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The existence of solutions for general variational inequality and applications in FC spaces

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Abstract

In this article, we prove the existence of solutions for the general variational inequality $\phi(x, y) \geq f(x) - f(y)$ and Minty type theorem by using the generalized KKM theorem in topological spaces without linear structure. Some properties of solutions set for the general variational inequality are studied by Minty type theorem. As applications, equivalence between Browder fixed point theorem and Ky Fan's minimax inequality are studied in topological spaces.

Keywords: generalized R-KKM mapping, general variational inequality, minty type theorem, fixed point

1 Introduction

The existence of solutions and the properties of solutions set for general variational inequality were studied by many authors in linear topological spaces (see [1-7]). These results depend deeply on the linear structure of spaces. In 1987, Horvath [8] used the concept of retractile subset family to replace the convexity assumptions and established the KKM theorem in H -space. Motivated by Horvath's work, the various extensions of KKM theorem and their applications in many aspects were investigated by various authors under the assumption without convexity. Recently, Ding [9,10] and Deng and Xia [11] introduced the class of finitely continuous topological spaces (in short FC -spaces) without any convexity and linear structure. The class of FC -spaces includes the classes of H -spaces, G -convex spaces, G - H -convex spaces, L -convex spaces and many topological spaces with convexity structure. The purpose of this article is to study the existence of solutions for the following general variational inequality (in short VI(1.1))

$$\phi(x, y) \geq f(x) - f(y), \quad (1)$$

Minty type theorem and the properties of the solutions set for general variational inequality(1.1) in FC -spaces.

2 Preliminaries

Let E be a topological space, X be a nonempty subset of E . Throughout this article, we assume that $\langle X \rangle$ denotes all nonempty finite subset of X , Δ_n denotes the standard n -simplex (e_0, \dots, e_n) in R^{n+1} , Δ_k denotes k -sub-simplex of Δ_n with vectors $(e_{i_0}, \dots, e_{i_k})$, $(0 \leq k \leq n)$, $R = (-\infty, +\infty)$, $\bar{R} = [-\infty, +\infty]$. The following definitions can be found in [4,7,9,11].

Definition 2.1. (E, ϕ_N) is said to be a finitely continuous space (in short *FC*-space) if E is a topological space and for each $N = \{x_0, \dots, x_n\} \subseteq \langle E \rangle$, there exists a continuous mapping $\phi_N: \Delta_n \rightarrow E$. A subset X of (E, ϕ_N) is said to be an *FC*-subspace of E if for each $N = \{x_0, \dots, x_n\} \subseteq \langle E \rangle$ and for any $\{x_{i_0}, \dots, x_{i_k}\} \subset X \cap N$, $\phi_N(\Delta_k) \subset X$, where $\Delta_k = \text{co}\{e_j : j = 0, \dots, k\}$.

Definition 2.2. Let X be an *FC*-subspace of (E, ϕ_N) .

(1) $f: X \rightarrow R$ is said to be an *R*-convex function if for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, $f(x_t) \leq \sum_{i=0}^n t_i f(x_i)$, where $x_t = \phi_N\left(\sum_{i=0}^n t_i e_i\right)$, $t_i \in [0, 1]$, $\sum_{i=0}^n t_i = 1$.

(2) $f: X \rightarrow R$ is said to be a strictly *R*-convex function if for each $N = \{x_0, x_1\} \in \langle X \rangle$ and $x_0 \neq x_1$, $f(x_t) < t f(x_0) + (1-t)f(x_1)$, where $x_t = \phi_N(t e_0 + (1-t)e_1)$, $t \in (0, 1)$.

(3) $\phi: X \times X \rightarrow \bar{R}$ is said to be λ -quasi-convex related to the second variant if subset $\{y \in X; \phi(x, y) < \lambda\}$ is an *FC*-subspace for any $x \in X$. ϕ is said to be *R*-quasi-convex related to the second variant if ϕ is λ -quasi-convex related to the second variant for any $x \in X$ and $\lambda \in R$.

(4) $\phi: X \times X \rightarrow \bar{R}$ is said to be an *R*-semi-continuous function if for any $N = \{x_0, x_1\} \in \langle X \rangle$,

$$\overline{\lim}_{t \rightarrow 0^+} \phi(x_t, x_1) \leq \phi(x_0, x_1),$$

where $x_t = \phi_N((1-t)e_0 + t e_1)$, $t \in [0, 1]$.

Definition 2.3. Let X be a nonempty subset of E . A function $\phi: X \times X \rightarrow \bar{R}$ is said to be monotone if $\phi(x, y) + \phi(y, x) \leq 0$ for any $x, y \in X$. In addition, if ϕ also satisfies the condition $\phi(x, y) + \phi(y, x) = 0$ if and only if $x = y$, then ϕ is said to be strictly monotone.

Definition 2.4. Let X be a nonempty set and (Y, ϕ_N) be an *FC*-space. For $\lambda \in R$, a function $\phi: X \times Y \rightarrow \bar{R}$ is said to be λ -generalized *R*-diagonally quasi-convex (resp., *R*-diagonally quasi-concave) related to the first variant if, for each $\{x_0, \dots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \dots, y_n\} \subseteq \langle Y \rangle$ such that, for each subset $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$ and $\max_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \geq \lambda$, (resp. $\min_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \leq \lambda$),

$$\max_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \geq \lambda, \quad (\text{resp. } \min_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \leq \lambda).$$

where Δ_k denotes k -sub-simplex of with vectors $(e_{i_0}, \dots, e_{i_k})$, $(0 \leq k \leq n)$ related to $\{y_{i_0}, \dots, y_{i_k}\}$.

Definition 2.5. Let X be a nonempty set and (Y, ϕ_N) be an *FC*-space. A mapping $T: X \rightarrow 2^Y$ is said to be a generalized *R*-KKM mapping if, for each finite subset $\{x_0, \dots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that, for each subset $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$, $\phi_N(\Delta_k) \subset \cup_{j=0}^k T(x_{i_j})$.

Lemma 2.1. ϕ is λ -generalized *R*-diagonally quasi-convex (resp., *R*-diagonally quasi-concave) related to the first variant if and only if the mapping $G: X \rightarrow 2^Y$ defined by

$$G(x) = \{y \in Y; \phi(x, y) \geq \lambda\}, \quad (\text{resp. } G(x) = \{y \in Y; \phi(x, y) \leq \lambda\})$$

is a generalized *R*-KKM mapping.

Proof If ϕ is λ -generalized *R*-diagonally quasi-convex related to the first variant, then for each $\{x_0, \dots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that, for each subset

$\{i_0, \dots, i_k\} \in \{0, \dots, n\}$ and $\bar{y} \in \varphi_N(\Delta_k)$, $\max_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \geq \lambda$. Hence, there exists some $j_0 \in \{0, \dots, k\}$ such that $\phi(x_{i_{j_0}}, \bar{y}) \geq \lambda$, i.e., $\bar{y} \in G(x_{i_{j_0}})$, therefore $\varphi_N(\Delta_k) \subset \cup_{j=0}^k G(x_{i_j})$, i.e., G is a generalized R-KKM mapping.

Conversely, suppose that G is a generalized R-KKM mapping. Then for each $\{x_0, \dots, x_n\} \in \langle X \rangle$, there exists $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ such that, for each subset $\{i_0, \dots, i_k\} \subset \{0, \dots, n\}$, $\varphi_N(\Delta_k) \subset \cup_{j=0}^k G(x_{i_j})$. Thus for any $\bar{y} \in \varphi_N(\Delta_k)$, there exists some $j_0 \in \{0, \dots, k\}$ such that $\bar{y} \in G(x_{i_{j_0}})$, i.e., $\phi(x_{i_{j_0}}, \bar{y}) \geq \lambda$. It follows that $\max_{0 \leq j \leq k} \phi(x_{i_j}, \bar{y}) \geq \lambda$. This completes the proof.

Definition 2.6. Let X, Y be two nonempty subsets of E . For $\lambda \in R$, a function $\phi : X \times Y \rightarrow \bar{R}$ is said to be λ -transfer compactly lower (resp., upper) semi-continuous relate to the second variant if, for each nonempty compact subset $K \subset Y$ and each $y \in K$, there exists $x \in X$ such that $\phi(x, y) > \lambda$ (resp., $\phi(x, y) < \lambda$) implies that there exists an open neighborhood $U(y)$ of y in K and a point $\bar{x} \in X$ such that $\phi(\bar{x}, z) > \lambda$ (resp., $\phi(\bar{x}, z) < \lambda$) for all $z \in U(y)$.

Definition 2.7. Let X, Y be two nonempty subsets of E . A mapping $T : X \rightarrow 2^Y$ is said to be transfer compactly closed valued on X if, for $x \in X$ and for each nonempty compact subset K of Y , $y \notin T(x) \cap K$ implies that there exists a point $\bar{x} \in X$ such that $y \notin cl_K(T(\bar{x}) \cap K)$, where $cl_K(T(\bar{x}) \cap K)$ is the closure of $T(\bar{x}) \cap K$ in K .

Lemma 2.2. Let $T : X \rightarrow 2^Y$ be a set-valued mapping defined by

$$T(x) = \{y \in Y; \phi(x, y) \leq \lambda\}, \text{ (resp. } T(x) = \{y \in Y; \phi(x, y) \geq \lambda\}).$$

Then T is transfer compactly closed valued if and only if ϕ is λ -transfer compactly lower (resp., upper) semi-continuous related to second variant.

Proof. Suppose that T is transfer compactly closed valued. If for each nonempty compact subset $K \subset Y$ and each $y \in K$, there exists $x \in X$ such that $\phi(x, y) > \lambda$ (resp., $\phi(\bar{x}, y) < \lambda$), then $y \notin T(x) \cap K$. Thus, there exists a point $\bar{x} \in X$ such that $y \notin cl_K(T(\bar{x}) \cap K)$. Hence, there exists an open neighborhood $U(y)$ of y in K such that $U(y) \cap T(\bar{x}) = \emptyset$, i.e., $\phi(\bar{x}, z) > \lambda$ (resp., $\phi(\bar{x}, z) < \lambda$) for all $z \in U(y)$.

Conversely, for each nonempty compact subset $K \subset Y$ and for any $y \notin T(x) \cap K$, we have (i) if $y \notin K$ then $y \notin cl_K(T(\bar{x}) \cap K)$; (ii) if $y \in K$ and $y \notin T(x)$ then $\phi(x, y) > \lambda$ (resp., $\phi(\bar{x}, z) < \lambda$). Hence, there exists an open neighborhood $U(y)$ of y in K and $\bar{x} \in X$ such that $\phi(\bar{x}, z) > \lambda$ (resp. $\phi(\bar{x}, z) < \lambda$) for all $z \in U(y)$, therefore $y \notin cl_K(T\bar{x})$. This implies that T is transfer compactly closed valued. The proof is completed.

3 Solutions of VI(1.1)

In the Sections 3 and 4, we assume that (E, ϕ_N) is an FC-space, X is a nonempty FC-subspace of E , $f : X \rightarrow \bar{R}$ is a function with $f \boxtimes +\infty$, and $\varphi : X \times X \rightarrow R$ is a function with $\varphi(x, x) \geq 0$ for all $x \in X$.

Theorem 3.1. Suppose that

(i) there exists a compact subset K of E and $x^* \in X \cap K$ such that

$$f(x) > \phi(x, x^*) + f(x^*) \quad (\forall x \in X \setminus K);$$

(ii) $f(y) + \varphi(x, y) - f(x)$ is 0-generalized R-diagonally quasi-convex related to the variant y ;

(iii) $f(x) - \varphi(x, y) - f(y)$ is 0-transfer compactly lower semi-continuous related to the variant x .

Then VI(1.1) has a solution in $X \cap K$, i.e., there exists $\bar{x} \in X \cap K$ such that

$$\phi(\bar{x}, \gamma) \geq f(\bar{x}) - f(\gamma) \quad (\forall \gamma \in X).$$

Proof. For any $\gamma \in X$, let

$$G(\gamma) = \{x \in X; f(\gamma) + \phi(x, \gamma) \geq f(x)\}. \tag{3.1}$$

Then G is a multi-valued mapping from X into itself. It is easy to see that, if $\bar{x} \in \bigcap_{\gamma \in X} G(\gamma) \neq \emptyset$, then \bar{x} is a solution of VI(1.1).

We now show that $\bigcap_{\gamma \in X} G(\gamma) \neq \emptyset$. In fact, condition (ii) and Lemma 2.1 imply that G is a generalized R-KKM mapping, condition (iii) and Lemma 2.2 imply that G is transfer compactly closed valued. It follows from condition (i) that $x \notin G(x^*)$ for all $x \in X \setminus K$, and so $G(x^*) \subset K$, i.e., $\overline{G(x^*)}$ is a compact subset of E . By Theorem 3.4 in [11], $\bigcap_{\gamma \in X} G(\gamma) \neq \emptyset$. In addition, $\bigcap_{\gamma \in X} G(\gamma) \subset G(x^*) \subset K$, and so the solutions of VI(1.1) is in $X \cap K$. This completes the proof.

Theorem 3.2. Suppose that

(i) there exist a compact subset K of E and $x^* \in X \cap K$ such that

$$f(x) > \phi(x, x^*) + f(x^*) \quad (\forall x^* \in X \setminus K);$$

(ii) $f(\gamma) + \phi(x, \gamma)$ is R-quasi-convex related to the variant γ ;

(iii) $f(x) - \phi(x, \gamma)$ is lower semi-continuous related to the variant x .

Then, VI(1.1) has a solution in $X \cap K$.

Proof. We first show the multi-valued mapping G defined by (3.1) is a generalized R-KKM mapping. Suppose that G is not a generalized R-KKM mapping, then there exist $N = \{x_0, \dots, x_n\} \in \langle X \rangle$, $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$ and $\gamma' \in \phi_N(\Delta_k)$ such that $\gamma' \notin \bigcup_{j=0}^k G(x_{i_j})$. Since $f(x) + \phi(x, \gamma)$ is R-quasi-convex related to the variant γ and $\{x_{i_0}, \dots, x_{i_k}\} \subset \{\gamma \in X : f(\gamma) + \phi(\gamma', \gamma) < f(\gamma')\}$, then $\phi_N(\Delta_k) \subset \{\gamma \in X : f(\gamma) + \phi(\gamma', \gamma) < f(\gamma')\}$. And $\gamma' \in \phi_N(\Delta_k)$, then $f(\gamma') + \phi(\gamma', \gamma') < f(\gamma')$, i.e., $\phi(\gamma', \gamma') < 0$, this contracts $\phi(\gamma', \gamma') \geq 0$. Therefore, G is a generalized R-KKM mapping.

Condition (i) implies that $x \notin G(x^*)$ for all $x \in X \setminus K$, so, $G(x^*) \subset K$, i.e., $\overline{G(x^*)}$ is a compact subset of E . It follows from condition (iii) that $G(\gamma)$ is a closed subset for each $\gamma \in X$. By Theorem 3.2 in [11], $\bigcap_{\gamma \in X} G(\gamma) \neq \emptyset$, and so VI(1.1) has a solution in $X \cap K$. This completes the proof.

By Theorems 3.1 and 3.2, it is easy to get Ky Fan's minimax inequality in FC -spaces.

Corollary 3.1. Assume that (E, ϕ_N) , X, f, ϕ satisfy the conditions (ii) and (iii) in Theorem 3.1.

Then there exists $\bar{x} \in X$ such that

$$f(\gamma) + \phi(\bar{x}, \gamma) \geq f(\bar{x}) \quad (\forall \gamma \in X).$$

Corollary 3.2 Assume that (E, ϕ_N) , X, f, ϕ satisfy the conditions (ii) and (iii) in Theorem 3.2.

Then there exists $\bar{x} \in X$ such that

$$f(\gamma) + \phi(\bar{x}, \gamma) \geq f(\bar{x}) \quad (\forall \gamma \in X).$$

Remark 3.1 Theorems 3.1 and 3.2, Corollaries 3.1 and 3.2 extend some results of Gwinner [1] and Ky Fan minimax inequality to FC -spaces without linear structure.

4 Minty theorem and monotone variational inequality

In this section, we present Minty type theorem in *FC*-spaces.

Theorem 4.1. Suppose that

- (i) $\varphi: X \times X \rightarrow R$ is a monotone and *R*-semi-continuous mapping;
- (ii) for each $x \in X$, $f(y) + \varphi(x, y)$ is an *R*-convex function related to the variant y ;
- (iii) for any $N = \{x_0, x_1\} \in \langle X \rangle$, $\liminf_{t \rightarrow 0^+} f(x_t) \geq f(x_0)$, where $x_t = \phi_N((1-t)e_0 + te_1)$, $t \in [0, 1]$.

Then there exists $\bar{x} \in X$ such that $f(y) + \varphi(\bar{x}, y) \geq f(\bar{x})$ for all $y \in X$ if and only if $f(y) - \varphi(y, \bar{x}) \geq f(\bar{x})$ for all $y \in X$.

Proof. For each $y \in X$, let

$$M(y) = \{x \in X; f(y) + \varphi(x, y) - f(x) \geq 0\}, \tag{4.1}$$

$$N(y) = \{x \in X; f(y) - \varphi(y, x) - f(x) \geq 0\}. \tag{4.2}$$

It is sufficient to prove that

$$\bigcap_{y \in X} M(y) = \bigcap_{y \in X} N(y). \tag{4.3}$$

Since φ is monotone, it is easy to see that $\bigcap_{y \in X} M(y) \subset \bigcap_{y \in X} N(y)$. We claim that

$$\bigcap_{y \in X} N(y) \subset \bigcap_{y \in X} M(y). \tag{4.4}$$

Suppose that (4.4) is not true. Then there exists $\bar{x} \in \bigcap_{y \in X} N(y)$ and $\bar{x} \notin \bigcap_{y \in X} M(y)$ i.e.,

$$f(\bar{x}) + \varphi(y, \bar{x}) \leq f(y) \quad (\forall y \in X), \tag{4.5}$$

and there exists $\bar{y} \in X$ such that

$$f(\bar{y}) + \varphi(\bar{x}, \bar{y}) < f(\bar{x}). \tag{4.6}$$

Let $A = \{\bar{x}, \bar{y}\}$, Setting $x_t = \phi_A((1-t)e_0 + te_1)$ for each $t \in [0, 1]$. By condition (iii), we have

$$\liminf_{t \rightarrow 0^+} f(x_t) \geq f(\bar{x}).$$

Since φ is *R*-semi-continuous, $\limsup_{t \rightarrow \sigma^+} \varphi(x_t, \bar{y}) \leq \varphi(\bar{x}, \bar{y})$, we have

$$\liminf_{t \rightarrow 0^+} (f(x_t) - \varphi(x_t, \bar{y})) \geq f(\bar{x}) - \varphi(\bar{x}, \bar{y}) > f(\bar{y}).$$

Thus, there exists $t^* \in [0, 1]$ such that

$$f(x_t) - \varphi(x_t, \bar{y}) > f(\bar{y}) \quad (\forall t \in (0, t^*]),$$

therefore, we have

$$f(\bar{y}) + \varphi(x_t, \bar{y}) < f(x_t) \quad (\forall t \in (0, t^*]). \tag{4.7}$$

Setting $y = x_t$ in (4.5), we get

$$f(\bar{x}) + \varphi(x_t, \bar{x}) \leq f(x_t). \tag{4.8}$$

Combining (4.7) and (4.8), we have

$$(1 - t)(f(\bar{y}) + \phi(x_t, \bar{y})) + t(f(\bar{x}) + \phi(x_t, \bar{x})) < f(x_t). \quad (4.9)$$

It follows from condition (ii) that

$$(1 - t)(f(\bar{y}) + \phi(x, \bar{y})) + t(f(\bar{x}) + \phi(x_t, \bar{x})) \geq f(x_t) + \phi(x_t, x_t). \quad (4.10)$$

Combining (4.9) and (4.10), we have $\phi(x_t, x_t) < 0$, this contradicts to $\phi(x_t, x_t) \geq 0$. Hence, (4.4) is true. The proof is completed.

Remark 4.1. Theorem 4.1 extends famous Minty theorem in *FC*-spaces without linear structure.

By using Minty type theorem (Theorem 4.1), we now discuss the properties of the solutions set for VI(1.1).

Theorem 4.2. Suppose that

(i) there exist a compact subset K of E and $x^* \in X \cap K$ such that

$$f(x) > \phi(x, x^*) + f(x^*) \quad (\forall x \in X \setminus K);$$

(ii) for each $x \in X$, $f(y) + \phi(x, y)$ is a *R*-convex function related to the variant y ;

(iii) $f(x) - \phi(x, y)$ is lower semi-continuous related to the variant x ;

(iv) $\phi(x, y)$ is a monotone and *R*-semi-continuous mapping;

(v) for any $N = \{x_0, x_1\} \in \langle X \rangle$, $\liminf_{t \rightarrow 0^+} f(x_t) \geq f(x_0)$, where $x_t = \phi_N((1 - t)e_0 + te_1)$, $t \in [0, 1]$.

Then the solution set for VI(1.1) is a nonempty compact *FC*-subspace of E in $X \cap K$.

Furthermore, either (I) ϕ is a strictly monotone function in (iv) or (II) $f(y) + \phi(x, y)$ is strictly *R*-convex function in (ii), then the solution of VI(1.1) is unique in $X \cap K$.

Proof Let S be the solution set for VI(1.1) in $X \cap K$. It is clear that S is nonempty by Theorem 3.2 and $S = \bigcap_{y \in X} M(y)$, where $M(y)$ is defined by (4.1). We show that S is a compact *FC*-subspace of E in $X \cap K$. In fact, from Theorem 4.1, we have $S = \bigcap_{y \in X} M(y) = \bigcap_{y \in X} N(y)$, where $N(y)$ is defined by (4.2). Since $f(y) + \phi(x, y)$ is a *R*-convex function related to the variant y , it is easy to see that $N(y)$ is an *FC*-subspace of E , then $S = \bigcap_{y \in X} N(y)$ is an *FC*-subspace of E . By condition (i) and (iii), we have $S \subset K$ and $S = \bigcap_{y \in X} M(y)$ is closed. Therefore, S is a nonempty compact *FC*-subspace of E in $X \cap K$.

We now prove that the solution of VI(1.1) is unique in $X \cap K$ under the condition (I). In fact, let $x_1, x_2 \in X \cap K$ be two solutions of VI(1.1) with $x_1 \neq x_2$. Then

$$f(y) + \phi(x_1, y) \geq f(x_1) \quad (\forall y \in X) \quad (4.11)$$

and

$$f(y) + \phi(x_2, y) \geq f(x_2) \quad (\forall y \in X). \quad (4.12)$$

Putting $y = x_2$ in (4.11) and $y = x_1$ in (4.12), respectively, we have

$$f(x_2) + \phi(x_1, x_2) \geq f(x_1), \quad (4.13)$$

$$f(x_1) + \phi(x_2, x_1) \geq f(x_2). \quad (4.14)$$

Adding (4.13) and (4.14), we have $\phi(x_1, x_2) + \phi(x_2, x_1) \geq 0$. This implies that $\phi(x_1, x_2) + \phi(x_2, x_1) = 0$ and so $x_1 = x_2$.

Now we prove that the solution of VI(1.1) is unique in $X \cap K$ under condition (II). In fact, suppose that $x_1, x_2 \in X \cap K$ be two solutions of VI(1.1) with $x_1 \neq x_2$. Let $N = \{x_1, x_2\}$ and $\bar{x} = \varphi_N(\frac{1}{2}e_0 + \frac{1}{2}e_1) \in X$. Then

$$\frac{1}{2}(f(x_1) + f(x_2) + \phi(x_1, x_2) + \phi(x_1, x_1)) > f(\bar{x}) + \phi(x_1, \bar{x}) \geq f(x_1)$$

Therefore, we have

$$\phi(x_1, x_1) + \phi(x_1, x_2) > f(x_1) - f(x_2). \tag{4.15}$$

Since $\varphi(x, y)$ is monotone, we know that $\varphi(x, y) + \varphi(y, x) \leq 0$ for any $x, y \in X$. Especially setting $x = y$, we have $\varphi(x, x) \leq 0$. Since $\varphi(x, x) \geq 0$ for all $x \in X$, it follows that $\varphi(x, x) = 0$. Therefore (4.15) implies that $\varphi(x_1, x_2) > f(x_1) - f(x_2)$. Analogously, we have $\varphi(x_2, x_1) > f(x_2) - f(x_1)$ and so $\varphi(x_1, x_2) + \varphi(x_2, x_1) > 0$, this contradicts to the fact that φ is monotone. Thus, the solution of VI(1.1) is unique. This completes the proof.

5 Applications

In 1968, Browder [12] established the fixed point theorem for set-valued mappings in Hausdorff topological vector spaces. This theorem plays an important role in solving quasi-variational inequality (see [3,13]). As applications, we now generalize the Browder fixed point theorem to *FC*-spaces by using Corollary 3.4 and prove the equivalent relation between Browder fixed point theorem and Ky Fan's minimax inequality in *FC*-spaces without linear structure.

Theorem 5.1. Let (E, ϕ_N) be an *FC*-space and X be a compact *FC*-subspace of E . Suppose that $T : X \rightarrow 2^X$ is a set-valued mapping satisfying one of the following conditions

- (i) for each $x \in X$, $T(x)$ is a nonempty *FC*-subspace of X and for each $y \in X$, $T^{-1}(y)$ is an open subset of X ;
- (ii) for each $x \in X$, $T(x)$ is an open subset of X , and for each $y \in X$, $T^{-1}(y)$ is a nonempty *FC*-subspace of X .

Then T has a fixed point in X .

Proof. Suppose that condition (i) is satisfied and T has no fixed point in X . For $x, y \in X$, let

$$\phi(x, y) = \begin{cases} -1, & ((x, y) \in \text{graph}(T)), \\ 0, & ((x, y) \notin \text{graph}(T)), \end{cases}$$

where $\text{graph}(T) = \{(x, y); y \in T(x)\}$. Since for each $x \in X$, $x \notin T(x)$ we have $\varphi(x, x) = 0$. Now we verify that all conditions of Corollary 3.2 are satisfied. In fact, for $\lambda \in R$ and $x \in X$, we have

$$\{y \in X; \phi(x, y) < \lambda\} = \begin{cases} X, & (\lambda > 0), \\ T(x), & (-1 < \lambda \leq 0), \\ \emptyset, & (\lambda \leq -1). \end{cases}$$

This implies that $\varphi(x, y)$ is R -quasi-convex related to the variant y . For each $y \in X$ and $\lambda \in R$,

$$\{x \in X; \phi(x, y) \geq \lambda\} = \begin{cases} \emptyset, & (\lambda \geq 0), \\ X \setminus T^{-1}(y), & (-1 < \lambda \leq 0), \\ X, & (\lambda \leq -1). \end{cases}$$

This implies that $\varphi(x, y)$ is upper semi-continuous related to the variant x . From Corollary 3.2, there exists $\bar{x} \in X$ such that $\phi(\bar{x}, y) \geq 0$ for all $y \in X$.

Since $\varphi(x, y) \leq 0$ for all $x, y \in X$, we know that $\phi(\bar{x}, y) = 0$ for all $y \in X$, i.e., $T(\bar{x}) = \emptyset$, which is a contradiction.

Now we suppose that the condition (ii) is satisfied. Let $H(x) = T^{-1}(x) = \{y \in X; x \in T(y)\}$ for $x \in X$. Then H satisfies condition (i). Hence, H has a fixed point in X and so T has a fixed point in X . This completes the proof.

Theorem 5.2. Browder type fixed point theorem (Theorem 5.1) is equivalent to Ky Fan's minimax inequality (Corollary 3.2).

Proof. Using the Ky Fan's minimax inequality to prove the Browder fixed point theorem has been shown by Theorem 5.1. Now we prove that Ky Fan's minimax inequality using Browder fixed point theorem. In fact, if the conclusion of Corollary 3.2 is not true, then for each $x \in X$, there exists $y \in X$ such that $f(y) + \varphi(x, y) < f(x)$. Set

$$T(x) = \{y \in X; f(y) + \phi(x, y) < f(x)\}.$$

Then $T(x)$ is a nonempty FC-subspace of X for any $x \in X$ and for any $y \in X$.

$$T^{-1}(y) = \{x \in X; f(y) + \phi(x, y) < f(x)\} = \{x \in X; f(x) - \phi(x, y) > f(y)\}$$

is an open subset of X . Therefore there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, i.e., $f(\bar{x}) + \phi(\bar{x}, \bar{x}) < f(\bar{x})$ and so $\phi(\bar{x}, \bar{x}) < 0$. This is a contraction. The proof is completed.

Competing interests

The author declares that they have no competing interests.

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