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Nonlinear $\mathcal{L}$-Fuzzy stability of cubic functional equations

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Abstract

We establish some stability results for the cubic functional equations

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y),$$

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x)$$

in the setting of various $\mathcal{L}$-fuzzy normed spaces that in turn generalize a Hyers-Ulam stability result in the framework of classical normed spaces. First, we shall prove the stability of cubic functional equations in the $\mathcal{L}$-fuzzy normed space under arbitrary t-norm which generalizes previous studies. Then, we prove the stability of cubic functional equations in the non-Archimedean $\mathcal{L}$-fuzzy normed space. We therefore provide a link among different disciplines: fuzzy set theory, lattice theory, non-Archimedean spaces, and mathematical analysis.

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1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and it was affirmatively answered for Banach spaces by Hyers [2]. Subsequently, the result of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The article [4] of Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. For more informations on such problems, refer to the papers [5-15].

The functional equations

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y), \tag{1.1}$$

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

and

$$f(3x+y)+f(3x-y)=3f(x+y)+3f(x-y)+48f(x) \tag{1.3}$$

are called the *cubic functional equations*, since the function $f(x) = cx^3$ is their solution. Every solution of the cubic functional equations is said to be a *cubic mapping*. The stability problem for the cubic functional equations was studied by Jun and Kim [16] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later a number of mathematicians worked on the stability of some types of cubic equations [4,17-19]. Furthermore, Mirmostafaee and Moslehian [20], Mirmostafaee et al. [21], Alsina [22], Miheț and Radu [23] and others [24-28] investigated the stability in the settings of fuzzy, probabilistic, and random normed spaces.

2. Preliminaries

In this section, we recall some definitions and results which are needed to prove our main results.

A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval [0,1], i.e., a function $T : [0,1] \times [0,1] \to [0,1]$ such that for all $a, b, c \in [0,1]$ the following four axioms are satisfied:

(i) $T(a, b) = T(b, a)$ (: commutativity);

(ii) $T(a, (T(b, c))) = T(T(a, b), c)$ (: associativity);

(iii) $T(a, 1) = a$ (: boundary condition);

(iv) $T(a, b) \le T(a, c)$ whenever $b \le c$ (: monotonicity).

Basic examples are the Lukasiewicz t-norm T_L, $T_L(a, b) = \max(a + b - 1, 0)$ $\forall a, b \in [0,1]$ and the t-norms T_P, T_M, T_D, where $T_P(a, b) := ab$, $T_M(a, b) := \min\{a, b\}$,

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if } \max(a,b) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If T is a t-norm then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T\left(x_T^{(n-1)}, x\right)$, if $n \ge 1$. A t-norm T is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $\left(x_T^{(n)}\right)_{n\in N}$ is equicontinuous at $x = 1$ (cf. [29]).

Other important triangular norms are (see [30]):

-the *Sugeno-Weber family* $\{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{\mathrm{SW}} = T_D$, $T_\infty^{\mathrm{SW}} = T_P$ and

$$T_\lambda^{\mathrm{SW}}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right)$$

if $\lambda \in (-1, \infty)$.

-the *Domby family* $\{T_\lambda^{\mathrm{D}}\}_{\lambda\in[0,\infty]}$, defined by T_{D}, if $\lambda = 0$, T_{M}, if $\lambda = \infty$ and

$$T_\lambda^{\mathrm{D}}(x,y) = \frac{1}{1+\left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

-the *Aczel-Alsina family* $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{AA}(x, y) = e^{-\left(|\log x|^\lambda + |\log y|^\lambda\right)^{1/\lambda}}$$

if $\lambda \in (0, \infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, \ldots, x_n) \in [0,1]^n$ the value $T(x_1, \ldots, x_n)$ defined by

$$\mathrm{T}_{i=1}^0 x_i = 1, \mathrm{T}_{i=1}^n x_i = \mathrm{T}\left(\mathrm{T}_{i=1}^{n-1} x_i, x_n\right) = T(x_1, \ldots, x_n).$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in [0,1] the value

$$\mathrm{T}_{i=1}^\infty x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^n x_i. \tag{2.1}$$

The limit on the right side of (2.1) exists, since the sequence $\{\mathrm{T}_{i=1}^n x_i\}_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Proposition 2.1. [30] (1) *For* $T \geq T_L$ *the following implication holds:*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

(2) *If* T *is of Hadžić-type then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1$$

for every sequence $\{x_n\}_{n\in N}$ *in* [0, 1] *such that* $\lim_{n\to\infty} x_n = 1$.

(3) *If* $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n)^\alpha < \infty.$$

(4) *If* $T \in \{T_\lambda^{sw}\}_{\lambda\in[-1,\infty)}$, *then*

$$\lim_{n\to\infty} \mathrm{T}_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

3. $\mathcal{L}$-Fuzzy normed spaces

The theory of fuzzy sets was introduced by Zadeh [31]. After the pioneering study of Zadeh, there has been a great effort to obtain fuzzy analogs of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [32-40,43-50]. One of the problems in $\mathcal{L}$ -fuzzy topology is to obtain an appropriate concept of $\mathcal{L}$ -fuzzy metric spaces and $\mathcal{L}$ -fuzzy normed spaces. Saadati and Park [40], respectively, introduced and studied a notion of intuitionistic fuzzy metric (normed) spaces and then Deschrijver et al. [41] generalized the concept of intuitionistic fuzzy metric (normed) spaces and studied a notion of $\mathcal{L}$ -fuzzy metric spaces and $\mathcal{L}$ -fuzzy normed spaces (also, see [41,42,51-55]). In this section, we give some definitions and related lemmas for our main results.

In this section, we give some definitions and related lemmas which are needed later.

Definition 3.1 ([43]). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and U be a non-empty set called universe. A *$\mathcal{L}$-fuzzy set* $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L$. For any $u \in U$, $\mathcal{A}(u)$ represents the *degree* (in L) to which u satisfies $\mathcal{A}$.

Lemma 3.2 ([44]). *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$ *for all* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 3.3 ([45]). An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

In Section 2, we presented the classical definition of t-norm, which can be easily extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 3.4. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) for any $x \in L, \mathcal{T}(x, 1_{\mathcal{L}}) = x$ (: boundary condition);
(ii) for any $(x, y) \in L^2, \mathcal{T}(x, y) = \mathcal{T}(y, x)$ (: commutativity);
(iii) for any $(x, y, z) \in L^3, \mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ (: associativity);
(iv) for any $(x, x', y, y') \in L^4$, $x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y')$(: monotonicity).

A t-norm can also be defined recursively as an $(n + 1)$-array operation ($n \in \mathrm{N} \setminus \{0\}$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_{(1)}, \ldots, x_{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x_{(1)}, \ldots, x_{(n)}), x_{(n+1)}), \quad \forall n \geq 2, \ x_{(i)} \in L.$$

The t-norm $\mathbf{M}$ defined by

$$\mathbf{M}(x, y) = \begin{cases} x \text{ if } x \leq_L y \\ y \text{ if } y \leq_L x \end{cases}$$

is a continuous t-norm.

Definition 3.5. A t-norm $\mathcal{T}$ on L^* is said to be *t-representable* if there exist a t-norm T and a t-conorm S on [0,1] such that

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)), \quad \forall x = (x_1, x_2), \ y = (y_1, y_2) \in L^*.$$

Definition 3.6. A *negation* on $\mathcal{L}$ is any strictly decreasing mapping $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*.

In this article, let $\mathcal{N} : L \to L$ be a given mapping. The negation N_s on ([0,1], ≤) defined as $N_s(x) = 1 - x$ for all $x \in [0, 1]$ is called the *standard negation* on ([0,1], ≤).

Definition 3.7. The 3-tuple $(V, \mathcal{P}, \mathcal{T})$ is said to be a *$\mathcal{L}$-fuzzy normed space* if V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is a $\mathcal{L}$ -fuzzy set on $V\times\,]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in\,]0, +\infty[$,

(i) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(ii) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(iv) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x+y, t+s)$;
(v) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(vi) $\lim_{t\to 0}\mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty}\mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called a *$\mathcal{L}$-fuzzy norm*. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable, then the 3-tuple $(V, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space*.

Definition 3.8. (1) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\varepsilon \in L\backslash\{0_{\mathcal{L}}\}$ and t >0, there exists a positive integer n_0 such that

$$\mathcal{N}(\varepsilon) <_L \mathcal{P}(x_{n+p} - x_n, t), \quad \forall n \geq n_0,\ p > 0.$$

(2) If every Cauchy sequence is convergent, then the $\mathcal{L}$ -fuzzy norm is said to be *complete* and the $\mathcal{L}$ -fuzzy normed space is called a *$\mathcal{L}$-fuzzy Banach space*, where $\mathcal{N}$ is an involutive negation.

(3) The sequence $\{x_n\}$ is said to be *convergent* to $x \in V$ in the $\mathcal{L}$ -fuzzy normed space $(V, \mathcal{P}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{P}} x$)if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all t >0.

Lemma 3.9 ([46]). *Let $\mathcal{P}$be a $\mathcal{L}$-fuzzy norm on V. Then*

(1) *For all $\times \in V$, $\mathcal{P}(x, t)$is nondecreasing with respect to t.*
(2) $\mathcal{P}(x - y, t) = \mathcal{P}(y - x, t)$*for all $x, y \in V$ and $t \in\,]0, +\infty[$.*

Definition 3.10. Let $(V, \mathcal{P}, \mathcal{T})$ be a $\mathcal{L}$ -fuzzy normed space. For any $t \in\,]0, +\infty[$, we define the *open ball* $B(x, r, t)$ with center $x \in V$ and radius $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as

$$B(x, r, t) = \{y \in V : \mathcal{N}(r) <_L \mathcal{P}(x - y, t)\}.$$

4. Stability result in $\mathcal{L}$ -fuzzy normed spaces

In this section, we study the stability of functional equations in $\mathcal{L}$ -fuzzy normed spaces.

Theorem 4.1. *Let X be a linear space and $(Y, \mathcal{P}, \mathcal{T})$be a complete $\mathcal{L}$-fuzzy normed space. If $f : \times \to Y$ is a mapping with $f(0) = 0$ and Q is a $\mathcal{L}$-fuzzy set on $X^2 \times (0, \infty)$ with the following property:*

$$\begin{aligned}\mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \\ \geq_L Q(x, y, t), \quad \forall x, y \in X,\ t > 0.\end{aligned} \tag{4.1}$$

If

$$\mathcal{T}_{i=1}^{\infty}(Q(3^{n+i-1}x, 0, 3^{3n+2i+1}t)) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

and

$$\lim_{n\to\infty} Q(3^n x, 3^n y, 3^{3n} t) = 1_{\mathcal{L}}, \quad \forall x, y \in X,\ t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}(Q(3^{i-1}x, 0, 3^{2i+2}t)), \quad \forall x \in X,\ t > 0. \tag{4.2}$$

Proof. We brief the proof because it is similar as the random case [47,27]. Putting $y = 0$ in (4.1), we have

$$\mathcal{P}\left(\frac{f(3x)}{27} - f(x), t\right) \geq_{L^*} Q(x, 0, 3^3 t), \quad \forall x \in X,\ t > 0.$$

Therefore, it follows that

$$\mathcal{P}\left(\frac{f(3^{k+1}x)}{3^{3(k+1)}} - \frac{f(3^k x)}{3^{3k}}, \frac{t}{3^{k+1}}\right) \geq_L Q(3^k x, 0, 3^{2(k+1)} t). \quad \forall k \geq 1,\ t > 0.$$

By the triangle inequality, it follows that

$$\mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\right) \geq_L \mathcal{T}_{i=1}^{n}(Q(3^{i-1}x, 0, 3^{2i+2}t)), \quad \forall x \in X,\ t > 0. \tag{4.3}$$

In order to prove the convergence of the sequence $\left\{\frac{f(3^n x)}{27^n}\right\}$, we replace x with $3^m x$ in (4.3) to find that, for all m, n >0,

$$\mathcal{P}\left(\frac{f(3^{n+m}x)}{27^{(n+m)}} - \frac{f(3^m x)}{27^m}, t\right) \geq_L \mathcal{T}_{i=1}^{n}(Q(3^{i+m-1}x, 0, 3^{2i+3m+2}t)), \quad \forall x \in X,\ t > 0.$$

Since the right-hand side of the inequality tends to $1_{\mathcal{L}}$ as m tends to infinity, the sequence $\left\{\frac{f(3^n x)}{3^{3n}}\right\}$ is a Cauchy sequence. Thus, we may define $C(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^{3n}}$ for all $x \in X$. Replacing x, y with $3^n x$ and $3^n y$, respectively, in (4.1), it follows that C is a cubic mapping. To prove (4.2), take the limit as $n \to \infty$ in (4.3). To prove the uniqueness of the cubic mapping C subject to (4.2), let us assume that there exists another cubic mapping C' which satisfies (4.2). Obviously, we have $C(3^n x) = 3^{3n}C(x)$ and $C'(3^n x) = 3^{3n}C'(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (4.2) that

$$\begin{aligned}
&\mathcal{P}\left(C(x) - C'(x), t\right) \\
&\geq_L \mathcal{P}\left(C(3^n x) - C'(3^n x), 3^{3n} t\right) \\
&\geq_L \mathcal{T}\left(\mathcal{P}\left(C(3^n x) - f(3^n x), 3^{3n-1} t\right), \mathcal{P}\left(f(3^n x) - C'(3^n x), 2^{3n-1} t\right)\right) \\
&\geq_L \mathcal{T}\left(\mathcal{T}_{i=1}^{\infty}(Q(3^{n+i-1}x, 0, 3^{3n+2i+1}t)), \mathcal{T}_{i=1}^{\infty}(Q(3^{n+i-1}x, 0, 3^{3n+2i+1}t))\right) \\
&= \mathcal{T}(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,
\end{aligned}$$

which proves the uniqueness of C. This completes the proof.

Theorem 4.2. *Let X be a linear space and* $(Y, \mathcal{P}, \mathcal{T})$*be a complete* $\mathcal{L}$*-fuzzy normed space. If* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *and Q is a* $\mathcal{L}$*-fuzzy set on* $X^2 \times (0, \infty)$ *with the following property:*

$$\mathcal{P}(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t) \tag{4.4}$$

$$\geq_L Q(x, y, t), \quad \forall x, y \in X,\ t > 0.$$

If

$$\mathcal{T}_{i=1}^{\infty}(Q(2^{n+i-1}x, 0, 2^{3n+2i+1}t)) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

and

$$\lim_{n\to\infty} Q(2^n x, 2^n y, 2^{3n}t) = 1_{\mathcal{L}}, \quad \forall x, y \in X,\ t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}(Q(2^{i-1}x, 0, 2^{2i+1}t)), \quad \forall x \in X,\ t > 0. \tag{4.5}$$

Proof. We omit the proof because it is similar as the last theorem and see [28].

Corollary 4.3. *Let* $(X, \mathcal{P}', \mathcal{T})$*be* $\mathcal{L}$*-fuzzy normed space and* $(Y, \mathcal{P}, \mathcal{T})$*be a complete* $\mathcal{L}$*-fuzzy normed space. If* $f : X \to Y$ *is a mapping such that*

$$\mathcal{P}(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t)$$

$$\geq_L \mathcal{P}'(x + y, t), \quad \forall x, y \in X,\ t > 0,$$

and

$$\lim_{n\to\infty} \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'(x, 2^{2n+i+2}t)) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{T}_{i=1}^{\infty}(\mathcal{P}'(x, 2^{i+2}t)), \quad \forall x \in X,\ t > 0.$$

Proof. See [28].

Now, we give an example to validate the main result as follows:

Example 4.4 ([28]). Let (X, $||\cdot||$) be a Banach space, $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ be an intuitionistic fuzzy normed space in which $\mathcal{T}_{\mathbf{M}}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ and

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + ||x||}, \frac{||x||}{t + ||x||}\right), \quad \forall x \in X,\ t > 0,$$

also $(Y, \mathcal{P}_{\mu,\nu}, \mathcal{T}_M)$ be a complete intuitionistic fuzzy normed space. Define a mapping $f : X \to Y$ by $f(x) = x^3 + x_0$ for all $x \in X$, where x_0 is a unit vector in X. A straightforward computation shows that

$$\begin{aligned}&\mathcal{P}_{\mu,\nu}(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t)\\ &\qquad \geq_{L^*} \mathcal{P}_{\mu,\nu}(x + y, t), \quad \forall x, y \in X,\ t > 0.\end{aligned}$$

Also, we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}_{M,i=1}^{\infty}(\mathcal{P}_{\mu,\nu}(x, 2^{2n+i+1}t)) &= \lim_{n\to\infty}\lim_{m\to\infty} \mathcal{T}_{M,i=1}^{m}(\mathcal{P}_{\mu,\nu}(x, 2^{2n+i+1}t))\\ &= \lim_{n\to\infty}\lim_{m\to\infty} \mathcal{P}_{\mu,\nu}(x, 2^{2n+2}t)\\ &= \lim_{n\to\infty} \mathcal{P}_{\mu,\nu}(x, 2^{2n+2}t)\\ &= 1_{L^*}.\end{aligned}$$

Therefore, all the conditions of Theorem 4.2 hold and so there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathcal{P}_{\mu,\nu}(x, 2^2 t), \quad \forall x \in X,\ t > 0.$$

5. Non-Archimedean *L*-fuzzy normed spaces

In 1897, Hensel [?] introduced a field with a valuation in which does not have the Archimedean property.

Definition 5.1. Let $\mathcal{K}$ be a field. A *non-Archimedean absolute value* on $\mathcal{K}$ is a function $|\cdot| : \mathcal{K} \to [0, +\infty[$such that, for any a, $b \in \mathcal{K}$,

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$;
(ii) $|ab| = |a|\ |b|$;
(iii) $|a + b| \leq \max\{|a|, |b|\}$ (: the strict triangle inequality).

Note that $|n| \leq 1$ for each integer $n \geq 1$. We always assume, in addition, that $|\cdot|$ is non-trivial, i.e., there exists $a_0 \in \mathcal{K}$ such that $|a_0| \neq 0, 1$.

Definition 5.2. A *non-Archimedean $\mathcal{L}$-fuzzy normed space* is a triple $(V, \mathcal{P}, \mathcal{T})$, where V is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is a $\mathcal{L}$ -fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all x, $y \in V$ and t, $s \in]0, +\infty[$,

(i) $0_{\mathcal{L}} <_L \mathcal{P}(x, t)$;
(ii) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(iii) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(vi) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, \max\{t, s\})$;
(v) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;
(vi) $\lim_{t\to 0}\mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty}\mathcal{P}(x, t) = 1_{\mathcal{L}}$.

Example 5.3. Let $(X, ||\cdot||)$ be a non-Archimedean normed linear space. Then the triple $(X, \mathcal{P}, \min)$, where

$$\mathcal{P}(x,\ t) = \begin{cases} 0, \text{ if } t \leq ||x||; \\ 1, \text{ if } t > ||x||, \end{cases}$$

is a non-Archimedean $\mathcal{L}$ -fuzzy normed space in which $L = [0,1]$.

Example 5.4. Let $(X, ||\cdot||)$ be a non-Archimedean normed linear space. Denote $\mathcal{T}_M(a,\ b) = (\min\{a_1,\ b_1\},\ \max\{a_2,\ b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $X \times]0, +\infty[$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,\ t) = \left(\frac{t}{t + ||x||},\ \frac{||x||}{t + ||x||}\right), \quad \forall x \in X,\ t \in \mathbb{R}^+.$$

Then $(X,\ \mathcal{P}_{\mu,\nu},\ \mathcal{T}_M)$ is a non-Archimedean intuitionistic fuzzy normed space.

6. $\mathcal{L}$-fuzzy Hyers-Ulam-Rassias stability for cubic functional equations in non-Archimedean $\mathcal{L}$-fuzzy normed space

Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y,\ \mathcal{P},\ \mathcal{T})$ be a non-Archimedean $\mathcal{L}$ -fuzzy Banach space over $\mathcal{K}$. In this section, we investigate the stability of the cubic functional equation (1.1).

Next, we define a $\mathcal{L}$ -fuzzy approximately cubic mapping. Let Ψ be a $\mathcal{L}$ -fuzzy set on $X \times X \times [0, \infty)$ such that $\Psi\ (x, y, \cdot)$ is nondecreasing,

$$\Psi(cx,\ cx,\ t) \geq_L \Psi\left(x,\ x,\ \frac{t}{|c|}\right), \quad \forall x \in X,\ c \neq 0$$

and

$$\lim_{t\to\infty} \Psi(x, y, t) = 1_{\mathcal{L}}, \quad \forall x, y \in X, t > 0.$$

Definition 6.1. A mapping $f : X \to Y$ is said to be Ψ-*approximately cubic* if

$$\begin{aligned} &\mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \\ &\qquad \geq_L \Psi(x, y, t), \quad \forall x, y \in X, t > 0. \end{aligned} \tag{6.1}$$

Here, we assume that $3 \neq 0$ in $\mathcal{K}$ (i.e., characteristic of $\mathcal{K}$ is not 3).

Theorem 6.2. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$: Let $f : X \to Y$ be a Ψ-approximately cubic mapping. If there exist a $\alpha \in \mathbb{R}$ ($\alpha > 0$) and an integer k, $k \geq 2$ with $|3^k| < \alpha$ and $|3| \neq 1$ such that*

$$\Psi(3^{-k}x, 3^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \quad \forall x, y \in X, t > 0, \tag{6.2}$$

and

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|3|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X, t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|3|^{ki}}\right), \quad \forall x \in X, t > 0, \tag{6.3}$$

where

$$\mathcal{M}(x, t) := \mathcal{T}(\Psi(x, 0, t), \Psi(3x, 0, t), \ldots, \Psi(3^{k-1}x, 0, t)), \quad \forall x \in X, t > 0.$$

Proof. First, we show, by induction on *j*, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(3^j x) - 27^j f(x), t) \geq_L \mathcal{M}_j(x, t) := \mathcal{T}(\Psi(x, 0, t), \ldots, \Psi(3^{j-1}x, 0, t)). \tag{6.4}$$

Putting $y = 0$ in (6.1), we obtain

$$\mathcal{P}(f(3x) - 27f(x), t) \geq_L \Psi(x, 0, t), \quad \forall x \in X, t > 0.$$

This proves (6.4) for $j = 1$. Let (6.4) hold for some $j > 1$. Replacing y by 0 and x by $3^j x$ in (6.1), we get

$$\mathcal{P}(f(3^{j+1}x) - 27f(3^j x), t) \geq_L \Psi(3^j x, 0, t), \quad \forall x \in X, t > 0.$$

Since $|27| \leq 1$, it follows that

$$\begin{aligned} &\mathcal{P}(f(3^{j+1}x) - 27^{j+1}f(x), t) \\ &\geq_L \mathcal{T}\left(\mathcal{P}(f(3^{j+1}x) - 27f(3^j x), t), \mathcal{P}(8f(3^j x) - 27^{j+1}f(x), t)\right) \\ &= \mathcal{T}\left(\mathcal{P}(f(2^{j+1}x) - 8f(2^j x), t), \mathcal{P}\left(f(3^j x) - 27^j f(x), \frac{t}{|27|}\right)\right) \\ &\geq_L \mathcal{T}\left(\mathcal{P}(f(3^{j+1}x) - 27f(3^j x), t), \mathcal{P}(f(3^j x) - 27^j f(x), t)\right) \\ &\geq_L \mathcal{T}(\Psi(3^j x, 0, t), \mathcal{M}_j(x, t)) \\ &= \mathcal{M}_{j+1}(x, t), \quad \forall x \in X, t > 0. \end{aligned}$$

Thus (6.4) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(3^k x) - 27^k f(x),\ t) \geq_L \mathcal{M}(x,\ t), \quad \forall x \in X,\ t > 0. \tag{6.5}$$

Replacing x by $3^{-(kn+k)}x$ in (6.5) and using the inequality (6.2), we obtain

$$\begin{aligned}\mathcal{P}\left(f\left(\frac{x}{3^{kn}}\right) - 27^k f\left(\frac{x}{3^{kn+k}}\right),\ t\right) &\geq_L \mathcal{M}\left(\frac{x}{3^{kn+k}},\ t\right)\\ &\geq_L \mathcal{M}(x,\ \alpha^{n+1}t) \quad \forall x \in X,\ t > 0,\ n \geq 0\end{aligned}$$

and so

$$\begin{aligned}&\mathcal{P}\left((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right) - (3^{3k})^{n+1} f\left(\frac{x}{(3^k)^{n+1}}\right),\ t\right)\\ &\geq_L \mathcal{M}\left(x,\ \frac{\alpha^{n+1}}{|(3^{3k})^n|}t\right)\\ &\geq_L \mathcal{M}\left(x,\ \frac{\alpha^{n+1}}{|(3^k)^n|}t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.\end{aligned}$$

Hence, it follow that

$$\begin{aligned}&\mathcal{P}\left((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right) - (3^{3k})^{n+p} f\left(\frac{x}{(3^k)^{n+p}}\right),\ t\right)\\ &\geq_L \mathcal{T}_{j=n}^{n+p}\left(\mathcal{P}\left(\left(3^{3k}\right)^j f\left(\frac{x}{(3^k)^j}\right) - (3^{3k})^{j+p} f\left(\frac{x}{(3^k)^{j+p}}\right),\ t\right)\right)\\ &\geq_L \mathcal{T}_{j=n}^{n+p}\mathcal{M}\left(x,\ \frac{\alpha^{j+1}}{|(3^k)^j|}t\right), \quad \forall x \in X,\ t > 0,\ n \geq 0.\end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x,\ \frac{\alpha^{j+1}}{|(3^k)^j|}t\right) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\left\{(3^{3k})^n f\left(\frac{x}{(3^k)^n}\right)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right) - C(x), t\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0. \tag{6.6}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\begin{aligned}&\mathcal{P}\left(f(x) - (3^{3k})^n f\left(\frac{x}{(3^k)^n}\right),\ t\right)\\ &= \mathcal{P}\left(\sum_{i=0}^{n-1}(3^{3k})^i f\left(\frac{x}{(3^k)^i}\right) - (3^{3k})^{i+1} f\left(\frac{x}{(3^k)^{i+1}}\right),\ t\right)\\ &\geq_L \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}\left(\left(3^{3k}\right)^i f\left(\frac{x}{(3^k)^i}\right) - (3^{3k})^{i+1} f\left(\frac{x}{(3^k)^{i+1}}\right),\ t\right)\right)\\ &\geq_L \mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x,\ \frac{\alpha^{i+1}t}{|3^k|^i}\right)\end{aligned}$$

and so

$$\begin{aligned}&\mathcal{P}(f(x)-C(x),\ t)\\&\geq_L \mathcal{T}\left(\mathcal{P}\left(f(x)-(3^{3k})^n f\left(\frac{x}{(3^k)^n}\right),\ t\right),\ \mathcal{P}\left((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right)-C(x),\ t\right)\right)\\&\geq_L \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1}\mathcal{M}\left(x,\ \frac{\alpha^{i+1}t}{|3^k|^i}\right),\ \mathcal{P}((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right)-C(x),\ t)\right).\end{aligned}\tag{6.7}$$

Taking the limit as $n \to \infty$ in (6.7), we obtain

$$\mathcal{P}(f(x)-C(x),\ t)\geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x,\ \frac{\alpha^{i+1}t}{|3^k|^i}\right),$$

which proves (6.3). As $\mathcal{T}$ is continuous, from a well known result in $\mathcal{L}$ -fuzzy (probabilistic) normed space (see, [51, Chap. 12]), it follows that

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{P}((27^k)^n f(3^{-kn}(x+3y))+(27^k)^n f(3^{-kn}(3x-y))-15(27^k)^n f(3^{-kn}(x+y))\\&\quad-15(27^k)^n f(3^{-kn}(x-y))-80(27^k)^n f(3^{-kn}y),\ t)\\&\quad=\mathcal{P}(C(x+3y)+C(3x-y)-15C(x+y)-15C(x-y)-80C(y),\ t),\quad \forall t>0.\end{aligned}$$

On the other hand, replacing x, y by $3^{-kn}x$, $3^{-kn}\ y$ in (6.1) and (6.2), we get

$$\begin{aligned}&\mathcal{P}((27^k)^n f(3^{-kn}(x+3y))+(27^k)^n f(3^{-kn}(3x-y))-15(27^k)^n f(3^{-kn}(x+y))\\&\quad-15(27^k)^n f(3^{-kn}(x-y))-80(27^k)^n f(3^{-kn}y),\ t)\\&\quad\geq_L \Psi\left(3^{-kn}x,\ 3^{-kn}y,\ \frac{t}{|3^{3k}|^n}\right)\\&\quad\geq_L \Psi\left(x,\ y,\ \frac{\alpha^n t}{|3^k|^n}\right),\quad \forall x,\ y\in X,\ t>0.\end{aligned}$$

Since $\lim_{n\to\infty}\Psi\left(x, y, \frac{\alpha^n t}{|3^k|^n}\right)=1_{\mathcal{L}}$, we infer that C is a cubic mapping.

For the uniqueness of C, let $C' : X \to Y$ be another cubic mapping such that

$$\mathcal{P}(C'(x)-f(x),\ t)\geq_L \mathcal{M}(x,\ t),\quad \forall x\in X,\ t>0.$$

Then we have, for all x, $y \in X$ and $t > 0$,

$$\begin{aligned}&\mathcal{P}(C(x)-C'(x),\ t)\\&\geq_L \mathcal{T}\left(\mathcal{P}\left(C(x)-(3^{3k})^n f\left(\frac{x}{(3^k)^n}\right),\ t\right),\ \mathcal{P}\left((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right)-C'(x),\ t\right),\ t)\right).\end{aligned}$$

Therefore, from (6.6), we conclude that $C = C'$. This completes the proof.

Corollary 6.3. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, \mathcal{T})$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$ under a t-norm $\mathcal{T} \in \mathcal{H}$. Let $f : X \to Y$ be a Ψ-approximately cubic mapping. If there exist $\alpha \in \mathbb{R}$ ($\alpha > 0$), $|3| \neq 1$ and an integer k, $k \geq 3$ with $|3^k| < \alpha$ such that*

$$\Psi(3^{-k}x,\ 3^{-k}y,\ t)\geq_L \Psi(x,\ y,\ \alpha t),\quad \forall x,\ y\in X,\ t>0,$$

then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}(f(x)-C(x),\ t)\geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x,\ \frac{\alpha^{i+1}t}{|3|^{ki}}\right),\quad \forall x\in X,\ t>0,$$

where

$$\mathcal{M}(x,\ t)\ := \mathcal{T}(\Psi(x,\ 0,\ t),\ \Psi(3x,\ 0,\ t),\ \ldots,\ \Psi(3^{k-1}x,\ 0,\ t)),\quad \forall x\in X,\ t>0.$$

Proof. Since

$$\lim_{n\to\infty}\mathcal{M}\left(x,\ \frac{\alpha^j t}{|3|^{kj}}\right)=1_{\mathcal{L}},\quad \forall x\in X,\ t>0,$$

and $\mathcal{T}$ is of Hadžić type, it follows from Proposition 2.1 that

$$\lim_{n\to\infty}\mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x,\ \frac{\alpha^j t}{|3|^{kj}}\right)=1_{\mathcal{L}},\quad \forall x\in X,\ t>0.$$

Now, if we apply Theorem 6.2, we get the conclusion.

Now, we give an example to validate the main result as follows:

Example 6.4. Let $(X,\ ||\cdot||)$ be a non-Archimedean Banach space, $(X,\ \mathcal{P}_{\mu,\nu},\ \mathcal{T}_M)$ be non-Archimedean $\mathcal{L}$ -fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x,\ t)=\left(\frac{t}{t+||x||},\ \frac{||x||}{t+||x||}\right),\quad \forall x\in X,\ t>0,$$

and $(Y,\ \mathcal{P}_{\mu,\nu},\ \mathcal{T}_M)$ be a complete non-Archimedean $\mathcal{L}$ -fuzzy normed space (intuitionistic fuzzy normed space) (see, Example 5.4). Define

$$\Psi(x,\ y,\ t)=\left(\frac{t}{1+t},\ \frac{1}{1+t}\right),\quad \forall x,\ y\in X,\ t>0.$$

It is easy to see that (6.2) holds for $\alpha = 1$. Also, since

$$\mathcal{M}(x,\ t)=\left(\frac{t}{1+t},\ \frac{1}{1+t}\right),\quad \forall x\in X,\ t>0,$$

we have

$$\begin{aligned}\lim_{n\to\infty}\mathcal{T}_{M,j=n}^{\infty}\mathcal{M}\left(x,\ \frac{\alpha^j t}{|3|^{kj}}\right)&=\lim_{n\to\infty}\left(\lim_{m\to\infty}\mathcal{T}_{M,j=n}^{m}\mathcal{M}\left(x,\ \frac{t}{|3|^{kj}}\right)\right)\\&=\lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+|3^k|^n},\ \frac{|2^k|^n}{t+|3^k|^n}\right)\\&=(1,0)=1_{L^*},\quad \forall x\in X,\quad t>0.\end{aligned}$$

Let $f: X\to Y$ be a Ψ-approximately cubic mapping. Therefore, all the conditions of Theorem 6.2 hold and so there exists a unique cubic mapping $C: X\to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x)-C(x),\ t)\geq_{L^*}\left(\frac{t}{t+|3^k|},\ \frac{|3^k|}{t+|3^k|}\right),\quad \forall x\in X,\ t>0.$$

Definition 6.5. A mapping $f: X\to Y$ is said to be Ψ*-approximately cubic I* if

$$\mathcal{P}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),\ t) \tag{6.8}$$

$$\geq_L\Psi(x,\ y,\ t),\quad \forall x,\ y\in X,\ t>0.$$

In this section, we assume that $2\neq 0$ in $\mathcal{K}$ (i.e., the characteristic of $\mathcal{K}$ is not 2).

Theorem 6.6. *Let* $\mathcal{K}$*be a non-Archimedean field*, X *be a vector space over* $\mathcal{K}$*and* $(Y,\mathcal{P},\mathcal{T})$*be a non-Archimedean* $\mathcal{L}$*-fuzzy Banach space over*$\mathcal{K}$. *Let* $f: X\to Y$ *be a* Ψ-

approximately cubic I mapping. If $|2| \neq 1$ *and for some* $\alpha \in \mathbb{R}$, $\alpha > 0$, *and some integer* k, $k \geq 2$ *with* $|2^k| < \alpha$,

$$\Psi(2^{-k}x,\ 2^{-k}y,\ t) \geq_L \Psi(x,\ y,\ \alpha t), \quad \forall x, y \in X,\ t > 0, \tag{6.9}$$

and

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x,\ \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x),\ t) \geq \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x,\ \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \quad \forall x \in X,\ t > 0, \tag{6.10}$$

where

$$\mathcal{M}(x,\ t) := \mathcal{T}(\Psi(x,\ 0,\ t),\ \Psi(2x,\ 0,\ t),\ \ldots,\ \Psi(2^{k-1}x,\ 0,\ t)), \quad \forall x \in X,\ t > 0.$$

Proof. We omit the proof because it is similar as the random case (see, [28]).

Corollary 6.7. *Let* $\mathcal{K}$*be a non-Archimedean field,* X *be a vector space over* $\mathcal{K}$*and* $(Y, \mathcal{P}, \mathcal{T})$*be a non-Archimedean* $\mathcal{L}$ *-fuzzy Banach space over* $\mathcal{K}$*under a t-norm* $\mathcal{T} \in \mathcal{H}$. *Let* $f : X \to Y$ *be a* Ψ*-approximately cubic I mapping. If there exist a* $\alpha \in \mathbb{R}$ $(\alpha > 0)$ *and an integer* k, $k \geq 2$ *with* $|2^k| < \alpha$ *such that*

$$\Psi(2^{-k}x,\ 2^{-k}y,\ t) \geq_L \Psi(x,\ y,\ \alpha t), \quad \forall x,\ y \in X,\ t > 0,$$

then there exists a unique cubic mapping $C : X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x),\ t) \geq_L \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x,\ \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \quad \forall x \in X,\ t > 0,$$

where

$$\mathcal{M}(x,\ t) := \mathcal{T}(\Psi(x,\ 0,\ t),\ \Psi(2x,\ 0,\ t),\ \ldots,\ \Psi(2^{k-1}x,\ 0,\ t)), \quad \forall x \in X,\ t > 0.$$

Proof. Since

$$\lim_{n\to\infty} \mathcal{M}\left(x,\ \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0,$$

and $\mathcal{T}$ is of Hadžić type, it follows from Proposition 2.1 that

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x,\ \frac{\alpha^j t}{|2|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0.$$

Now, if we apply Theorem 6.2, we get the conclusion.

Now, we give an example to validate the main result as follows:

Example 6.8. Let (X, $||\cdot||$ be a non-Archimedean Banach space, $(X,\ \mathcal{P}_{\mu,\nu},\ \mathcal{T}_M)$ be non-Archimedean $\mathcal{L}$ -fuzzy normed space (intuitionistic fuzzy normed space) in which

$$\mathcal{P}_{\mu,\nu}(x,\ t) = \left(\frac{t}{t + ||x||},\ \frac{||x||}{t + ||x||}\right), \quad \forall x \in X,\ t > 0,$$

and $(Y,\ \mathcal{P}_{\mu,\nu},\ \mathcal{T}_M)$ be a complete non-Archimedean $\mathcal{L}$ -fuzzy normed space (intuitionistic fuzzy normed space) (see, Example 5.4). Define

$$\Psi(x,\ y,\ t) = \left(\frac{t}{1+t},\ \frac{1}{1+t}\right), \quad \forall x,\ y \in X,\ t > 0.$$

It is easy to see that (6.9) holds for $\alpha = 1$. Also, since

$$\mathcal{M}(x,\ t) = \left(\frac{t}{1+t},\ \frac{1}{1+t}\right), \quad \forall x \in X,\ t > 0,$$

we have

$$\begin{aligned}\lim_{n\to\infty} \mathcal{T}^{\infty}_{M,j=n}\mathcal{M}\left(x,\ \frac{\alpha^j t}{|2|^{kj}}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty} \mathcal{T}^{m}_{M,j=n}\mathcal{M}\left(x,\ \frac{t}{|2|^{kj}}\right)\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\left(\frac{t}{t+|2^k|^n},\ \frac{|2^k|^n}{t+|2^k|^n}\right) \\ &= (1,0) = 1_{L^*}, \quad \forall x \in X,\ t > 0.\end{aligned}$$

Let $f : X \to Y$ be a Ψ-approximately cubic I mapping. Therefore, all the conditions of Theorem 6.6 hold and so there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - C(x),\ t) \geq_{L^*} \left(\frac{t}{t+|2^k|},\ \frac{|2^k|}{t+|2^k|}\right), \quad \forall x \in X,\ t > 0.$$

Definition 6.9. A mapping $f : X \to Y$ is said to be Ψ*-approximately cubic II* if

$$\begin{aligned}&P\left(f\left(3x + y\right) + f\left(3x - y\right) - 3f\left(x + y\right) - 3f\left(x - y\right) - 48f\left(x\right),\ t\right) \\ &\quad \geq_L \Psi\left(x,\ y,\ t\right), \quad \forall x,\ y \in X,\ t > 0.\end{aligned} \tag{6.11}$$

Here, we assume that $3 \neq 0$ in $\mathcal{K}$ (i.e., the characteristic of $\mathcal{K}$ is not 3).

Theorem 6.10. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and $(Y, \mathcal{P}, T)$ be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$. Let $f : X \to Y$ be a Ψ-approximately cubic II function. If* $|3| \neq 1$ *and, for some* $\alpha \in \mathbb{R}$, $\alpha > 0$, *and some integer* k, $k \geq 3$, *with* $|3^k| < \alpha$,

$$\Psi\left(3^{-k}x,\ 3^{-k}y,\ t\right) \geq_L \Psi\left(x,\ y,\ \alpha t\right), \quad \forall x,\ y \in X,\ t > 0, \tag{6.12}$$

and

$$\lim_{n\to\infty} \mathcal{T}^{\infty}_{j=n}\mathcal{M}\left(x,\ \frac{\alpha^j t}{|3|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X,\ t > 0, \tag{6.13}$$

then there exists a unique cubic mapping $C : X \to Y$ *such tha*

$$\mathcal{P}\left(f\left(x\right) - C\left(x\right),\ t\right) \geq T^{\infty}_{i=1}\mathcal{M}\left(x,\ \frac{\alpha^{i+1}\,t}{|3|^{ki}}\right), \tag{6.14}$$

for all $x \in X$ *and* $t > 0$, *where*

$$\mathcal{M}\left(x,\ t\right) := \mathcal{T}\left(\Psi\left(x,\ 0,\ 2t\right),\ \Psi\left(3x,\ 0,\ 2t\right),\ \ldots,\ \Psi\left(3^{k-1}x,\ 0,\ 2t\right)\right), \quad \forall x \in X,\ t > 0.$$

Proof. First, we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}\left(f\left(3^j x\right) - 27^j f\left(x\right),\ t\right) \geq_L \mathcal{M}_j\left(x,\ t\right) := \mathcal{T}\left(\Psi\left(x,\ 0,\ 2t\right),\ \ldots,\ \Psi\left(3^{j-1}x,\ 0,\ 2t\right)\right). \tag{6.15}$$

Put $y = 0$ in (6.11) to obtain

$$\mathcal{P}\left(f(3x) - 27f(x), t\right) \geq_L \Psi(x, 0, 2t), \quad \forall x \in X, \ t > 0. \tag{6.16}$$

This proves (6.15) for $j = 1$. Let (6.15) hold for some $j > 1$. Replacing y by 0 and x by $3^j x$ in (6.16), we get

$$\mathcal{P}\left(f\left(3^{j+1}x\right) - 27f\left(3^j x\right), t\right) \geq_L \Psi\left(3^j x, 0, 2t\right), \quad \forall x \in X, \ t > 0.$$

Since $|27| \leq 1$, then we have

$$\begin{aligned}
&\mathcal{P}\left(f\left(3^{j+1}x\right) - 27^{j+1}f(x), t\right) \\
&\geq_L \mathcal{T}\left(\mathcal{P}\left(f\left(3^{j+1}x\right) - 27f\left(3^j x\right), t\right), \mathcal{P}\left(27f\left(3^j x\right) - 27^{j+1}f(x), t\right)\right) \\
&= \mathcal{T}\left(\mathcal{P}\left(f\left(3^{j+1}x\right) - 27f\left(3^j x\right), t\right), \mathcal{P}\left(f\left(3^j x\right) - 27^j f(x), \frac{t}{|27|}\right)\right) \\
&\geq_L \mathcal{T}\left(\mathcal{P}\left(f\left(3^{j+1}x\right) - 27f\left(3^j x\right), t\right), \mathcal{P}\left(f\left(3^j x\right) - 27^j f(x), t\right)\right) \\
&\geq_L \mathcal{T}\left(\Psi\left(3^j x, 0, 2t\right), \mathcal{M}_j(x, t)\right) \\
&= \mathcal{M}_{j+1}(x, t), \quad \forall x \in X.
\end{aligned}$$

Thus (6.15) holds for all $j \geq 1$. In particular, it follows that

$$\mathcal{P}\left(f\left(3^k x\right) - 27^k f(x), t\right) \geq_L \mathcal{M}(x, t), \quad \forall x \in X, \ t > 0. \tag{6.17}$$

Replacing x by $3^{-(kn+k)}x$ in (6.17) and using inequality (6.12) we obtain

$$\begin{aligned}
&\mathcal{P}\left(f\left(\frac{x}{3^{kn}}\right) - 27^k f\left(\frac{x}{3^{kn+k}}\right), t\right) \geq_L \mathcal{M}\left(\frac{x}{3^{kn+k}}, t\right) \\
&\geq_L \mathcal{M}\left(x, \alpha^{n+1}t\right), \quad \forall x \in X, \ t > 0, \ n \geq 0.
\end{aligned} \tag{6.18}$$

Then we have

$$\begin{aligned}
&\mathcal{P}\left(\left(3^{3k}\right)^n f\left(\frac{x}{\left(3^{3k}\right)^n}\right) - \left(3^{3k}\right)^{n+1} f\left(\frac{x}{\left(3^{3k}\right)^{n+1}}\right), t\right) \\
&\geq_L \mathcal{M}\left(x, \frac{\alpha^{n+1}}{\left|\left(3^{3k}\right)^n\right|} t\right), \quad \forall x \in X, \ t > 0, \ n \geq 0.
\end{aligned} \tag{6.19}$$

Hence we have

$$\begin{aligned}
&\mathcal{P}\left(f\left(\frac{x}{3^{kn}}\right) - 27^k f\left(\frac{x}{3^{kn+k}}\right), t\right) \geq_L \mathcal{M}\left(\frac{x}{3^{kn+k}}, t\right) \\
&\geq_L \mathcal{M}\left(x, \alpha^{n+1}t\right), \quad \forall x \in X, \ t > 0, \ n \geq 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{j+1}}{\left|\left(3^{3k}\right)^j\right|} t\right) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\{k^n f(k^{--n} x)\}_{n \lfloor N}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$ -fuzzy Banach space $(Y, \mathcal{P}, \mathcal{T})$. Hence we can define a mapping $C : X \to Y$ such that

$$\lim_{n\to\infty} \mathcal{P}\left((3^{3k})nf\left(\frac{x}{\left(3^k\right)^n}\right) - C(x), t\right) = 1_{\mathcal{L}}, \quad \forall x \in X, \ t > 0. \tag{6.20}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$,

$$\begin{aligned}
&\mathcal{P}\left(f(x) - (3^{3k})^n f\left(\frac{x}{(3^k)^n}\right), t\right) \\
&= \mathcal{P}\left(\sum_{i=0}^{n-1} (3^{3k})^i f\left(\frac{x}{(3^k)^i}\right) - (3^{3k})^{i+1} f\left(\frac{x}{(3^k)^{i+1}}\right), t\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1}\left(\mathcal{P}\left((3^{3k})^i f\left(\frac{x}{(3^k)^i}\right) - (3^{3k})^{i+1} f\left(\frac{x}{(3^k)^{i+1}}\right), t\right)\right) \\
&\geq_L \mathcal{T}_{i=0}^{n-1} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|3^{3k}|^i}\right).
\end{aligned} \tag{6.21}$$

Therefore, we have

$$\begin{aligned}
&\mathcal{P}\left(f(x) - C(x), t\right) \\
&\geq_L \mathcal{T}\left(\mathcal{P}\left(f(x) - \left(3^{3k}\right)^n f\left(\frac{x}{(3^k)^n}\right), t\right), \mathcal{P}\left(\left(3^{3k}\right)^n f\left(\frac{x}{(3^k)^n}\right) - C(x), t\right)\right) \\
&\geq_L \mathcal{P}\left(\mathcal{T}_{i=0}^{n-1} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|3^{3k}|^i}\right), \mathcal{P}\left(\left(3^{3k}\right)^n f\left(\frac{x}{(3^k)^n}\right) - C(x), t\right)\right).
\end{aligned}$$

By letting $n \to \infty$ in the above inequality, we obtain

$$\mathcal{P}\left(f(x) - C(x), t\right) \geq_L \mathcal{T}_{i=1}^{\infty} \mathcal{M}\left(x, \frac{\alpha^{i+1} t}{|3^{3k}|^i}\right), \quad \forall x \in X,\ t < 0.$$

This proves (6.14). Since $\mathcal{T}$ is continuous, from the well known result in $\mathcal{L}$ -fuzzy (probabilistic) normed space (see, [51, Chap. 12]), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{P}\left(\left(3^k\right)^n f\left(3^{-kn}(3x+y)\right) + \left(3^k\right)^n f\left(3^{-kn}(3x-y)\right) - 3\left(3^k\right)^n f\left(3^{-kn}(x+y)\right)\right. \\
&\quad -3\left(3^k\right)^n f\left(3^{-kn}(x-y)\right) - 48\left(3^k\right)^n f\left(3^{-kn}x\right), t) \\
&\quad = \mathcal{P}\left(C(3x+y) + C(3x-y) - 3C(x+y) - 3C(x-y) - 48C(x), t\right), \quad \forall t > 0.
\end{aligned}$$

On the other hand, replace x, y by $3^{-kn}x$, $3^{-kn}y$ in (6.11) and (6.12) to get

$$\begin{aligned}
&\mathcal{P}\left(3^k\right)^n f\left(3^{-kn}(3x+y)\right) + \left(3^k\right)^n f\left(3^{-kn}(3x-y)\right) - 3\left(3^k\right)^n f\left(3^{-kn}(x+y)\right) \\
&\quad -3\left(3^k\right)^n f\left(3^{-kn}(x-y)\right) - 48\left(3^k\right)^n f\left(3^{-kn}y\right), t) \\
&\quad \geq_L \Psi\left(3^{-kn}x, 3^{-kn}y, \frac{t}{|3^k|^n}\right) \\
&\quad \geq_L \Psi\left(x, y, \frac{\alpha^n t}{|3^k|^n}\right), \quad \forall x, y \in X,\ t > 0.
\end{aligned}$$

Since $\lim_{n\to\infty} \Psi\left(x, y, \frac{\alpha^n t}{|3^k|^n}\right) = 1_{\mathcal{L}}$, we infer that C is a cubic mapping.

If $C' : X \to Y$ is another cubic mapping such that $\mathcal{P}\left(C'(x) - f(x), t\right) \geq_L \mathcal{M}(x, t)$ for all $x \in X$ and $t > 0$, then, for all $n \geq 1$, $x \in X$ and $t > 0$,

$$\begin{aligned}
&\mathcal{P}\left(C(x) - C'(x), t\right) \\
&\geq_L \mathcal{T}\left(\mathcal{P}(C(x) - \left(3^{3k}\right)^n f\left(\frac{x}{(3^k)^n}\right), t), \mathcal{P}((3^{3k})^n f\left(\frac{x}{(3^k)^n}\right) - C'(x), t), t)\right).
\end{aligned}$$

Thus, from (6.20), we conclude that $C = C'$. This completes the proof.

Corollary 6.11. *Let $\mathcal{K}$be a non-Archimedean field, X be a vector space over $\mathcal{K}$and $(Y, \mathcal{P}, \mathcal{T})$.be a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathcal{K}$ under a t-norm $T \in \mathcal{H}$. Let $f: X \to Y$ be* a *Ψ-approximately cubic II function. If, for some $\alpha \in \mathbb{R}$, $\alpha > 0$ and an integer k, $k \geq 3$, with $|3^k| < \alpha$,*

$$\Psi\left(3^{-k}x, 3^{-k}y, t\right) \geq_L \Psi\left(x, y, \alpha t\right), \quad \forall x \in X, \ t > 0,$$

then there exists a unique cubic mapping $C: X \to Y$ such that, for all $\times \in X$ and $t > 0$,

$$\mathcal{P}\left(f(x) - C(x), t\right) \geq_L \mathcal{T}_{i=1}^{\infty}\mathcal{M}\left(x, \frac{\alpha^{i+1}t}{|3|^{ki}}\right),$$

Where

$$\mathcal{M}(x, t) := \mathcal{T}\left(\Psi(x, 0, 2t), \Psi(3x, 0, t), \ldots, \Psi\left(3^{k-1}x, 0, 2t\right)\right), \quad \forall x \in X, \ t > 0.$$

Proof. Since

$$\lim_{n\to\infty} \mathcal{M}\left(x, \frac{\alpha^j t}{|3|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X, \ t > 0,$$

and T is of Hadžić type, from Proposition 2.1, it follows that

$$\lim_{n\to\infty} \mathcal{T}_{j=n}^{\infty}\mathcal{M}\left(x, \frac{\alpha^j t}{|3|^{kj}}\right) = 1_{\mathcal{L}}, \quad \forall x \in X, \ t > 0.$$

Thus, if we apply Theorem 6.10, then we can get the conclusion. This completes the proof.

7. Conclusion

We established the Hyers-Ulam-Rassias stability of the cubic functional equations (1.1), (1.2), and (1.3) in various fuzzy spaces. In Section 4, we proved the stability of functional equations (1.1), (1.2), and (1.3) in a $\mathcal{L}$ -fuzzy normed space under arbitrary t-norm which is a generalization of [26]. In Section 6, we proved the stability of functional equations (1.1), (1.2), and (1.3) in a non-Archimedean $\mathcal{L}$ -fuzzy normed space. We therefore provided a link among three various discipline: fuzzy set theory, lattice theory, and mathematical analysis.

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Authors' contributions

All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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