each $p_i > s'$, by Theorem 1.4, Hölder's inequality and Lemma 2.1, we have

$$
\begin{aligned}
\left\|\mathcal{M}_{\prod \vec{b},\Omega,\alpha}(\vec{f})\right\|_{L^p(R^n)} &= \left(\int_{R^n} \left|\mathcal{M}_{\prod \vec{b},\Omega,\alpha}(\vec{f})(x)\right|^p dx\right)^{\frac{1}{p}} \\
&\le C\left(\int_{R^n} \left|\prod_{i=1}^m \left[M^{s'}_{1,\frac{\alpha s'}{m},b_i} f_i^{s'}\right]^{\frac{1}{s'}}\right|^p dx\right)^{\frac{1}{p}} \\
&\le C\prod_{i=1}^m \left(\int_{R^n} \left|M^{s'}_{1,\frac{\alpha s'}{m},b_i} f_i^{s'}\right|^{\frac{q_i}{s'}} dx\right)^{\frac{1}{q_i}} \\
&\le C\prod_{i=1}^m \left\|f_i^{s'}\right\|^{1/s'}_{L^{p_i/s'}(R^n)} \\
&= C\prod_{i=1}^m \|f_i\|_{L^{p_i}(R^n)},
\end{aligned}
$$

where $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha}{mn}$.

To prove (ii), we choose a small positive number ∊ with $0 < \varepsilon < \min\left\{\alpha, \frac{mn}{s'} - \alpha, \frac{n}{p}\right\}$. One can then see from the condition of Theorem 1.6 that $1 \le s' < p_i < \frac{mn}{\alpha + \varepsilon}$ and $1 \le s' < p_i < \frac{mn}{\alpha - \varepsilon}$, and let

$$
\begin{aligned}
\frac{1}{q_1} &= \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha + \varepsilon}{n} = \frac{1}{p} - \frac{\varepsilon}{n} > 0, \\
\frac{1}{q_2} &= \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha - \varepsilon}{n} = \frac{1}{p} + \frac{\varepsilon}{n} > 0.
\end{aligned}
$$

Now if each $p_i > s'$, then conclusion (i) implies that

$$
\begin{aligned}
\left\|\mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})\right\|_{L^{q_1}(R^n)} &\le \|f_i\|_{L^{p_i}(R^n)}, \\
\left\|\mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})\right\|_{L^{q_2}(R^n)} &\le \|f_i\|_{L^{p_i}(R^n)}.
\end{aligned}
$$

Noting that $\frac{p}{2q_1} + \frac{p}{2q_2} = 1$. Using Lemma 2.2, Hölder's inequality and the above inequalities, we have

$$\begin{aligned}
&\left\|I_{\prod\vec{b},\Omega,\alpha}(\vec{f})\right\|_{L^p(R^n)}\\
&= \left(\int_{R^n}\left|I_{\prod\vec{b},\Omega,\alpha}(\vec{f})\right|^p dx\right)^{\frac{1}{p}}\\
&\leq C\left(\int_{R^n}\left[\mathcal{M}_{\prod\vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x)\right]^{\frac{p}{2}}\left[\mathcal{M}_{\prod\vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x)\right]^{\frac{p}{2}}dx\right)^{\frac{1}{p}}\\
&\leq C\left(\int_{R^n}\left[\mathcal{M}_{\prod\vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x)\right]^{q_1}dx\right)^{\frac{1}{2q_1}}\left(\int_{R^n}\left[\mathcal{M}_{\prod\vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x)\right]^{q_2}dx\right)^{\frac{1}{2q_2}}\\
&\leq C\left\|\mathcal{M}_{\prod\vec{b},\Omega,\alpha+\varepsilon}(\vec{f})\right\|^{1/2}_{L^{q_1}(R^n)}\left\|\mathcal{M}_{\prod\vec{b},\Omega,\alpha-\varepsilon}(\vec{f})\right\|^{1/2}_{L^{q_2}(R^n)}\\
&\leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(R^n)}.
\end{aligned}$$

Thus, this complete the proof of Theorem 1.6. □

Lemma 2.3. [10]*Suppose that* $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and that* $u(x)^{s'}$, $v(x)^{s'} \in A(p/s', q/s')$. *Then there is an* $\in > 0$ *such that*

$$\varepsilon < \alpha < \alpha + \varepsilon < n,$$

$$1/p > (\alpha + \varepsilon)/n, 1/q < (n - \varepsilon)/n,$$

and $u(x)^{s'}$, $v(x)^{s'} \in A(p/s', q_{\llcorner}/s')$, $u(x)^{s'}, v(x)^{s'} \in A(p/s', \tilde{q}_\varepsilon/s')$ *hold at the same time, where* $1/q_\varepsilon = 1/p - (\alpha + \varepsilon)/n, 1/\tilde{q}_\varepsilon = 1/p - (\alpha - \varepsilon)/n$.

The proof of Theorem 1.7.

Proof. We prove conclusion (i) first. It is easy to see that $\frac{1}{p/s'} = \frac{1}{p_1/s'} + \cdots + \frac{1}{p_m/s'} - \frac{\alpha s'}{n}, \frac{1}{q_i/s'} = \frac{1}{p_i/s'} - \frac{\alpha s'/m}{n}$ $u_i(x)^{s'}$, $v_i(x)^{s'} \llcorner A(p_i/s', q_i/s')$. By Lemma 2.1 and Theorem 1.4, we have

$$\begin{aligned}
\left\|\mathcal{M}_{\prod\vec{b},\Omega,\alpha}(\vec{f})\right\|_{L^p_{v^p}} &= \left(\int_{R^n}\left|\mathcal{M}_{\prod\vec{b},\Omega,\alpha}(\vec{f})(x)v(x)\right|^p dx\right)^{1/p}\\
&\leq C\left(\int_{R^n}\left|\prod_{i=1}^{m}\left[M^{s'}_{1,\frac{\alpha s'}{m},b_i}f_i^{s'}(x)v_i(x)^{s'}\right]^{\frac{1}{s'}}\right|^p dx\right)^{1/p}\\
&\leq C\prod_{i=1}^{m}\left(\int_{R^n}\left|M^{s'}_{1,\frac{\alpha s'}{m},b_i}f_i^{s'}v_i(x)^{s'}\right|^{\frac{q_i}{s'}}dx\right)^{\frac{1}{q_i}}\\
&\leq C\prod_{i=1}^{m}\left(\int_{R^n}\left|f_i(x)^{s'}u_i(x)^{s'}\right|^{p_i/s'}dx\right)^{1/p_i}\\
&= C\prod_{i=1}^{m}\left(\int_{R^n}|f_i(x)u_i(x)|^{p_i}dx\right)^{1/p_i}.
\end{aligned}$$

Now we prove (ii), note that under the condition of Theorem 1.7, by Lemma 2.3, there is an $\in >0$ such that

$$\varepsilon < \alpha < \alpha + \varepsilon < mn,$$

$$1/p_i > (\alpha + \varepsilon)/mn, 1/q_i < (mn - \varepsilon)/mn,$$

and $u_i(x)^{s'}$, $v_i(x)^{s'} \in A(p_i/s', q_{i\in}/s')$, $u_i(x)^{s'}, v_i(x)^{s'} \in A(p_i/s', \tilde{q}_{i\varepsilon}/s')$ hold at the same time, where $1/q_{i\varepsilon} = 1/p_i - (\alpha + \varepsilon)/mn, 1/\tilde{q}_{i\varepsilon} = 1/p_i - (\alpha - \varepsilon)/mn$. Let

$$\frac{1}{\beta_1} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha + \varepsilon}{n} = \frac{1}{p} - \frac{\varepsilon}{n} > 0,$$
$$\frac{1}{\beta_2} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha - \varepsilon}{n} = \frac{1}{p} + \frac{\varepsilon}{n} > 0.$$

The boundedness of $\mathcal{M}_{\prod \vec{b},\Omega,\alpha}$ implies

$$\left\| \mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f}) \right\|_{L^{\beta_1}(v^{\beta_1})} \le \|f_i\|_{L^{p_i}(u_i^{p_i})},$$
$$\left\| \mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f}) \right\|_{L^{\beta_2}(v^{\beta_2})} \le \|f_i\|_{L^{p_i}(u_i^{p_i})}.$$

Now by Lemma 2.2, Hölder's inequality and the inequalities above, we get

$$\left(\int_{R^n} \left| I_{\prod \vec{b},\Omega,\alpha}(\vec{f})(x) \right|^p v(x)^p dx \right)^{\frac{1}{p}}$$
$$\le C \left(\int_{R^n} \left[\mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f})(x) v(x) \right]^{\frac{p}{2}} \left[\mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f})(x) v(x) \right]^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$
$$\le C \left\| \mathcal{M}_{\prod \vec{b},\Omega,\alpha+\varepsilon}(\vec{f}) \right\|_{L^{\beta_1}(v^{\beta_1})}^{1/2} \left\| \mathcal{M}_{\prod \vec{b},\Omega,\alpha-\varepsilon}(\vec{f}) \right\|_{L^{\beta_2}(v^{\beta_2})}^{1/2}$$
$$\le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(u_i^{p_i})}.$$

Thus, Theorem 1.7 is proved. □

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Author details
[1]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China [2]School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China [3]Department of fundamental Courses, Zhejiang Pharmaceutical College, Ningbo, Zhejiang 315100, China

Authors' contributions
All authors contributed in all parts in equal extent, and read and approved the final manuscript.

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