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Full invariants on certain class of abstract dual systems

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Abstract
In this article, we study the convergence of sequences of operators on some classical vector-valued sequence spaces in the frame of abstract dual systems. Several consequences of full invariants are obtained.

Keywords: sequential-evaluation convergence, abstract dual system, full invariant, uniformly exhaustive, essentially bounded

1 Introduction

A dual system in linear analysis consists of a linear space and a collection of linear functionals defined on the linear space. Theory of locally convex spaces is exactly about theory of dual systems which plays a crucial role in many fields of mathematical analysis. In fact, many people working in different fields of mathematics have been devoting themselves on the research of some special dual systems such as measure system $(\Sigma, Ca(\Sigma, X))$, abstract function system $(\Omega, C(\Omega, X))$ and operator system $(X, L(X, Y))$ as well as fuzzy system $(U, F(U))$ etc.

In recent years, a few consequences have enlarged the known invariant ranges of boundedness, subseries convergence, and sequential-evaluation convergence etc., and theory of spaces has achieved substantial development. It is no doubtful that invariant principles make great influence on the trend of different fields of mathematical analysis.

This article is motivated by the problems considered by the authors in [1-7]. And the purpose of this article is to study full invariant of the convergence in $(Y^{l^p(X)}, l^p(X))$-topology on the dual system $(Y^{l^p(X)}, l^p(X))$, $(0 < p \leq \infty)$. We obtain that the above mentioned convergence is full invariant. Throughout the article, we adapt the definitions and notations from [8].

We denote (X, Y) is a dual system as usual.

Definition 1.1 [8]]. Let τ be a locally convex topology on X. If $(X, \tau)^* = Y$, then the topology τ is called a compatible topology on X with respect to the dual system (X, Y).

It is easy to know that the weakest compatible topology on X is the weak topology $\sigma(X, Y)$ on X and the strongest compatible topology is Mackey topology [1].

Definition 1.2 [8]. Let $\mathscr{B}$ be a family of $\sigma(Y, X)$-bounded sets of X, and let $\beta(X, Y)$ be a $\mathscr{B}$ convergent topology on X, $\sigma(X, Y)$ be the weak topology on X. If $\beta(X, Y) \supseteq \sigma(X, Y)$, then $\beta(X, Y)$ is called an admissible polar topology on X with respect to the dual system (X,Y).

It is not difficult to know that not every admissible polar topology $\beta(X,Y)$ is a compatible topology on X wrt the dual system (X, Y).

Definition 1.3 [9]. If a property P of X is shared by all admissible polar topologies lying between $\sigma(X,Y)$ and $\beta(X,Y)$, then P is called a full invariant.

We denote (E, F) an abstract dual system, where $E \neq \emptyset$, $F \subset X^E$ and X is a locally convex space.

A topology τ on E is called a (E, F)-topology if there exists $\mathscr{F} \subset 2^F$ such that $\cup_{M \in \mathscr{F}} M = F$ and τ is a topology of uniform convergence on each set $M \in \mathscr{F}$, which means that a sequence $\{u_n\} \subset (E,\tau)$ is said to converge to a point $u \in (E,\tau)$ if and only if for every $M \in \mathscr{F}$, $\{f(u_n)\}$ converges uniformly to $f(u)$ for all $f \in M$ [10]. We use $\sigma(E,F)$ to represent the weakest (E,F)-topology, i.e., in the topological space $(E,\sigma(E,F))$, a sequence $\{u_n\} \subset (E,\sigma(E,F))$ converges to a point $u \in (E, \sigma(E, F))$ if and only if for every $f \in F$, $\{f(u_n)\}$ converges to $f(u)$.

Similarly, for an abstract dual system (F,E), we can also give the concept of (F, E)-topology, denoted by $\tau^*(F, E)$. The weakest (F, E)-topology is denoted by $\sigma^*(F, E)$, called weak star topology. For the convenience, the sequence $\{f_n\}$ converges to f in $(F, \sigma^*(F, E))$ and in $(F, \tau^*(F, E))$ is denoted by $f_n \xrightarrow{\sigma^*(F,E)} f$ and $f_n \xrightarrow{\tau^*(F,E)} f$, respectively. In addition, (F, E)-topology $\beta^*(F,E)$ is called strong star topology if $f_n \xrightarrow{\beta^*(F,E)} f$ implies that $f_n \to f$ for all (F, E)-topology, say $\tau^*(F, E)$ on F. Obviously, $\beta^*(F, E)$ is the strongest (F, E)-topology on F. Full invariant of (F, E) are analog to those of (E,F).

2 Full invariant on the dual system $(Y^{l^p(X)}, l^p(X))$ (0< p <∞)

In the sequel, X and Y will always denote Banach spaces unless specified otherwise.

Let X^N be the set of all X-valued sequences,

$$l^p(X) = \left\{ (x_j) \in X^N : \sum_j \|x_j\|^p < +\infty \right\},$$

Y^X and $Y^{l^p(X)}$ represent the sets of Y - valued operators with domains X and $l^p(X)$, respectively. Let

$$l^p(X)^{\beta Y} = \left\{ (A_j) \subset Y^X : \sum_j A_j(x_j) \text{ is convergent for all } (x_j) \in l^p(X) \right\},$$

$l^p(X)^{\beta Y}$ is called the β-dual space of $l^p(X)$ [4].

Suppose that $(A_j) \in l^p(X)^{\beta Y}$, for each $(x_j) \in l^p(X)$, we define $f_{(A_j),n} : l^p(X) \to Y, n \in N$ and $f_{(A_j)} : l^p(X) \to Y$ by

$$f_{(A_j),n}\left[(x_j)\right] = \sum_{1\le j\le n} A_j(x_j),$$

and

$$f_{(A_j)}\left[(x_j)\right] = \sum_j A_j(x_j).$$

Obviously, $\{f_{(A_j),n}\} \subset Y^{l^p(X)}$, $f_{(A_j)} \in Y^{l^p(X)}$ and

$$f_{(A_j),n} \xrightarrow{\sigma^*\left(Y^{l^p(X)},l^p(X)\right)} f_{(A_j)}\ (n \to \infty)$$

in the dual system $\left(Y^{l^p(X)},\ l^p(X)\right)$.

In the following, we shall first characterize a collection of sets in $l^p(X)$ generating strong star topology $\beta^*\left(Y^{l^p(X)},\ l^p(X)\right)$.

Definition 2.1 [2]. If $M \subset l^p(X)$ and $\lim_{n\to\infty} \sum_{j\ge n} \|x_j\|^p = 0$ uniformly for $(x_j) \in M$, that is, for all $\varepsilon > 0$, there exists $j_\varepsilon \in N$ such that

$$\sup_{(x_j)\in M} \sum_{j\ge j_\varepsilon} \|x_j\|^p \le \varepsilon,$$

then M is called uniformly exhaustive set.

Denote $M[l^p(X)] = \{M \subset l^p(X) : M \text{ is uniformly exhaustive}\}$.

Lemma 2.1 [2]. Let $M \subset l^p(X)$, $0 < p < \infty$, $(A_j) \in l^p(X)^{\beta Y}$. If $\Sigma_j A_j(x_j)$ is uniformly convergent for all $(x_j) \in M$, then M is uniformly exhaustive.

Theorem 2.1. Let $\left(Y^{l^p(X)},\ l^p(X)\right)$ be a dual system, $\tau^*\left(Y^{l^p(X)},\ l^p(X)\right)$ be a topology on $Y^{l^p(X)}$ generated by $\mathscr{M}[l^p(X)]$. Then for every $(A_j) \in l^p(X)^{\beta Y}$, $f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^p(X)},l^p(X))} f_{(A_j)}\ (n \to \infty)$ if and only if

$$f_{(A_j),n} \xrightarrow{\tau^*\left(Y^{l^p(X)},l^p(X)\right)} f_{(A_j)}(n \to \infty).$$

Proof Necessity. Otherwise, we suppose that

$$f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^p(X)},l^p(X))} f_{(A_j)}\ (n \to \infty)$$

but

$$f_{(A_j),n} \xrightarrow{\tau^*\left(Y^{l^p(X)},l^p(X)\right)} f_{(A_j)}$$

doesn't hold. By the definition of $\left(Y^{l^p(X)},\ l^p(X)\right)$-topology, there must exist $M \in \mathscr{M}[l^p(X)]$ such that $\lim_n f_{(A_j),n}\left[(x_j)\right] = f_{(A_j)}\left[(x_j)\right]$ not uniformly for all $(x_j) \in M$, i.e., the series $\Sigma_j A_j(x_j)$ is not uniformly convergent with respect to all $(x_j) \in M$. Thus, there is $\varepsilon > 0$ such that for any $m_0 \in N$, there exist $m > m_0$ and $(x_j) \in M$ satisfying

$$\left\|\sum_{j\ge m} A_j(x_j)\right\| \ge 2\varepsilon.$$

Since $(A_j) \in l^p(X)^{\beta Y}$, $\Sigma_j A_j(x_j)$ is convergent and there exists $n > m$ such that

$$\left\|\sum_{j\ge n+1} A_j(x_j)\right\| < \varepsilon.$$

Hence

$$\left\|\sum_{m\le j\le n} A_j(x_j)\right\| = \left\|\sum_{j\ge m} A_j(x_j) - \sum_{j\ge n+1} A_j(x_j)\right\| \ge \left\|\sum_{j\ge m} A_j(x_j)\right\| - \left\|\sum_{j\ge n+1} A_j(x_j)\right\| > 2\varepsilon - \varepsilon = \varepsilon.$$

Since M is uniformly exhaustive, there exists $j_1 \in N$ such that for every $(x_j) \in M$,

$$\sum_{j\ge j_1} \|x_j\|^p < \frac{1}{2}.$$

Thus, there exist integers $n_1 > m_1 > j_1$ and $(x_{1j}) \in M$ such that

$$\left\|\sum_{m_1\le j\le n_1} A_j(x_{1j})\right\| > \varepsilon.$$

Again using the uniform exhaustion of M, we can get $j_2 > n_1$ such that for every $(x_j) \in M$,

$$\sum_{j\ge j_2} \|x_j\|^p < \frac{1}{2^2}.$$

So there exist $n_2 > m_2 > j_2$ and $(x_{2j}) \in M$ such that

$$\left\|\sum_{m_2\le j\le n_2} A_j(x_{2j})\right\| > \varepsilon.$$

In the same procedure, we can get a sequence of integers $m_1 < n_1 < m_2 < n_2 <\ldots$ and $\{(x_{kj})_{j=1}^{\infty} : k \in N\} \subset M$: satisfying

$$\sum_{m_k\le j\le n_k} \|x_{kj}\|^p \le \sum_{j\ge j_k} \|x_{kj}\|^p < \frac{1}{2^k}$$

and

$$\left\|\sum_{m_k\le j\le n_k} A_j(x_{kj})\right\| > \varepsilon,\ k = 1, 2, \ldots$$

Let

$$y_j = \begin{cases} x_{kj}, & m_k \le j \le n_k, k = 1, 2, \ldots, \\ 0, & \text{otherwise}. \end{cases}$$

Then $(y_j) \in l^p(X)$. In fact,

$$\sum_j \|y_j\|^p = \sum_k \sum_{m_k\le j\le n_k} \|x_{kj}\|^p < \sum_k \frac{1}{2^k} = 1$$

However

$$\left\| \sum_{m_k \le j \le n_k} A_j(y_j) \right\| = \left\| \sum_{m_k \le j \le n_k} A_j(x_{kj}) \right\| > \varepsilon, \ k = 1, 2, \ldots.$$

This is a contradiction with $(A_j) \in l^p(X)^{\beta Y}$.

Sufficiency. It is trivial, we omit it.

Remark. For any $(Y^{l^p(X)}, l^p(X))$-topology, let $\mathcal{N}[l^p(X)]$ be a collection of sets generating the topology. Then by Lemma 2.1 we can show that for any $N \in \mathcal{N}[l^p(X)]$, N is uniformly exhaustive, that is, $\mathcal{N}[l^p(X)] \subset \mathcal{M}[l^p(X)]$. So $\tau^*(Y^{l^p(X)}, l^p(X))$-topology is exactly the strongest $\beta^*(Y^{l^p(X)}, l^p(X))$-topology. Therefore, the following consequence of Theorem 2.1 is immediate.

Corollary 2.1. Let $(Y^{l^p(X)}, l^p(X))$ be a dual system, $\tau^*(Y^{l^p(X)}, l^p(X))$ a $(Y^{l^p(X)}, l^p(X))$-topology on $Y^{l^p(X)}$ generated by $\mathscr{M}[l^p(X)]$,$(A_j) \in l^p(X)^{\beta Y}$, then $f_{(A_j),n} \xrightarrow{\tau^*(Y^{l^p(X)}, l^p(X))} f_{(A_j)} (n \to \infty)$ if and only if

$$f_{(A_j),n} \xrightarrow{\beta^*(Y^{l^p(X)}, l^p(X))} f_{(A_j)} (n \to \infty).$$

The Corollary 2.1 shows that the convergence of $\tau^*(Y^{l^p(X)}, l^p(X))$ is equivalent to that of $\beta^*(Y^{l^p(X)}, l^p(X))$. Combining this with Theorem 2.1, we obtain the next consequence which means that the convergence of $\{f_{(A_j),n}\}$ is full invariant.

Theorem 2.2. Let $(Y^{l^p(X)}, l^p(X))$ be a dual system, $\tau^*(Y^{l^p(X)}, l^p(X))$ be a $(Y^{l^p(X)}, l^p(X))$-topology on $Y^{l^p(X)}$ generated by $\mathscr{M}[l^p(X)]$, $(A_j) \in l^p(X)^{\beta Y}$, then $f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^p(X)}, l^p(X))} f_{(A_j)} (n \to \infty)$ if and only if

$$f_{(A_j),n} \xrightarrow{\beta^*(Y^{l^p(X)}, l^p(X))} f_{(A_j)} (n \to \infty).$$

3 Full invariant on the dual system $(Y^{l^\infty(X)}, l^\infty(X))$

Let X^N be the set of all X-valued sequences,

$$l^\infty(X) = \left\{ (x_j) \in X^N : \sup_{j \in N} \|x_j\| < +\infty \right\}$$

and Y^X, $Y^{l^\infty(X)}$ represent the sets of Y-valued operators with domains X and $l^\infty(X)$, respectively.

Let

$$l^\infty(X)^{\beta Y} = \left\{ (A_j) \subset Y^X : \sum_j A_j(x_j) \text{ is convergent for all } (x_j) \in l^\infty(X) \right\},$$

then $l^\infty(X)^{\beta Y}$ is called the β-dual space of $l^\infty(X)$ [4].

Suppose that $(A_j) \in l^\infty(X)^{\beta Y}$, for every $(x_j) \in l^\infty(X)$, we define $f_{(A_j),n} : l^\infty(X) \to Y$, $n \in N$ and $f_{(A_j)} : l^\infty(X) \to Y$by

$$f_{(A_j),n}\left[(x_j)\right] = \sum_{1\le j\le n} A_j(x_j)$$

and

$$f_{(A_j)}\left[(x_j)\right] = \sum_j A_j(x_j).$$

Obviously, $\{f_{(A_j),n}\} \subset Y^{l^\infty(X)}$, $f_{(A_j)} \in Y^{l^\infty(X)}$, and

$$f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$$

in the dual system $(Y^{l^\infty(X)}, l^\infty(X))$.

Next we shall first find a collection of sets in $l^\infty(X)$ generating strong star topology $\beta^*(Y^{l^\infty(X)}, l^\infty(X))$. And then we prove that $f_{(A_j),n} \xrightarrow{\beta^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}\ (n \to \infty)$. Furthermore, we specify that $f_{(A_j),n} \to f_{(A_j)}(n \to \infty)$ $(Y^{l^\infty(X)}, l^{\infty(X)})$-topology admit full invariant.

Definition 3.1 [2]. $M \subset l^\infty(X)$ is called essential bounded set if there is $j_0 \in N$ such that

$$\sup_{(x_j)\in M, j\ge jo} \|x_j\| < \infty$$

Denote $\mathscr{M}[l^\infty(X)] = \{M \subset l^\infty(X) : M$ is essential bounded$\}$.

Lemma 3.1 [2]. Let X and Y be Banach spaces, $M \subset l^\infty(X)$, $(A_j) \in l^\infty(X)^{\beta Y}$. If $\sum_j A_j(x_j)$ is uniformly convergent for all $(x_j) \in M$, then M is essential bounded.

Theorem 3.1. Let $(Y^{l^\infty(X)}, l^\infty(X))$ be a dual system, $\tau^*(Y^{l^\infty(X)}, l^\infty(X))$ be a $(Y^{l^\infty(X)}, l^\infty(X))$-topology on $Y^{l^\infty(X)}$ generated by $\mathscr{M}[l^\infty(X)]$, $(A_j) \in l^\infty(X)^{\beta Y}$, then $f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$ if and only if

$$f_{(A_j),n} \xrightarrow{\tau^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty).$$

Proof. Necessity. Otherwise, we suppose that

$$f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$$

but

$$f_{(A_j),n} \xrightarrow{\tau^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$$

doesn't hold. Then there must exist $M \in \mathscr{M}[l^\infty(X)]$ such that $\lim_n f_{(A_j),n}[(x_j)] = f_{(A_j)}[(x_j)]$ not uniformly for all $(x_j) \in M$, i.e., the series $\sum_j A_j\ (x_j)$ is not uniformly convergent with respect to $(x_j) \in M$. Thus, there exist $\varepsilon > 0$, a sequence of integers $m_1 < n_1 < m_2 < n_2 < \ldots$ and $\{(x_{kj}) : k \in N\} \subset M$ satisfying

$$\left\| \sum_{m_k\le j\le n_k} A_j(x_{kj}) \right\| \ge \varepsilon,\ k = 1, 2, \ldots.$$

Let

$$y_j = \begin{cases} x_{kj}, \ m_k \leq j \leq n_k, k = 1, 2, \ldots, \\ 0, \quad \text{otherwise}. \end{cases}$$

Since M is essential bounded, there is $j_0 \in N$ such that

$$\sup_{j \geq j_0} \|y_j\| \leq \sup_{(z_j) \in M, j \geq j_0} \|z_j\| < +\infty.$$

Then $(y_j) \in l^\infty(X)$. But we have

$$\left\| \sum_{m_k \leq j \leq n_k} A_j(y_i) \right\| = \left\| \sum_{m_k \leq j \leq n_k} A_j(x_{kj}) \right\| \geq \varepsilon, k = 1, 2, \ldots.$$

This is a contradiction with $(A_j) \in l^\infty(X)^{\beta Y}$.

Necessary. Trivial.

The Lemma 3.1 leads us to the following consequences.

Corollary 3.1. Let $(Y^{l^\infty(X)}, l^\infty(X))$ be a dual system, $\tau^*(Y^{l^\infty(X)}, l^\infty(X))$ be a $(Y^{l^\infty(X)}, l^\infty(X))$-topology on $Y^{l^\infty(X)}$ generated by $\mathscr{M}[l^\infty(X)]$, $(A_j) \in l^\infty(X)^{\beta Y}$, then $f_{(A_j),n} \xrightarrow{\tau^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$. if and only if

$$f_{(A_j),n} \xrightarrow{\beta^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)} \ (n \to \infty)$$

Theorem 3.2. Let $(Y^{l^\infty(X)}, l^\infty(X))$ be a dual system, $\tau^*(Y^{l^\infty(X)}, l^\infty(X))$ be a $(Y^{l^\infty(X)}, l^\infty(X))$-topology on $Y^{l^\infty(X)}$ generated by $\mathscr{M}[l^\infty(X)]$, $(A_j) \in l^\infty(X)^{\beta Y}$, then $f_{(A_j),n} \xrightarrow{\sigma^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty)$ if and only if

$$f_{(A_j),n} \xrightarrow{\beta^*(Y^{l^\infty(X)}, l^\infty(X))} f_{(A_j)}(n \to \infty).$$

4 Conclusion

In a frame of some certain class of dual system, we discuss a few problems on convergence of the sequences of operators in classical vector-valued sequence spaces and we obtain some consequences on full invariants. Precisely, sequences of operators in the spaces $Y^{l^p(X)}$ and $Y^{l^\infty(X)}$ are studied, respectively. We not only find the strongest sequential-evaluation convergence but also obtain a series of theorems about full invariants on the sequential evaluation convergence.

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Authors' contributions

FW conceived of the study. FW and CH participated in the design of the proof and drafted the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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