Feng and Meng *Journal of Inequalities and Applications* 2012, **2012**:74
http://www.journalofinequalitiesandapplications.com/content/2012/1/74

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables

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Abstract

In this article, we establish some new Ostrowski type integral inequalities on time scales involving functions of two independent variables for k^2 points, which on one hand unify continuous and discrete analysis, on the other hand extend some known results in the literature. The established results can be used in the estimate of error bounds for some numerical integration formulae, and some of the results are sharp.

Mathematical Subject Classification 2010: 26E70; 26D15; 26D10.

Keywords: Ostrowski type inequality, time scales, numerical integration, error bound, sharp bound

1. Introduction

Recently many authors have studied various inequalities, among which the Ostrowski type inequalities have attracted much attention in the literature. The Ostrowski inequality was originally presented in [1] (see also in [[2], pp. 468]) as stated in the following theorem.

Theorem 1.1. Let $f : I \to R$ be a differentiable mapping in the interior Int$I$ of I, where $I \subset R$ is an interval, and let $a, b \in IntI$. $a < b$. If $|f'(t)| \leq M$, $\forall t \in [a, b]$, then we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M,$$

for $x \in [a, b]$.

In recent years, various generalizations of the Ostrowski inequality including continuous and discrete versions have been established (for example, see [3-14] and the references therein). On the other hand, Hilger [15] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type inequalities on time scales (see [16-24]). The established Ostrowski type inequalities on time scales unify continuous and discrete analysis, and can be used to provide explicit error bounds for some known and some new numerical quadrature formulae.

In this article, we will establish some new Ostrowski type inequalities on time scales involving functions of two independent variables for k^2 points, which on one hand extend some known results in the literature, on the other hand unify continuous and discrete analysis.

First we will give some preliminaries on time scales. A time scale is an arbitrary nonempty closed subset of the real numbers. If $\mathbb{T}$ denotes an arbitrary time scale, then on $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \rho(t) = \sup(s \in \mathbb{T}, s > t)$.

Definition 1.2. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 1.3. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.

Definition 1.4. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous at right-dense points and if the left-sided limits exist at left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$, where $\mu(t) = \sigma(t) - t$. C_{rd} denotes the set of rd-continuous functions.

Definition 1.5. For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f at t is denoted by $f^\Delta(t)$ (provided it exists) with the property such that for every $\varepsilon > 0$ there exists a neighborhood $\mathfrak{U}$ of t satisfying

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon \left|\sigma(t) - s\right| \quad \text{for all } s \in \mathfrak{U}.$$

Remark 1.6. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t + 1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 1.7. If $F^\Delta(t) = f(t)$, $t \in \mathbb{T}^\kappa$, then F is called an *antiderivative* of f, and the Cauchy integral of f is defined by

$$\int_a^b f(t)\boldsymbol{\Delta} t = \mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a}).$$

The following two theorem include some important properties for *delta derivative* on time scales.

Theorem 1.8. If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then

(i) $\int_a^b [f(t) + g(t)]\boldsymbol{\Delta} t = \int_a^b f(t)\Delta t + \int_a^b g(t)\boldsymbol{\Delta} t$,

(ii) $\int_a^b (\alpha f)(t)\boldsymbol{\Delta} t = \alpha \int_a^b f(t)\Delta t$,

(iii) $\int_a^b f(t)\boldsymbol{\Delta} t = -\int_b^a f(t)\boldsymbol{\Delta} t$,

(iv) $\int_a^b f(t)\boldsymbol{\Delta} t = \int_a^c f(t)\Delta t + \int_c^b f(t)\boldsymbol{\Delta} t$,

(v) $\int_a^a f(t)\boldsymbol{\Delta} t = 0$,

(vi) if $f(t) \geq 0$ for all $a \leq t \leq b$, then $\int_a^b f(t)\boldsymbol{\Delta} t \geq 0$.

Definition 1.9. $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k = 0, 1, 2 \ldots$ are defined by

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau,\ \forall s,t\in\mathbb{T},$$

where $h_0(t, s) = 1$.

For more details about the calculus of time scales, we refer to [25].

Throughout this article, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$, while $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}_0$ denotes the set of nonnegative integers. For a function f and two integers m_0, m_1, we have $\sum_{s=m_0}^{m_1} f = 0$ provided $m_0 > m_1$. $\mathbb{T}_1, \mathbb{T}_2$ denote two arbitrary time scales, and for an interval $[a,b], [a,b]_{\mathbb{T}_i} := [a,b]\cap\mathbb{T}_i, i = 1,2$. Finally, for the sake of convenience, we denote the forward jump operators on $\mathbb{T}_1, \mathbb{T}_2$ by σ uniformly.

2. Main results

Theorem 2.1. Let $a,b\in\mathbb{T}_1, c,d\in\mathbb{T}_2, f :\in C_{rd}([a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2},\mathbb{R})$ such that the partial delta derivative of order 2 exists and there exists a constant K with $\sup_{a<s<b,\ c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\right| = K$. Suppose that $x_i\in[a,b]_{\mathbb{T}_1}, y_i\in[c,d]_{\mathbb{T}_2}, i = 0,1,\ldots,k$. $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a,b]_{\mathbb{T}_1}$, while $J_k : c = y_0 < y_1 < \ldots < y_{k-1} < y_k = d$ is a division of the interval $[c,d]_{\mathbb{T}_2}$. $\alpha_i\in[x_{i-1},x_i]_{\mathbb{T}_1}, \beta_i\in[y_{i-1},y_i]_{\mathbb{T}_2}, i = 1,2,\ldots,k$. Then we have

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)\,f(x_i,y_j) + \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)\\
&\quad-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\
&\quad+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j) + (x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k) - (x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\
&\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_i) - (x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k) + (x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\
&\quad-\sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s - \int_a^b[(y_k-\beta_k)f(\sigma(s),y_k)-(y_0-\beta_1)f(\sigma(s),y_0)]\Delta_1 s\\
&\quad-\sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i\sigma(t))\Delta_2 t - \int_c^d[(x_k-\alpha_k)f(x_k,\sigma(t))-(x_0-\alpha_1)f(x_0,\sigma(t))]\Delta_2 t\\
&\quad+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\Bigg|\\
&\quad\le K\left\{\sum_{i=0}^{k-1}[h_2(x_i,\alpha_{i+1})+h_2(x_{i+1},\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_j,\beta_{j+1})+h_2(y_{j+1},\beta_{j+1})]\right\}.
\end{aligned}\tag{1}$$

The inequality (1) is sharp in the sense that the right side of (1) can not be replaced by a smaller one.

Proof. Define

$$H(s,t,I_k,J_k) = (s-\alpha_{i+1})(t-\beta_{j+1}),\ (s,t)\in[x_i,x_{i+1})\times[y_j,y_{j+1}),\quad i,j = 0,1,\ldots,k-1. \tag{2}$$

Then we obtain

$$\begin{aligned}
&\int_a^b\int_c^d H(s,t,I_k,J_k)\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(s-\alpha_{i+1})(t-\beta_{j+1})\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\Delta_2 t\Delta_1 s\\
&=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}(s-\alpha_{i+1})\left[(y_{j+1}-\beta_{j+1})\frac{\partial f(s,y_{j+1})}{\Delta_1 s}-(y_j-\beta_{j+1})\frac{\partial f(s,y_j)}{\Delta_1 s}-\int_{y_j}^{y_{j+1}}\frac{\partial f(s,\sigma(t))}{\Delta_1 s}\Delta_2 t\right]\\
&\Delta_1 s\\
&=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\{[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_{j+1})-(x_i-\alpha_{i+1})f(x_i,y_{j+1})](y_{j+1}-\beta_{j+1})\\
&\quad-[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_j)-(x_i-\alpha_{i+1})f(x_i,y_j)](y_j-\beta_{j+1})\\
&\quad-\int_{x_i}^{x_{i+1}}[(y_{j+1}-\beta_{j+1})f(\sigma(s),y_{j+1})-(y_j-\beta_{j+1})f(\sigma(s),y_j)]\Delta_1 s\\
&\quad-\int_{y_j}^{y_{j+1}}[(x_{i+1}-\alpha_{i+1})f(x_{i+1},\sigma(t))-(x_i-\alpha_{i+1})f(x_i,\sigma(t))]\Delta_2 t\\
&\quad+\left.\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\right\}\\
&=\sum_{i=1}^{k-1}\sum_{j=0}^{k-1}(\alpha_{i+1}-\alpha_i)(y_{j+1}-\beta_{j+1})f(x_i,y_{j+1})+\sum_{j=0}^{k-1}(x_k-\alpha_k)(y_{j+1}-\beta_{j+1})f(x_k,y_{j+1})\\
&\quad-\sum_{j=0}^{k-1}(x_0-\alpha_1)(y_{j+1}-\beta_{j+1})f(x_0,y_{j+1})\\
&\quad-\sum_{i=1}^{k-1}\sum_{j=0}^{k-1}(\alpha_{i+1}-\alpha_i)(y_j-\beta_{j+1})f(x_i,y_j)-\sum_{j=0}^{k-1}(x_k-\alpha_k)(y_j-\beta_{j+1})f(x_k,y_j)\\
&\quad+\sum_{j=0}^{k-1}(x_0-\alpha_1)(y_j-\beta_{j+1})f(x_0,y_j)\\
&\quad-\sum_{j=0}^{k-1}\int_a^b[(y_{j+1}-\beta_{j+1})f(\sigma(s),y_{j+1})-(y_j-\beta_{j+1})f(\sigma(s),y_j)]\Delta_1 s\\
&\quad-\sum_{i=0}^{k-1}\int_c^d[(x_{i+1}-\alpha_{i+1})f(x_{i+1},\sigma(t))-(x_i-\alpha_{i+1})f(x_i,\sigma(t))]\Delta_2 t+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&=\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_i)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\
&\quad+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j)+(x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k)-(x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\
&\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j)-(x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k)+(x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\
&\quad-\sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s-\int_a^b[(y_k-\beta_k)f(\sigma(s),y_k)-(y_0-\beta_1)f(\sigma(s),y_0)]\Delta_1 s\\
&\quad-\sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i\sigma(t))\Delta_2 t-\int_c^d[(x_k-\alpha_k)f(x_k,\sigma(t))-(x_0-\alpha_1)f(x_0,\sigma(t))]\Delta_2 t\\
&\quad+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s.
\end{aligned}\tag{3}$$

On the other hand, we have

$$\begin{aligned}
&\int_a^b\int_c^d\left|H(s,t,I_k,J_k)\right|\Delta_2 t\Delta s=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}\left|(s-\alpha_{i+1})(t-\beta_{j+1})\right|\Delta_2 t\Delta_1 s\\
&=\left[\sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}\left|(s-\alpha_{i+1})\right|\Delta_1 s\right]\left[\sum_{j=0}^{k-1}\int_{y_j}^{y_{j+1}}\left|(t-\beta_{i+1})\right|\Delta_2 t\right]\\
&=\left\{\sum_{i=0}^{k-1}\left[\int_{x_i}^{\alpha_{i+1}}(\alpha_{i+1}-s)\Delta_1 s+\int_{\alpha_{i+1}}^{x_{i+1}}(s-\alpha_{i+1})\Delta_1 s\right]\right\}\left\{\sum_{j=0}^{k-1}\left[\int_{y_j}^{\beta_{j+1}}(\beta_{j+1}-t)\ \Delta_2 t+\int_{\beta_{j+1}}^{y_{j+1}}\right.\right.\\
&(t-\beta_{j+1})\Delta_2 t]\}\\
&=\sum_{i=0}^{k-1}[h_2(x_i,\alpha_{i+1})+h_2(x_{i+1},\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_j,\beta_{j+1})+h_2(y_{j+1},\beta_{j+1})].
\end{aligned}\tag{4}$$

Combining (3) and (4) we get the desired inequality (1).

In order to prove the sharpness of (1), we take $k = 1$, $\alpha_1 = b$, $\beta_1 = d$, $f(s, t) = st$. Then the left side of (1) becomes

$$\begin{aligned}
&\left|\int_a^b\int_c^d \sigma(s)\sigma(t)\Delta_2t\Delta_1s-(d-c)\int_a^b c\sigma(s)\Delta_1s-(b-a)\int_c^b a\sigma(t)\Delta_2t+(d-c)(b-a)ac\right|\\
&=\left|\int_a^b\int_c^d [\sigma(s)\sigma(t)-c\sigma(s)-a\sigma(t)+ac]\Delta_2t\Delta_1s\right|=\left|\int_a^b\int_c^d[\sigma(s)-a][\sigma(t)-c]\Delta_2t\Delta_1s\right|\\
&=\left|\int_a^b\int_c^d \{[(s-a)^2]_s^{\Delta}[(t-c)^2]_t^{\Delta}-[(\sigma(s)-a)(t-c)+(\sigma(t)-c)(s-a)+(s-a)(t-c)]\}\Delta_2t\Delta_1s\right|\\
&=\left|\int_a^b\int_c^d \{[(s-a)^2]_s^{\Delta}[(t-c)^2]_t^{\Delta}-[[(s-a)^2]_s^{\Delta}(t-c)+[(t-c)^2]_t^{\Delta}(s-a)-(t-c)(s-a)]\}\Delta_2t\Delta_1s\right|\\
&=\left|(b-a)^2(d-c)^2-(b-a)^2\int_c^d(t-c)\Delta_2t-(d-c)^2\int_a^b(s-a)\Delta_1s+\int_a^b\int_c^d(t-c)(s-a)\Delta_2t\Delta_1s\right|\\
&=\left|\int_a^b\int_c^d(t-d)(s-b)\boldsymbol{\Delta_2t\Delta_1s}\right|=\int_a^b\int_c^d(d-t)(b-s)\Delta_2t\Delta_1s=\int_b^a(s-b)\Delta_1s\int_d^c(t-d)\Delta_2t\\
&=h_2(a,b)h_2(c,d).
\end{aligned}$$

On the other hand, Using $K = 1$, the right side of (1) reduces to $h_2(a, b)h_2(c, d)$, which implies (2) holds for equality form, and then the sharpness of (1) is proved.

Remark 2.2. Theorem 2.1 is the 2D extension of [[24], Theorem 3].

From Theorem 2.1 we can obtain some particular Ostrowski type inequalities on time scales. For example, if we take $k = 1$, $\alpha_1 = b$, $\beta_1 = d$, then we have

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s-(d-c)\int_a^b f(\sigma(s),y_0)\Delta_1s-(b-a)\int_c^d f(x_0,\sigma(t))\Delta_2t+(d-c)(b-a)f(x_0,y_0)\right|\\
&\quad\le K\,h_2(a,b)h_2(c,d).
\end{aligned}$$

If we take $k = 1$, $\alpha_1 = a$, $\beta_1 = c$, then we have

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s-(d-c)\int_a^b f(\sigma(s),\mathbf{y_1})\Delta_1s-(b-a)\int_c^d f(\mathbf{x_1},\sigma(t))\Delta_2t+(d-c)(b-a)f(\mathbf{x_1},\mathbf{y_1})\right|\\
&\quad\le K\,h_2(b,a)h_2(d,c).
\end{aligned}$$

If we take $k = 1, \alpha_1 = \dfrac{a+b}{2}, \beta_1 = \dfrac{c+d}{2}$, then we have

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s-\frac{d-c}{2}\int_a^b[f(\sigma(s),\mathbf{y_1})+f(\sigma(s),y_0)]\Delta_1s-\frac{b-a}{2}\int_c^d[f(\mathbf{x_1},\sigma(t))+f(x_0,\sigma(t))]\Delta_2t\\
&\qquad+\frac{(b-a)(d-c)}{4}[f(\mathbf{x_1},\mathbf{y_1})+f(\mathbf{x_1},y_0)+f(x_0,\mathbf{y_1})+f(x_0,y_0)]\Bigg|\\
&\quad\le K\left[h_2\left(a,\frac{a+b}{2}\right)+h_2\left(b,\frac{a+b}{2}\right)\right]\left[h_2\left(c,\frac{c+d}{2}\right)+h_2\left(d,\frac{c+d}{2}\right)\right].
\end{aligned}$$

If we take $k = 2$, $\alpha_1 = a$, $\alpha_2 = b$, $\beta_1 = c$, $\beta_2 = d$, $x_1 = x$, $y_1 = y$, then we have

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2t\Delta_1s-(d-c)\int_a^b f(\sigma(s),y)\,\Delta_1s-(b-a)\int_c^d f(x,\sigma(t))\Delta_2t\\
&\quad+(b-a)(d-c)f(x,y)\Bigg|\\
&\quad\le K[h_2(x,a)+h_2(x,b)][h_2(y,c)+h_2(y,d)].
\end{aligned}$$

If we take $k = 2,\ \alpha_1 = \frac{a+x}{2}, \alpha_2 = \frac{x+b}{2}, \beta_1 = \frac{c+y}{2}, \beta_2 = \frac{y+d}{2}, x_1 = x, y_1 = y$, then we have

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(\sigma(s),\sigma(t))\,\Delta_2 t\Delta_1 s - \frac{(d-c)}{2}\int_a^b f(\sigma(s),y)\Delta_1 s - \frac{(b-a)}{2}\int_c^d f(x,\sigma(t))\Delta_2 t \\
&- \int_a^b\left[\frac{(d-y)}{2}f(\sigma(s),\mathbf{y}_2) + \frac{(y-c)}{2}f(\sigma(s),y_0)\right]\Delta_1 s - \int_c^d\left[\frac{(b-x)}{2}f(\mathbf{x}_2,\sigma(t)) + \frac{(x-a)}{2}f(x_0,\sigma(t))\right]\Delta_2 t \\
&+ \frac{(b-a)(d-c)}{4}f(x,y) + \frac{(b-a)(d-y)}{4}f(x,\mathbf{y}_2) + \frac{(b-a)(y-c)}{4}f(x,y_0) + \frac{(d-c)(b-x)}{4}f(\mathbf{x}_2,y) \\
&+ \frac{(x-a)(d-c)}{4}f(x_0,y) + \frac{(d-y)(b-x)}{4}f(\mathbf{x}_2,\mathbf{y}_2) + \frac{(y-c)(b-x)}{4}f(\mathbf{x}_2,y_0) \\
&+ \frac{(d-y)(x-a)}{4}f(x_0,\mathbf{y}_2) + \frac{(y-c)(x-a)}{4}f(x_0,y_0)\Bigg| \\
&\le K\left[h_2\left(a,\frac{a+x}{2}\right) + h_2\left(x,\frac{a+x}{2}\right) + h_2\left(x,\frac{x+b}{2}\right) + h_2\left(b,\frac{x+b}{2}\right)\right]\times \\
&\left[h_2\left(c,\frac{c+y}{2}\right) + h_2\left(y,\frac{c+y}{2}\right) + h_2\left(y,\frac{y+d}{2}\right) + h_2\left(d,\frac{y+d}{2}\right)\right].
\end{aligned}$$

If we furthermore take $x = \frac{b+a}{2}, y = \frac{d+c}{2}$ in the inequality above, then we obtain the time scale version of Simpson's inequality [26], which is omitted here.

In Theorem 2.1, if we take $\mathbb{T}_1, \mathbb{T}_2$ for some special time scales, then we immediately obtain the following three corollaries.

Corollary 2.3 (Continuous case). Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 2.1, then $h_2(t,s) = \frac{(t-s)^2}{2}$, and we obtain

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j) + \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k) \\
&- \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0) \\
&+ \sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j) + (x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k) - (x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0) \\
&- \sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j) - (x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k) + (x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0) \\
&- \sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(s,y_j)ds - \int_a^b[(y_k-\beta_k)f(s,y_k) - (y_0-\beta_1)f(s,y_0)]ds \\
&- \sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,t)dt - \int_c^d[(x_k-\alpha_k)f(x_k,t) - (x_0-\alpha_1)f(x_0,t)]dt + \int_a^b\int_c^d f(s,t)dtds\Bigg| \\
&\le K\left\{\sum_{i=0}^{k-1}\left[\frac{(x_i-\alpha_{i+1})^2}{2} + \frac{(x_{i+1}-\alpha_{i+1})^2}{2}\right]\sum_{j=0}^{k-1}\left[\frac{(y_i-\beta_{j+1})^2}{2} + \frac{(y_{j+1}-\beta_{j+1})^2}{2}\right]\right\},
\end{aligned} \tag{5}$$

where $K = \sup_{a<s<b,\ c<t<d}\left|\frac{\partial^2 f(s,t)}{\partial s\partial t}\right|$.

Corollary 2.4 (Discrete case). Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z},\ a = m_1,\ b = m_2,\ c = n_1,\ d = n_2$ in Theorem 2.1. Suppose that $x_i \in [m_1, m_2]_{\mathbb{Z}}$, $y_i \in [n_1, n_2]_{\mathbb{Z}}$, $i = 0,1, \ldots, k$. $I_k : m_1 = x_0 < x_1 < \ldots < x_{k-1} < x_k = m_2$ is a division of $[m_1, m_2]_{\mathbb{Z}}$, while $J_k : n_1 = y_0 < y_1 < \ldots < y_{k-1} < y_k = n_2$ is a division of $[n_1, n_2]_{\mathbb{Z}}$. $\alpha_i \in [x_{i-1}, x_i]_{\mathbb{Z}}$, $\beta_i \in [y_{i-1}, y_i]_{\mathbb{Z}}$, $i = 1,2,\ldots,k$. Then we have

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)\\
&-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\
&\quad+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j)+(x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k)-(x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\
&\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j)-(x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k)+(x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\
&\quad-\sum_{j=1}^{k-1}\sum_{s=m_1+1}^{m_2}(\beta_{j+1}-\beta_j)f(s,y_j)-\sum_{s=m_1+1}^{m_2}[(y_k-\beta_k)f(s,y_k)-(y_0-\beta_1)f(s,y_0)]\\
&\quad-\sum_{i=1}^{k-1}\sum_{t=n_1+1}^{n_2}(\alpha_{i+1}-\alpha_i)f(x_i,t)-\sum_{t=n_1+1}^{n_2}[(x_k-\alpha_k)f(x_k,t)-(x_0-\alpha_1)f(x_0,t)]+\sum_{s=m_1+1}^{m_2}\sum_{t=n_1+1}^{n_2}f(s,t)\Bigg|\\
&\le K\left\{\sum_{i=0}^{k-1}\left[\frac{(x_i-\alpha_{i+1})(x_i-\alpha_{i+1}-1)}{2}+\frac{(x_{i+1}-\alpha_{i+1})(x_{i+1}-\alpha_{i+1}-1)}{2}\right]\times\right.\\
&\quad\left.\sum_{j=0}^{k-1}\left[\frac{(y_j-\beta_{j+1})(y_j-\beta_{j+1}-1)}{2}+\frac{(y_{j+1}-\beta_{j+1})(y_{j+1}-\beta_{j+1}-1)}{2}\right]\right\},
\end{aligned}\tag{6}$$

where K denotes the maximum value of the absolute value of the difference $\Delta_1\Delta_2 f$ over $[m_1, m_2\text{-}1]_{\mathbb{Z}} \times [n_1, n_2\text{-}1]_{\mathbb{Z}}$.

As long as we notice $h_2(t,s)=\dfrac{(t-s)(t-s-1)}{2}$ for $\forall t,\ s\in\mathbb{Z}$, we can easily get the desired result.

Corollary 2.5 (Quantum calculus case). Let $\mathbb{T}_1=q_1^{\mathbb{N}_0}, \mathbb{T}_2=q_2^{\mathbb{N}_0}$ in Theorem 2.1, where m_1, m_2, n_1, n_2 $\in$ $\mathbb{N}_0$ and $q_i > 1$, $i = 1, 2$. Suppose that $x_i\in[q_1^{m_1},q_1^{m_2}]_{q_1^{\mathbb{N}_0}}, y_j\in[q_2^{n_1},q_2^{n_2}]_{q_2^{\mathbb{N}_0}}, i=0,1,\ldots,k.$ $I_k: q_1^{m_1}=x_0<x_1<\cdots<x_{k-1}<x_k=q_1^{m_2}$ is a division of $[q_1^{m_1},q_1^{m_2}]_{\mathbb{N}_0}$, while $J_k: q_2^{n_1}=y_0<y_1<\cdots<y_{k-1}<y_k=q_2^{n_2}$ is a division of $[q_2^{n_1},q_2^{n_2}]_{q^{\mathbb{N}_0}}\alpha_i\in[x_{i-1},x_i]_{q_1^{\mathbb{N}_0}}, \beta_i\in[y_{i-1},y_j]_{q_2^{\mathbb{N}_0}}, i=1,2,\ldots,k$. Then we have

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)\\
&-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\
&\quad+\sum_{j=0}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j)+(x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k)-(x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\
&\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j)-(x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k)+(x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\
&\quad-\mathbf{q}_1^{\mathbf{m}_1}(q_1-1)\left\{\sum_{j=1}^{k-1}\sum_{s=m_1}^{m_2-1}q_1^{s-m_1}(\beta_{j+1}-\beta_j)f(q^{s+1},y_j)-\sum_{s=m_1}^{m_2-1}q_1^{s-m_1}[(y_k-\beta_k)f(q^{s+1},y_k)\right.\\
&\quad\left.-(y_0-\beta_1)f(q^{s+1},y_0)]\right\}\\
&\quad-\mathbf{q}_2^{\mathbf{n}_1}(q_2-1)\left\{\sum_{i=1}^{k-1}\sum_{t=n_1}^{n_2-1}q_2^{t-n_1}(\alpha_{i+1}-\alpha_i)f(x_i,q^{t+1})-\sum_{t=n_1}^{n_2-1}q_2^{t-n_1}[(x_k-\alpha_k)f(x_k,q^{t+1})\right.\\
&\quad\left.-(x_0-\alpha_1)f(x_0,q^{t+1})]\right\}\\
&\quad+\mathbf{q}_1^{\mathbf{m}_1}\mathbf{q}_2^{\mathbf{n}_1}(q_1-1)(q_2-1)\sum_{s=m_1}^{m_2-1}\sum_{t=n_1}^{n_2-1}q_1^{s-m_1}q_2^{t-n_1}f(q^{s+1},q^{t+1})\Bigg|\\
&\le K\left\{\sum_{i=0}^{k-1}\left[\frac{(x_i-\alpha_{i+1})(x_i-q_1\alpha_{i+1})}{1+q_1}+\frac{(x_{i+1}-\alpha_{i+1})(x_{i+1}-q_1\alpha_{i+1})}{1+q_1}\right]\times\right.\\
&\quad\left.\sum_{j=0}^{k-1}\left[\frac{(y_j-\beta_{j+1})(y_j-q_2\beta_{j+1})}{1+q_2}+\frac{(y_{j+1}-\beta_{j+1})(y_{j+1}-q_2\beta_{j+1})}{1+q_2}\right]\right\},
\end{aligned}\tag{7}$$

where K denotes the maximum value of the absolute value of the $q_1\ q_2$-difference $D_{q_1q_2}f(t,s)$ over $\left[q_1^{m_1}, q_1^{m_2-1}\right]_{q_1^{\mathbb{N}_0}} \times [q_2^{n_1}, q_2^{n_2-1}]_{q_2^{\mathbb{N}_0}}$.

Proof. Since $h_k(t,s) = \prod_{n=0}^{k-1} \frac{t - q_i^n s}{\sum_{\mu=0}^{n} q_i^{\mu}}$ for $\forall s,t \in q_i^{\mathbb{N}_0}, i = 1,2$, we have

$$h_2(t,s) = \frac{(t-s)(t-q_i s)}{1+q_i}, \quad i = 1,2. \tag{8}$$

Substituting (8) into (1) we get the desired result.

Theorem 2.6. Under the conditions of Theorem 2.1, if there exist constants K_1, K_2 such that $K_1 \le \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \le K_2$, then we have the following inequality

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j) + \sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)\\
&-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\
&\quad+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j) + (x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k) - (x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\
&\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j) - (x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k) + (x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\
&\quad-\sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s - \int_a^b[(y_k-\beta_k)f(\sigma(s),y_k)-(y_0-\beta_1)f(\sigma(s),y_0)]\Delta_1 s\\
&\quad-\sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i\sigma(t))\Delta_2 t - \int_c^d[(x_k-\alpha_k)f(x_k,\sigma(t))-(x_0-\alpha_1)f(x_0,\sigma(t))]\Delta_2 t\\
&\quad+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\
&-\frac{K_1+K_2}{2}\sum_{i=0}^{k-1}[h_2(x_{i+1},\alpha_{i+1})-h_2(x_i,\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_{j+1},\beta_{j+1})-h_2(y_j,\beta_{j+1})]\Bigg|\\
&\le\frac{K_2-K_1}{2}\left\{\sum_{i=0}^{k-1}[h_2(x_i,\alpha_{i+1})+h_2(x_{i+1},\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_j,\beta_{j+1})+h_2(y_{j+1},\beta_{j+1})]\right\}.
\end{aligned}\tag{9}$$

Proof. We notice that

$$\begin{aligned}
\int_a^b\int_c^d H(s,t,I_k,J_k)\Delta_2 t\Delta_1 s &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(s-\alpha_{i+1})(t-\beta_{j+1})\Delta_2 t\Delta_1 s\\
&= \sum_{i=0}^{k-1}\int_{x_i}^{x_{i+1}}(s-\alpha_{i+1})\Delta_1 s\sum_{j=0}^{k-1}\int_{y_j}^{y_{j+1}}(t-\beta_{j+1})\Delta_2 t \qquad (10)\\
&= \sum_{i=0}^{k-1}[h_2(x_{i+1},\alpha_{i+1})-h_2(x_i,\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_{j+1},\beta_{j+1})-h_2(y_j,\beta_{j+1})],
\end{aligned}$$

We also have $\left|\dfrac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} - \dfrac{K_1+K_2}{2}\right| \le \dfrac{K_2-K_1}{2}$, and

$$\begin{aligned}&\left|\int_a^b\int_c^d H(s,t,I_k,J_k)\left(\frac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}-\frac{K_1+K_2}{2}\right)\Delta_2 t\Delta_1 s\right|\\ &\le \frac{K_2-K_1}{2}\int_a^b\int_c^d \left|H(s,t,I_k,J_k)\right|\Delta_2 t\Delta_1 s.\end{aligned} \tag{11}$$

Then combining (4), (10) and (11) we obtain the desired inequality (9).

Remark 2.7. If we take $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$, $k=2$, $\alpha_1=a+\lambda\frac{b-a}{2}$, $\alpha_2=b-\lambda\frac{b-a}{2}$, $\beta_1=c+\lambda\frac{d-c}{2}$, $\beta_2=d-\lambda\frac{d-c}{2}$, $x_1=x$, $y_1=y$, where $\lambda\in[0,1]$, then Theorem 2.6 reduces to [[11], Theorem 4].

In the following, we will establish a generalized Ostrowski-Grüss type integral inequality on time scales based on the result of Theorem 2.1.

Lemma 2.8 (2D Grüss' inequality on time scales). Let $f,g\in C_{rd}([a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2},\mathbb{R})$ such that $\varphi\le f(x,y)\le\Phi$ and $\gamma\le g(x,y)\le\Gamma$ for all $x\in[a,b]_{\mathbb{T}_1}, y\in[c,d]_{\mathbb{T}_2}$, where φ, Φ, γ, Γ are constants. Then we have

$$\begin{aligned}&\left|\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(s,t)g(s,t)\Delta_2 t\Delta_1 s-\frac{1}{(d-c)(b-a)}\int_a^b\int_c^d f(s,t)\Delta_2 t\Delta_1 s\frac{1}{(d-c)(b-a)}\right.\\ &\left.\int_a^b\int_c^d g(s,t)\Delta_2 t\Delta_1 s\right|\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma).\end{aligned} \tag{12}$$

The proof for Lemma 2.6 is similar to [[27], pp. 295-296], and we omit it here.

Theorem 2.9. Under the conditions of Theorem 2.1, if there exist constants K_1, K_2 such that $K_1\le\dfrac{\partial^2 f(s,t)}{\Delta_1 s\Delta_2 t}\le K_2$, then we have the following inequality

$$\begin{aligned}&\left|\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_k-\beta_k)f(x_i,y_k)\right.\\ &\quad-\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)(y_0-\beta_1)f(x_i,y_0)\\ &\quad+\sum_{j=1}^{k-1}(x_k-\alpha_k)(\beta_{j+1}-\beta_j)f(x_k,y_j)+(x_k-\alpha_k)(y_k-\beta_k)f(x_k,y_k)-(x_k-\alpha_k)(y_0-\beta_1)f(x_k,y_0)\\ &\quad-\sum_{j=1}^{k-1}(x_0-\alpha_1)(\beta_{j+1}-\beta_j)f(x_0,y_j)-(x_0-\alpha_1)(y_k-\beta_k)f(x_0,y_k)+(x_0-\alpha_1)(y_0-\beta_1)f(x_0,y_0)\\ &\quad-\sum_{j=1}^{k-1}\int_a^b(\beta_{j+1}-\beta_j)f(\sigma(s),y_j)\Delta_1 s-\int_a^b[(y_k-\beta_k)f(\sigma(s),y_k)-(y_0-\beta_1)f(\sigma(s),y_0)]\Delta_1 s\\ &\quad-\sum_{i=1}^{k-1}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,\sigma(t))\Delta_2 t-\int_c^d[(x_k-\alpha_k)f(x_k,\sigma(t))-(x_0-\alpha_1)f(x_0,\sigma(t))]\Delta_2 t\\ &\quad+\int_a^b\int_c^d f(\sigma(s),\sigma(t))\Delta_2 t\Delta_1 s\\ &-\frac{[f(b,d)-f(b,c)-f(a,d)+f(a,c)]}{(b-a)(d-c)}\sum_{i=0}^{k-1}[h_2(x_{i+1},\alpha_{i+1})-h_2(x_i,\alpha_{i+1})]\sum_{j=0}^{k-1}[h_2(y_{j+1},\beta_{j+1})\\ &\left.-h_2(y_j,\beta_{j+1})]\right|\\ &\le\frac{[(b-a)(d-c)]^2}{4}(K_2-K_1).\end{aligned} \tag{13}$$

Proof. From the definition of $H(s, t, I_k, J_k)$ in (2) we observe that $max(H(s, t, I_k, J_k))$ - $min(H(s, t, I_k, J_k)) \leq (b - a)(d - c)$, and on the other hand,

$$\int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s = f(b,d) - f(b,c) - f(a,d) + f(a,c). \tag{14}$$

So by Lemma 2.8 we have

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d H(s,t,I_k,J_k) \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right. \\ &\quad \left. - \frac{1}{[(b-a)(d-c)]^2} \int_a^b \int_c^d H(s,t,I_k,J_k) \Delta_2 t \Delta_1 s \int_a^b \int_c^d \frac{\partial^2 f(s,t)}{\Delta_1 s \Delta_2 t} \Delta_2 t \Delta_1 s \right| \\ &\leq \frac{[\max(H(s,t,I_k,J_k)) - \min(H(s,t,I_k,J_k))]}{4} (K_2 - K_1) \leq \frac{(b-a)(d-c)}{4} (K_2 - K_1). \end{aligned} \tag{15}$$

Then combining (3), (10), (14), (15) we get the desired result.

Remark 2.10. If we take $k = 2$, then Theorem 2.9 becomes the 2D extension on time scales of [[20], Theorem 4]. If we take $k = 2, \mathbb{T} = \mathbb{R}$, then Theorem 2.9 becomes the 2D extension in the continuous case of [[12], Theorem 2.1].

Remark 2.11. For Theorems 2.6 and 2.9, we can also obtain similar results as shown in Corollaries 2.3-2.5, which are omitted here.

3. Conclusions

In this article, we establish some new generalized Ostrowski type inequalities on time scales involving functions of two independent variables for k^2 points, which unify continuous and discrete analysis. We note that the presented inequalities in Theorems 2.6 and 2.9 are not sharp, and the sharp version of them are supposed to further research.

Acknowledgements
This study was partially supported by the National Natural Science Foundation of China (11171178), Natural Science Foundation of Shandong Province (ZR 2009 AM011) (China) and Specialized Research Fund for the Doctoral Program of Higher Education (20103705110003)(China). The authors thank the referees very much for their careful comments and valuable suggestions on this article.

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Authors' contributions
QF carried out the main part of this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 14 December 2011 Accepted: 29 March 2012 Published: 29 March 2012

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doi:10.1186/1029-242X-2012-74
Cite this article as: Feng and Meng: **Generalized Ostrowski type inequalities for multiple points on time scales involving functions of two independent variables.** *Journal of Inequalities and Applications* 2012 **2012**:74.