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Berry-Esseen bounds for wavelet estimator in semiparametric regression model with linear process errors

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Abstract

Consider the semiparametric regression model $Y_i = x_i \beta + g(t_i) + \varepsilon_i$, $i = 1, \ldots, n$, where the linear process errors $\varepsilon_i = \sum_{j=-\infty}^{\infty} a_j e_{i-j}$ with $\sum_{j=-\infty}^{\infty} |a_j| < \infty$, and $\{e_i\}$ are identically distributed and strong mixing innovations with zero mean. Under appropriate conditions, the Berry-Esseen type bounds of wavelet estimators for β and $g(\cdot)$ are established. Our results obtained generalize the results of nonparametric regression model by Li et al. to semiparametric regression model.
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1 Introduction

Regression analysis is one of the most mature and widely applied branches of statistics. For a long time, however, its main theory has concerned parametric and nonparametric regressions. Recently, semiparametric regressions have received more and more attention. This is mainly because semiparametric regression reduces the high risk of misspecification relating to a fully parametric model and avoids some serious drawbacks of fully nonparametric methods.

In 1986, Engle et al. [1] first introduced the following semiparametric regression model:

$$Y_i = x_i\beta + g(t_i) + \varepsilon_i, \quad i = 1, \ldots, n, \tag{1.1}$$

where β is an unknown parameter of interest, $\{(x_i, t_i)\}$ are nonrandom design points, $\{y_i\}$ are the response variables, $g(\cdot)$ is an unknown function defined on the closed interval $[0, 1]$, and $\{\varepsilon_i\}$ are random errors.

The model (1.1) has been extensively studied. When the errors $\{\varepsilon_i\}$ are independent and identically distributed (i.i.d.) random variables, Chen and Shiah [2], Donald and Dewey [3], and Hamilton and Truong [4] used various estimation methods to obtain estimators of the unknown quantities in (1.1) and discussed the asymptotic properties of these estimators. When $\{\varepsilon_i\}$ are MA (∞) errors with the form $\varepsilon_i = \sum_{j=0}^{\infty} a_j e_{i-j}$, where $\{e_i\}$ are i.i.d. random variables, $\{a_j\}$ satisfy $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ and $\sup_n n \sum_{j=n}^{\infty} |a_j| < \infty$,

the law of the iterated logarithm for the semiparametric least square estimator (SLSE) of β and strong convergence rates of the nonparametric estimator of $g(\cdot)$ were discussed by Sun et al. [5]. The Berry-Esseen type bounds for estimators of β and $g(\cdot)$ in model (1) under the linear process errors $\varepsilon_i = \sum_{j=-\infty}^{\infty} a_j e_{i-j}$ with identically distributed and negatively associated random variables $\{e_i\}$ were derived by Liang and Fan [6].

Let us now recall briefly the definition of strong-mixing dependence. A sequence $\{e_i, i \in \mathbf{Z}\}$ is said to be strong mixing (or α-mixing) if $\alpha(n) \to 0$ as $n \to \infty$, where $\alpha(n) = \sup\left\{\left|P(AB) - P(A)P(B)\right| : A \in \mathcal{F}_{-\infty}^{m}, B \in \mathcal{F}_{m+n}^{\infty}\right\}$, and $\mathcal{F}_n^m$ denotes the σ-field generated by $\{e_i : m \leq i \leq n\}$.

For the properties of strong-mixing, one can read the book of Lin and Liu [7]. Recently, Yang and Li [8-10] and Xing et al. [11-13] established moment bounds and maximal moment inequality for partial sums for strong mixing sequences and their application. In this article, we study the Berry-Esseen type bounds for wavelet estimators of β and $g(\cdot)$ in model (1.1) based linear process errors $\{\varepsilon_i\}$ satisfying the following basic assumption (A1). Our results obtained generalize the results in [14] to semiparametric regression model.

(A1) (i) Let $\varepsilon_i = \sum_{j=-\infty}^{\infty} a_j e_{i-j}$, where $\sum\limits_{j=-\infty}^{\infty} |a_j| < \infty$, $\{e_j, j = 0, \pm 1, \pm 2, \ldots\}$ are identically distributed and strong mixing random variables with zero mean.

(ii) For $\delta > 0$, $E|e_0|^{2+\delta} < \infty$ and mixing coefficients $\alpha(n) = O(n^{-\lambda})$ for $\lambda > (2 + \delta)/\delta$.

Now, we introduce wavelet estimators of β and g for model (1.1). Let β be given, since $Ee_i = 0$, we have $g(t_i) = E(y_i - x_i\beta)$, $i = 1, \ldots, n$. Hence a natural estimator of $g(\cdot)$ is

$$g_n(t, \beta) = \sum_{j=1}^{n} (Y_j - x_j\beta) \int_{A_j} E_m(t, s)\, ds,$$

where $A_j = [s_{j-1}, s_j]$ are intervals that partition $[0, 1]$ with $t_j \in A_j$ and $0 \leq t_1 \leq \cdots \leq t_n \leq 1$, and wavelet kernel $E_m(t, s)$ can be defined by

$$E_m(t, s) = 2^m E_0(2^m t, 2^m s), \quad E_0(t, s) = \sum_{j \in z} \varphi(t - j)\varphi(s - j),$$

where $m = m(n) > 0$ is a integer depending only on n, $\phi(\cdot)$ is father wavelet with compact support. Set

$$\tilde{x}_i = x_i - \sum_{j=1}^{n} x_j \int_{A_j} E_m(t_i, s)\, ds, \quad \tilde{y}_i = Y_i - \sum_{j=1}^{n} Y_j \int_{A_j} E_m(t_i, s)\, ds, \quad S_n^2 = \sum_{i=1}^{n} \tilde{x}_i^2. \tag{1.2}$$

In order to estimate β, we seek to minimize

$$SS(\beta) = \sum_{i=1}^{n} \left[Y_i - x_i\beta - g_n(t_i, \beta)\right]^2 = \sum_{i=1}^{n} (\tilde{y}_i - \tilde{x}_i\beta)^2. \tag{1.3}$$

The minimizer to (1.3) is found to be

$$\hat{\beta}_n = S_n^{-2} \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i. \tag{1.4}$$

So, a plug-in estimator of the nonparametric component $g(\cdot)$, based on $\hat{\beta}_n$, is given by

$$\hat{g}_n(t) \triangleq \hat{g}_n(t, \hat{\beta}_n) = \sum_{i=1}^{n} (Y_i - x_i \hat{\beta}_n) \int_{A_i} E_m(t, s)\, ds. \tag{1.5}$$

In the following, the symbols c, C, C_1, C_2, ... denote positive constants whose values may change from one place to another, $b_n = O(a_n)$ means $b_n \le c a_n$, $[x]$ denotes the integral part of x, $||e_i||_r := (E|e_i|^r)^{1/r}$, $\Phi(u)$ represents the standard normal distribution function.

The article is organized as follows. In Section 2, we give some assumptions and main results. Sections 3 and 4 are devoted to the proofs of preliminary results. Proofs of theorems will be provided in Section 5. Some known results used in the proofs of preliminary and main results are appended in Appendix.

2 Assumptions, notations and results

At first we list some assumptions used in this article.

(A2) There exists a function $h(\cdot)$ defined on $[0, 1]$ such that $x_i = h(t_i) + u_i$ and

(i) $\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} u_i^2 = \sum_0 \quad (0 < \sum_0 < \infty)$, (ii) $\max_{1 \le i \le n} |u_i| = O(1)$.

(iii) For any permutation $(j_1, \ldots, j_n)$ of the integers $(1, \ldots, n)$,

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n} \log n} \max_{1 \le m \le n} \left| \sum_{i=1}^{m} u_{j_i} \right| < \infty.$$

(A3) The spectral density $f(\omega)$ of $\{\varepsilon_i\}$ satisfies $0 < c_1 \le f(\omega) \le c_2 < \infty$, for $\omega \in (-\pi, \pi]$.

(A4) Let $g(\cdot)$ and $h(\cdot)$ satisfy the *Lipschitz condition* of order 1 on $[0, 1]$, and $h(\cdot) \in H^\nu$, $\nu > \frac{3}{2}$, where H^ν is the *Sobolev space* of order ν.

(A5) Scaling function $\phi(\cdot)$ is γ-regular (γ is a positive integer) and has a compact support, satisfies the *Lipschitz condition* of order 1 and $|\hat{\varphi}(\xi) - 1| = O(\xi)$ as $\xi \to 0$, where $\hat{\varphi}$ denotes the Fourier transform of ϕ.

(A6) $\max_{1 \le i \le n} |s_i - s_{i-1}| = O(n^{-1})$.

(A7) There exists a positive constant d_1, such that $\min_{1 \le i \le n} (t_i - t_{i-1}) \ge d_1 \cdot \frac{1}{n}$.

For the sake of convenience, we use the following notations. Let $p = p(n)$, $q = q(n)$ denote positive integers such that $p + q \le 3n$ and $qp^{-1} \le c < \infty$. Set

$$\sigma_{n1}^2 = \text{Var}\left(\sum_{i=1}^{n} u_i \varepsilon_i \right), \quad \sigma_{n2}^2 = \text{Var}\left(\sum_{i=1}^{n} \varepsilon_i \int_{A_i} E_m(t, s)\, ds \right), \quad u(n) = \sum_{j=n}^{\infty} \alpha^{\delta/(2+\delta)}(j);$$

$$\gamma_{1n} = qp^{-1}, \quad \gamma_{2n} = pn^{-1}, \quad \gamma_{3n} = n\left(\sum_{|j|>n} |a_j| \right)^2, \quad \gamma_{4n} = np^{-1}\alpha(q);$$

$$\lambda_{1n} = qp^{-1}2^m, \quad \lambda_{2n} = pn^{-1}2^m, \quad \lambda_{3n} = \gamma_{3n}, \quad \lambda_{4n} = \gamma_{4n}, \quad \lambda_{5n} = 2^{-m/2} + \sqrt{2^m/n}\log n;$$

$$\mu_n(\rho, p) = \sum_{i=1}^{3} \gamma_{in}^{1/3} + u(q) + \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4};$$

$$\nu_n(m) = 2^{-\frac{2m}{3}} + (2^m/n)^{1/3}\log^{2/3} n + 2^{-m}\log n + n^{1/2}2^{-2m}.$$

After these assumptions and notations we can formulate the main results as follows:

Theorem 2.1. Suppose that (A1)-(A7) hold. If ρ satisfies

$$0 < \rho \le 1/2, \quad \rho < \min\left\{\frac{\delta}{2}, \frac{\delta\lambda - (2+\delta)}{2\lambda + (2+\delta)}\right\},\tag{2.1}$$

then

$$\sup_u\left|P\left(\frac{S_n^2(\hat{\beta}_n - \beta)}{\sigma_{n1}} \le u\right) - \Phi(u)\right| \le C_1(\mu_n(\rho, p) + v_n(m)).$$

Corollary 2.1 Under the same conditions as in Theorem 2.1, if $\rho = 1/3$, $2^m = O(n^{2/5})$, $\sup_{n\ge 1} n^{7/8}(\log n)^{-9/8}\sum_{|j|>n}|a_j| < \infty$ and $\delta > 1/3$, $\lambda \ge \max\{\frac{2+\delta}{\delta}, \frac{7\delta+14}{6\delta-2}\}$, then for each $t \in [0, 1]$, we have

$$\left|P\left(\frac{S_n^2(\hat{\beta}_n - \beta)}{\sigma_{n1}} \le u\right) - \Phi(u)\right| \le C_2\left(n^{-\frac{\lambda}{6\lambda+7}}\right).\tag{2.2}$$

Theorem 2.2. Suppose that the conditions in Theorem 2.1 are satisfied. Let $n^{-1}2^m \to 0$, then for each $t \in [0, 1]$

$$\sup_u\left|P\left(\frac{\hat{g}_n(t) - E\hat{g}_n(t)}{\sigma_{n2}} \le u\right) - \Phi(u)\right| \le C_3\left(\sum_{i=1}^3 \lambda_{in}^{1/3} + u(q) + \lambda_{2n}^{\rho} + \lambda_{4n}^{1/4} + \lambda_{5n}^{(2+\delta)/(3+\delta)}\right).$$

Corollary 2.2. Under the conditions of Theorem 2.2 with $\rho = 1/3$, $\delta > 2/3$, if $n^{-1}2^m = O(n^{-\theta})$ with $\frac{\lambda+1}{2\lambda+1} < \theta \le 3/4$, and $\lambda > \frac{(2+\delta)(9\theta-2)}{2\theta(3\delta-2)+2}$, then

$$\sup_u\left|P\left(\frac{\hat{g}_n(t) - E\hat{g}_n(t)}{\sigma_{n2}} \le u\right) - \Phi(u)\right| \le C_4\left(n^{-\min\left\{\frac{\lambda(2\theta-1)+(\theta-1)}{6\lambda+7}, \frac{4\lambda+4}{22\lambda+11}\right\}}\right).\tag{2.3}$$

Remark 2.1. Let $\tilde{h}(t) = h(t) - \sum_{j=1}^n h(t_j)\int_{A_j} E_m(t, s)ds$, under the assumptions (A4)-(A7) and by the relation (11) of the proof of Theorem 3.2 in [15], we obtain $\sup_t|\tilde{h}(t)| = O(n^{-1} + 2^{-m})$. Similarly, let $\tilde{g}(t) = g(t) - \sum_{j=1}^n g(t_j)\int_{A_j} E_m(t, s)ds$, then $\sup_t|\tilde{g}(t)| = O(n^{-1} + 2^{-m})$.

Remark 2.2. (i) By Corollary 2.1, the Berry-Esseen bound of the wavelet estimator $\hat{\beta}_n$ is near $O(n^{-\frac{1}{6}})$ for sufficiently large λ, which is faster than the one in [16]) that can get $O(n^{-\delta/4}\log n)$ for $\delta \le 1/2$ or $O(n^{-1/8})$ for $\delta > 1/2$ for strong mixing sequence, but slower than the one in [6] for weighted estimate that can get $O(n^{-1/4}(\log n)^{3/4})$.

(ii) From Corollary 2.2, the Berry-Esseen bound of the wavelet estimator $\hat{g}_n(\cdot)$ is near $O(n^{-1/12})$ for sufficiently large λ and $\theta = 3/4$.

3 Some preliminary lemmas for $\hat{\beta}_n$

From the definition of $\hat{\beta}_n$ in (1.4), we write

$$S_{n\beta} := \sigma_{n1}^{-1}S_n^2(\hat{\beta}_n - \beta) = \sigma_{n1}^{-1}\left[\sum_{i=1}^n \tilde{x}_i\varepsilon_i - \sum_{i=1}^n \tilde{x}_i\sum_{j=1}^n \varepsilon_j\int_{A_j} E_m(t_i, s)\,ds + \sum_{i=1}^n \tilde{x}_i\tilde{g}_i\right]\tag{3.1}$$

$$:= S_{n1} + S_{n2} + S_{n3},$$

where

$$S_{n1} = \sigma_{n1}^{-1} \sum_{i=1}^{n} \tilde{x}_i \varepsilon_i = \sigma_{n1}^{-1} \sum_{i=1}^{n} u_i \varepsilon_i + \sigma_{n1}^{-1} \sum_{i=1}^{n} \tilde{h}_i \varepsilon_i - \sigma_{n1}^{-1} \sum_{i=1}^{n} \varepsilon_i \left(\sum_{j=1}^{n} u_j \int_{A_j} E_m(t_i, s) ds \right) \quad (3.2)$$

$$= S_{n11} + S_{n12} + S_{n13},$$

$$|S_{n2}| = \left| \sigma_{n1}^{-1} \sum_{i=1}^{n} \tilde{x}_i \sum_{j=1}^{n} \varepsilon_j \int_{A_j} E_m(t_i, s) ds \right|$$

$$\leq \left| \sigma_{n1}^{-1} \sum_{i=1}^{n} u_i \left(\sum_{j=1}^{n} \varepsilon_j \int_{Aj} E_m(t_i, s) ds \right) \right| + \left| \sigma_{n1}^{-1} \sum_{i=1}^{n} \tilde{h}_i \left(\sum_{j=1}^{n} \varepsilon_j \int_{A_j} E_m(t_i, s) ds \right) \right| \quad (3.3)$$

$$+ \left| \sigma_{n1}^{-1} \sum_{i=1}^{n} \left(\sum_{l=1}^{n} u_l \int_{A_l} E_m(t_i, s) ds \right) \left(\sum_{j=1}^{n} \varepsilon_j \int_{A_j} E_m(t_i, s) ds \right) \right|$$

$$=: S_{n21} + S_{n22} + S_{n23},$$

$$S_{n3} = \sigma_{n1}^{-1} \sum_{i=1}^{n} \tilde{x}_i \tilde{g}_i. \quad (3.4)$$

For S_{n11}, we can write

$$S_{n11} = \sigma_{n1}^{-1} \sum_{i=1}^{n} u_i \varepsilon_i = \sigma_{n1}^{-1} \sum_{i=1}^{n} u_i \sum_{j=-n}^{n} a_j e_{i-j} + \sigma_{n1}^{-1} \sum_{i=1}^{n} u_i \sum_{|j|>n} a_j e_{i-j} := S_{n111} + S_{n112}. \quad (3.5)$$

It is not difficult to see that

$$S_{n111} = \sum_{l=1-n}^{2n} \sigma_{n1}^{-1} \left(\sum_{i=\max\{1,l-n\}}^{\min\{n,l+n\}} u_i a_{i-l} \right) e_l \triangleq \sum_{l=1-n}^{2n} Z_{nl}.$$

Let $k = [3n/(p + q)]$, then S_{n111} may be split as

$$S_{n111} = S'_{n111} + S''_{n111} + S'''_{n111}, \quad (3.6)$$

where

$$S'_{n111} = \sum_{w=1}^{k} \gamma_{1nw}, \quad S''_{n111} = \sum_{w=1}^{k} \gamma'_{1nw}, \quad S'''_{n111} = \gamma'_{1nk+1},$$

$$\gamma_{1nw} = \sum_{i=k_w}^{k_w+p-1} Z_{ni}, \quad \gamma'_{1nw} = \sum_{i=l_w}^{l_w+q-1} Z_{ni}, \quad \gamma'_{1nk+1} = \sum_{i=k(p+q)-n+1}^{2n} Z_{ni},$$

$$k_w = (w - 1)(p + q) + 1 - n, l_w = (w - 1)(p + q) + p + 1 - n, w = 1, ..., k.$$

From (3.1) to (3.6), we can write that

$$S_{n\beta} = S'_{n111} + S''_{n111} + S'''_{n111} + S_{n112} + S_{n12} + S_{n13} + S_{n2} + S_{n3}.$$

Now, we establish the following lemmas with its proofs.

Lemma 3.1. Suppose that (A1), (A2)(i), and (A3) hold, then

$$c_1 \pi n \leq \sigma_{n1}^2 \leq c_2 \pi n, \quad c_3 n^{-1} 2^m \leq \sigma_{n2}^2 \leq c_4 n^{-1} 2^m.$$

Proof. According to the proofs of (3.4) and Theorem 2.3 in [17], for any sequence $\{\gamma_l\}_{l \in N}$, we have

$$2c_1 \pi \sum_{l=1}^n \gamma_l^2 \leq E \left(\sum_{l=1}^n \gamma_l \varepsilon_l \right)^2 \leq 2c_2 \pi \sum_{l=1}^n \gamma_l^2,$$

which implies the desired results by Lemma A.4 and assumption(A2)(i). ♣

Lemma 3.2. Let assumptions (A1)-(A3), (A5), and (A6) be satisfied, then

$$E\left(S_{n111}''\right)^2 \leq C\gamma_{1n}, \; E\left(S_{n111}'''\right)^2 \leq C\gamma_{2n}, \; E(S_{n112})^2 \leq C\gamma_{3n}; \tag{3.7}$$

$$P\left(|S_{n111}''| \geq \gamma_{1n}^{1/3}\right) \leq C\gamma_{1n}^{1/3}, \; P\left(|S_{n111}'''| \geq \gamma_{2n}^{1/3}\right) \leq C\gamma_{2n}^{1/3}, \; P\left(|S_{n112}| \geq \gamma_{3n}^{1/3}\right) \leq C\gamma_{3n}^{1/3}. \tag{3.8}$$

Proof. By Lemmas 3.1 and A.1(i), and assumptions (A1)(i) and (A2)(i), we have

$$
\begin{aligned}
E\left(S_{n111}''\right)^2 &\leq C \sum_{w=1}^k \sum_{i=l_w}^{l_w+q-1} \sigma_{n1}^{-2} \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} u_i a_{j-i} \right)^2 \|e_i\|_{2+\delta}^2 \\
&\leq C \sum_{w=1}^k \sum_{i=1_w}^{l_w+q-1} n^{-1} \left(\max_{1 \leq i \leq n} |u_i| \right)^2 \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} |a_{j-i}| \right)^2 \\
&\leq Ckqn^{-1} \left(\sum_{i=-\infty}^\infty |a_j| \right)^2 \leq Cqp^{-1} = C\gamma_{1n},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
E\left(S_{n111}'''\right)^2 &\leq C \sum_{i=k(p+q)-n+1}^{2n} \sigma_{n1}^{-2} \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} u_i a_{j-i} \right)^2 \|e_i\|_{2+\delta}^2 \\
&\leq C \sum_{i=k(p+q)-n+1}^{2n} n^{-1} \left(\max_{1 \leq i \leq n} |u_i| \right)^2 \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} |a_{j-i}| \right)^2 \\
&\leq C[3n - k(p+q)]n^{-1} \left(\sum_{j=-\infty}^\infty |a_j| \right)^2 \leq Cpn^{-1} = C\gamma_{2n},
\end{aligned}
\tag{3.10}
$$

and by the Cauchy-inequality

$$
\begin{aligned}
E(S_{n112})^2 &\leq C\sigma_{n1}^{-2} \left(\sum_{i=1}^n |u_i|^2 \right) E \left(\sum_{i=1}^n \left(\sum_{|j|>n} a_j e_{i-j} \right)^2 \right) \leq C\sigma_{n1}^{-2} n \sum_{i=1}^n \left(\sum_{|j|>n} |a_j| \right)^2 \|e_i\|_{2+\delta}^2 \\
&\leq Cn \left(\sum_{|j|>n} |a_j| \right)^2 = C\gamma_{3n}.
\end{aligned}
\tag{3.11}
$$

Then, from (3.9) to (3.11), the proof of (3.7) is complete, which implies the desired result (3.8) by the Markov-inequality. ♣

Lemma 3.3. Let assumptions (A1)-(A7) be satisfied, then

$$(a)\ P\left\{|S_{n12}| \geq C(n^{-1} + 2^{-m})^{2/3}\right\} \leq C(n^{-1} + 2^{-m})^{2/3}.$$

$$(b)\ P\left\{|S_{n13}| \geq C\left(2^m n^{-1} \log^2 n\right)^{1/3}\right\} \leq C\left(2^m n^{-1} \log^2 n\right)^{1/3}.$$

$$(c)\ P\left\{|S_{n21}| \geq C\left(2^m n^{-1} \log^2 n\right)^{1/3}\right\} \leq C\left(2^m n^{-1} \log^2 n\right)^{1/3}.$$

$$(d)\ P\left\{|S_{n22}| \geq C(n^{-1} + 2^{-m})^{2/3}\right\} \leq C(n^{-1} + 2^{-m})^{2/3}.$$

$$(e)\ P\left\{|S_{n23}| \geq C\left(2^m n^{-1} \log^2 n\right)^{1/3}\right\} \leq C\left(2^m n^{-1} \log^2 n\right)^{1/3}.$$

$$(f)\ S_{n3} \leq C\left(2^{-m} \log n + n^{1/2} 2^{-2m}\right).$$

(3.12)

Proof. (a) By assumption (A2), Remark 2.1 and Lemma 3.1, we get

$$E\left(S_{n12}^2\right) \leq c_2 \pi \sigma_{n1}^{-2} \sum_{i=1}^{n} \left(\tilde{h}_i\right)^2 \leq C(n^{-1} + 2^{-m})^2,$$

By this and the Markov inequality, we have

$$P\left\{|S_{n12}| \geq C(n^{-1} + 2^{-m})^{2/3}\right\} \leq C(n^{-1} + 2^{-m})^{2/3}.$$

(3.13)

(b) Applying Lemmas 3.1, A.4, and A.5, we get that

$$ES_{n13}^2 \leq \sigma_{n1}^{-2} \cdot c_2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} u_j \int_{A_j} E_m(t_i, s)\, ds\right)^2$$

$$\leq c_2 \sigma_{n1}^{-2} \cdot \max_{1 \leq i,j \leq n} \left|\int_{A_j} E_m(t_i, s)\, ds\right| \cdot \max_{1 \leq j \leq n} \sum_{i=1}^{n} \left|\int_{A_j} E_m(t_i, s)\, ds\right| \cdot \left(\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} u_{j_i}\right|\right)^2$$

$$\leq C 2^m n^{-1} \log^2 n.$$

Therefore

$$P\left\{|S_{n13}| \geq C\left(2^m n^{-1} \log^2 n\right)^{1/3}\right\} \leq C\left(2^m n^{-1} \log^2 n\right)^{1/3}.$$

(3.14)

(c) Changing the order of summation in $\{S_{n21}\}$, similarly to the calculation for ES_{n13}^2,

$$ES_{n21}^2 \leq \sigma_{n1}^{-2} \cdot c_2 \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i \int_{A_j} E_m(t_i, s)\, ds\right)^2$$

$$\leq c_2 \sigma_{n1}^{-2} \cdot \max_{1 \leq i,j \leq n} \left|\int_{A_j} E_m(t_i, s)\, ds\right| \cdot \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left|\int_{A_j} E_m(t_i, s)\, ds\right| \cdot \left(\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} u_{j_i}\right|\right)^2$$

$$\leq C 2^m n^{-1} \log^2 n.$$

Therefore, we obtain that

$$P\left\{|S_{n21}| \geq C\left(2^m n^{-1} \log^2 n\right)^{1/3}\right\} \leq C\left(2^m n^{-1} \log^2 n\right)^{1/3}.$$

(3.15)

(d) Similarly, by Lemmas 3.1, A.4, A.5, and Remark 2.1, we get that

$$
\mathrm{E}S_{n22}^2 \le c_2 \sigma_{n1}^{-2} \cdot \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \tilde{h}_j \int_{A_i} E_m(t_j, s)\, ds \right)^2
$$

$$
\le c_2 \sigma_{n1}^{-2} \cdot \left(\sup_{t_j} \left| \tilde{h}_j \right| \right)^2 \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left| \int_{A_i} E_m(t_j, s)\, ds \right| \right) \left(\sum_{j=1}^{n} \left| \int_{A_i} E_m(t_i, s)\, ds \right| \right)
$$

$$
\le c_2 \sigma_{n1}^{-2} \cdot \left(\sup_{t_j} \left| \tilde{h}_j \right| \right)^2 \cdot n \le C(n^{-1} + 2^{-m})^2.
$$

Thus, we have

$$
P\{|S_{n22}| \ge C(n^{-1} + 2^{-m})^{2/3}\} \le C(n^{-1} + 2^{-m})^{2/3}. \tag{3.16}
$$

(e) We write that

$$
S_{n23} = \sigma_{n1}^{-1} \sum_{j=1}^{n} \left\{ \sum_{i=1}^{n} \int_{A_j} E_m(t_i, s)\, ds \left(\sum_{l=1}^{n} \int_{A_l} E_m(t_i, s)\, u_l ds \right) \right\} \varepsilon_j.
$$

Similarly to the calculation for $\mathrm{E}S_{n13}^2$ by (3.13), Lemmas 3.1, A.4, and A.5, we obtain that

$$
\mathrm{E}S_{n23}^2 \le c_2 \pi \sigma_{n1}^{-2} \sum_{j=1}^{n} \left[\sum_{i=1}^{n} \int_{A_j} E_m(t_i, s)\, ds \left(\sum_{l=1}^{n} \int_{A_l} E_m(t_i, s)\, u_l ds \right) \right]^2
$$

$$
\le c_2 \pi \sigma_{n1}^{-2} \max_{1 \le i, j \le n} \int_{A_i} E_m(t_j, s)\, ds \cdot \max_{1 \le j \le n} \sum_{i=1}^{n} \int_{A_i} E_m(t_j, s)\, ds
$$

$$
\cdot \left(\max_{1 \le l \le n} \sum_{j=1}^{n} \int_{A_l} E_m(t_j, s)\, ds \cdot \max_{1 \le m \le n} \left| \sum_{i=1}^{m} u_{j_i} \right| \right)^2 \le C \cdot 2^m n^{-1} \log^2 n.
$$

Hence, we have

$$
P \left\{ |S_{n23}| \ge C\left(2^m n^{-1} \log^2 n\right)^{1/3} \right\} \le C\left(2^m n^{-1} \log^2 n\right)^{1/3}. \tag{3.17}
$$

(f) By assumption (A2), Remarks 2.1, Lemma A.5, and the Abel inequality, we have

$$
\sigma_{n1} S_{n3} \le \left| \sum_{i=1}^{n} u_i \tilde{g}_i \right| + \left| \sum_{i=1}^{n} \tilde{h}_i \tilde{g}_i \right| + \left| \sum_{i=1}^{n} \left(\left| \sum_{j=1}^{n} u_j \right| \int_{A_j} E_m(t_i, s)\, ds \right) \tilde{g}_i \right|
$$

$$
\le c \left\{ \max_{1 \le i \le n} |\tilde{g}_i| \max_{1 \le k \le n} \left| \sum_{i=1}^{k} u_{ji} \right| + n \max_{1 \le i \le n} |\tilde{h}_i| \max_{1 \le i \le n} |\tilde{g}_i| \right.
$$

$$
\left. + \max_{1 \le i \le n} |\tilde{g}_i| \max_{1 \le j \le n} \sum_{i=1}^{n} \int_{A_j} |E_m(t_i, s)|\, ds \max_{1 \le k \le n} \sum_{i=1}^{k} |u_{j_i}| \right\}
$$

$$
= C_1 \left(n^{-1} + 2^{-m}\right) \sqrt{n} \log n + C_2 n \left(n^{-1} + 2^{-m}\right)^2.
$$

Thus, by Lemma 3.1 we obtain

$$S_{n3} \le C_1 \left(n^{-1} + 2^{-m} \right) \log n + C_2 \sqrt{n} \left(n^{-1} + 2^{-m} \right)^2 \le C \left(2^{-m} \log n + n^{1/2} 2^{-2m} \right). \quad (3.18)$$

Therefore, the desired result (3.12) follows from (3.13)-(3.18) immediately. ♣

Lemma 3.4. Suppose that (A1)-(A3), (A5), and (A6) hold. Set $s_n^2 \triangleq \sum_{w=1}^k \mathrm{Var} \left(y_{1nw} \right)$, then

$$\left| s_n^2 - 1 \right| \le C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} + u(q) \right).$$

Proof. Let $\Gamma_n = \sum_{1 \le i < j \le k} \mathrm{Cov}(y_{ni}, y_{nj})$, then $s_n^2 = \mathrm{E} \left(S_{n111}' \right)^2 - 2\Gamma_n$. By (3.5) and (3.6), it is easy to verify that $\mathrm{E}(S_{n11})^2 = 1$, and

$$\mathrm{E} \left(S_{n111}' \right)^2 = 1 + \mathrm{E} \left(S_{n111}'' + S_{n111}''' + S_{n112} \right)^2 - 2\mathrm{E} \left[S_{n11} \left(S_{n111}'' + S_{n111}''' + S_{n112} \right) \right].$$

According to Lemma 3.2, the C_r-inequality and the Cauchy-Schwarz inequality

$$\mathrm{E} \left(S_{n111}'' + S_{n111}''' + S_{n112} \right)^2 \le C \left(\gamma_{1n} + \gamma_{2n} + \gamma_{3n} \right),$$

and

$$\left| \mathrm{E} \left[S_{n11} \left(S_{n111}'' + S_{n111}''' + S_{n112} \right) \right] \right| \le C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} \right).$$

Thus, we obtain

$$\left| \mathrm{E} \left(S_{n111}' \right)^2 - 1 \right| \le C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} \right). \quad (3.19)$$

On the other hand, from Lemma 1.2.4 in [7], Lemmas 3.1 and A.4(iv), we can estimate

$$
\begin{aligned}
|\Gamma_n| &\le \sum_{1 \le i < j \le k} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{t_1 = k_j}^{k_j + p - 1} |\mathrm{Cov}(Z_{ns_1}, Z_{nt_1})| \\
&\le \frac{C}{n} \sum_{1 \le i < j \le k} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{t_1 = k_j}^{k_j + p - 1} \sum_{u = \max\{1, s_1 - n\}}^{\min\{n, s_1 + n\}} \sum_{v = \max\{1, t_1 - n\}}^{\min\{n, t_1 + n\}} |u_{u-s_1} u_{v-t_1}| \, |a_{u-s_1} a_{v-t_1}| \, |\mathrm{Cov}(e_{s_1}, e_{t_1})| \\
&\le \frac{C}{n} \sum_{1 \le i < j \le k} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{t_1 = k_j}^{k_j + p - 1} \sum_{u = \max\{1, s_1 - n\}}^{\min\{n, s_1 + n\}} \sum_{v = \max\{1, t_1 - n\}}^{\min\{n, t_1 + n\}} |a_{u-s_1} a_{v-t_1}| \, \alpha^{\delta/(2+\delta)}(t_1 - s_1) \|e_{t_1}\|_{2+\delta} \cdot \|e_{s_1}\|_{2+\delta} \\
&\le \frac{C}{n} \sum_{i=1}^{k-1} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{u = \max\{1, s_1 - n\}}^{\min\{n, s_1 + n\}} \sum_{j = i+1}^{k} \sum_{t_1 = k_j}^{k_j + p - 1} \sum_{v = \max\{1, t_1 - n\}}^{\min\{n, t_1 + n\}} \alpha^{\delta/(2+\delta)}(t_1 - s_1) |a_{u-s_1} a_{v-t_1}| \\
&\le \frac{C}{n} \sum_{i=1}^{k-1} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{j = i+1}^{k} \sum_{t_1 = k_j}^{k_j + p - 1} \alpha^{\delta/(2+\delta)}(t_1 - s_1) \\
&\le \frac{C}{n} \sum_{i=1}^{k-1} \sum_{s_1 = k_i}^{k_i + p - 1} \sum_{t_1 : |t_1 - s_1| \ge q} \alpha^{\delta/(2+\delta)}(t_1 - s_1) \le Ckpn^{-1} u(q) \le Cu(q).
\end{aligned}
\quad (3.20)
$$

Therefore, by (3.19) and (3.20), it follows that

$$\left| s_n^2 - 1 \right| \le \left| \mathrm{E} \left(S_{1n}' \right)^2 - 1 \right| + 2 |\Gamma_n| \le C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} + u(q) \right).$$

♣

Assume that $\{\eta_{1nw} : w = 1, \ldots, k\}$ are independent random variables, and its distribution is the same as that of $\{y_{1nw}, w = 1, \ldots, k\}$. Set $T_n = \sum_{w=1}^{k} \eta_{1nw}$, $B_{n1}^2 = \sum_{w=1}^{k} \mathrm{Var}(\eta_{1nw})$. Clearly $B_{n1}^2 = s_{n1}^2$. Then, we have the following lemmas:

Lemma 3.5. Let assumptions (A1)-(A3), (A5), (A6), and (2.1) hold, the

$$\sup_u \left| P\left(T_n / B_{n1} \leq u\right) - \Phi(u) \right| \leq C\gamma_{2n}^{\rho}.$$

Proof. By the Berry-Esseen inequality (see [18], Theorem 5.7]), we have

$$\sup_u |P\left(T_n / B_{n1} \leq u\right) - \Phi(u)| \leq C \frac{\sum_{w=1}^{k} \mathrm{E}|\gamma_{1nw}|^r}{B_{n1}^r}, \quad \text{for } 2 < r \leq 3. \tag{3.21}$$

From (2.1), we have $0 < 2\rho \leq 1$, $0 < 2\rho < \delta$, and, $(2 + \delta)/\delta < (1 + \rho)(2 + \delta)/(\delta - 2\rho) < \lambda$. Let $r = 2(1 + \rho)$, $\tau = \delta - 2\rho$, then $r + \tau = 2 + \delta$, and $\frac{r(r+\tau)}{2\tau} = \frac{(1+\rho)(2+\delta)}{\delta-2\rho} < \lambda$. According to Lemmas 3.1 and A.1(ii), and the C_r-inequality, taking $\varepsilon = \rho$, we get that

$$\sum_{w=1}^{k} \mathrm{E}-\gamma_{1nw}|^r \leq C \sum_{w=1}^{k} p^{\rho} \sum_{j=k_w}^{k_w+p-1} \left| \sum_{i=\max\{1,j-n\}}^{\min\{n,j+n\}} \sigma_{n1}^{-1} u_i a_{i-j} \right|^r \mathrm{E}|e_j|^r$$

$$+ \left[\sum_{j=k_w}^{k_w+p-1} \left(\sum_{i=\max\{1,j-n\}}^{\min\{n,j+n\}} \sigma_{n1}^{-1} u_i a_{i-j} \right)^2 ||e_i||_{2+\delta}^2 \right]^{r/2} \tag{3.22}$$

$$\leq C\sigma_{n1}^{-r} k p^{1+\rho} \leq C\gamma_{2n}^{\rho}.$$

Therefore, from Lemma 3.4, relations (3.21) and (3.22), we obtain the result. ♣

Lemma 3.6. Suppose that the conditions in Lemma 3.5 are satisfied, then

$$\sup_u |P\left(S'_{n111} \leq u\right) - P(T_n \leq u)| \leq C \left\{ \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} \right\}.$$

Proof. Let $\varphi_1(t)$ and $\psi_1(t)$ be the characteristic functions of S'_{n111} and T_{n1}, respectively.

Since

$$\psi_1(t) = \mathrm{E}\left(\exp\{itT_{n1}\}\right) = \prod_{w=1}^{k} \mathrm{E}\exp\{it\eta_{1nw}\} = \prod_{w=1}^{k} \mathrm{E}\exp\{it\gamma_{1nw}\},$$

then from Lemmas A.1(i), A.2, and 3.1, it follows that

$$|\phi_1(t) - \psi_1(t)| \leq C|t|\alpha^{1/2}(q) \sum_{w=1}^{k} ||\gamma_{1nw}||_2$$

$$\leq C|t|\alpha^{1/2}(q) \sum_{w=1}^{k} \left\{ \mathrm{E}\left[\sum_{i=k_w}^{k_w+p-1} \sigma_n^{-1} \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} u_i a_{j-i} \right) e_i \right]^2 \right\}^{1/2}$$

$$\leq C|t|\alpha^{1/2}(q) \sum_{w=1}^{k} \left[\sum_{i=k_w}^{k_w+p-1} \sigma_n^{-2} \left(\sum_{j=\max\{1,i-n\}}^{\min\{n,i+n\}} |u_i a_{j-i}| \right)^2 \left(\mathrm{E}|e_i|^{2+\delta} \right)^{2/2+\delta} \right]^{1/2}$$

$$\leq C|t|\alpha^{1/2}(q) \left[k \sum_{w=1}^{k} \sum_{i=k_w}^{k_w+p-1} \sigma_n^{-2} \right]^{1/2} \leq C|t|\alpha^{1/2}(q) k p^{1/2} n^{-1/2}$$

$$\leq C|t| (k\alpha(q))^{1/2} = C|t|\gamma_{4n}^{1/2}.$$

Therefore

$$\int_{-T}^{T} \left| \frac{\phi_1(t) - \psi_1(t)}{t} \right| dt \leq C\gamma_{4n}^{1/2} T. \tag{3.23}$$

As in the calculation of (4.7) in [14], using Lemma 3.5, we have

$$T \sup_{u} \int_{|\gamma| \leq c/T} |P(T_n \leq u + \gamma) - P(T_n \leq u)| d\gamma \leq C \left\{ \gamma_{2n}^{\rho} + 1/T \right\}. \tag{3.24}$$

Therefore, combining (3.23) and (3.24), choosing $T = \gamma_{4n}^{-1/4}$, and using the Esseen inequality (see [[18], Theorem 5.3]), we conclude that

$$\sup_{u} |P(S'_{n111} \leq u) - P(T_n \leq u)|$$

$$\leq \int_{-T}^{T} \left| \frac{\phi_1(t) - \psi_1(t)}{t} \right| dt + T \sup_{u} \int_{|\gamma| \leq c/T} |\tilde{G}_n(u + \gamma) - \tilde{G}_n(u)| d\gamma$$

$$= C \left\{ \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} \right\}.$$

♣

4 Some preliminary lemmas for $\hat{g}_n(t)$

From the definition of $\hat{g}_n(t)$ in (1.5), We can decompose the sum into three parts:

$$S_{ng} := \sigma_{n2}^{-1}(\hat{g}_n(t) - E\hat{g}_n(t)) = \sigma_{n2}^{-1} \sum_{i=1}^{n} \varepsilon_i \int_{A_i} E_m(t,s) ds + \sigma_{n2}^{-1} \sum_{i=1}^{n} x_i(\beta - \hat{\beta}_n) \int_{A_i} E_m(t,s) ds$$

$$- \sigma_{n2}^{-1} \sum_{i=1}^{n} x_i(\beta - E\hat{\beta}_n) \int_{A_i} E_m(t,s) ds =: H_{1n} + H_{2n} + H_{3n}.$$

Let us decompose the vector H_{1n} into two parts:

$$H_{1n} = \sigma_{n2}^{-1} \sum_{i=1}^{n} \int_{A_i} E_m(t,s) ds \left(\sum_{j=-n}^{n} a_j e_{i-j} \right) + \sigma_{n2}^{-1} \sum_{i=1}^{n} \int_{A_i} E_m(t,s) ds \left(\sum_{|j|>n} a_j e_{i-j} \right) =: H_{11n} + H_{12n},$$

Where

$$H_{11n} = \sigma_{n2}^{-1} \sum_{l=1-n}^{2n} \left(\sum_{i=\max(1,l-n)}^{\min(n,n+l)} a_{i-l} \int_{A_i} E_m(t,s) ds \right) e_l = \sum_{l=1-n}^{2n} M_{nl}.$$

Similar to S_{n111} in (3.6), H_{11n} can be split as $H_{11n} = H'_{11n} + H''_{11n} + H'''_{11n}$, where

$$H'_{11n} = \sum_{w=1}^{k} \gamma_{2nw}, \quad H''_{11n} = \sum_{w=1}^{k} \gamma'_{2nw}, \quad H'''_{11n} = \gamma'_{2nk+1},$$

$$\gamma_{2nw} = \sum_{i=k_w}^{k_w+p-1} M_{ni}, \quad \gamma'_{2nw} = \sum_{i=l_w}^{l_w+q-1} M_{ni}, \quad \gamma'_{2nk+1} = \sum_{i=k(p+q)-n+1}^{2n} M_{ni}. \tag{4.1}$$

Then

$$S_{ng} = H'_{11n} + H''_{11n} + H'''_{11n} + H_{2n} + H_{3n}. \tag{4.2}$$

Set $T_{n2} = \sum_{w=1}^{k} \eta_{2nw}$, $B_{n2}^2 = \sum_{w=1}^{k} \text{Var}(\eta_{2nw})$. Similarly to Lemmas 3.2-3.6, we have the following lemmas without proofs, except for Lemma 4.2.

Lemma 4.1. Suppose that the conditions in Theorem 2.2 are satisfied, then

$$E\left(H''_{11n}\right)^2 \leq C\lambda_{1n}, \ E\left(H'''_{11n}\right)^2 \leq C\lambda_{2n}, \ EH_{12n}^2 \leq C\lambda_{3n};$$
$$P\left(\left|H''_{11n}\right| \geq \lambda_{1n}^{1/3}\right) \leq C\lambda_{1n}^{1/3}, \ P\left(\left|H'''_{11n}\right| \geq \lambda_{2n}^{1/3}\right) \leq C\lambda_{2n}^{1/3}, \ P\left(\left|H_{12n}\right| \geq \lambda_{3n}^{1/3}\right) \leq C\lambda_{3n}^{1/3}.$$

Lemma 4.2. Let assumptions (A1)-(A7) be satisfied, then

$$E|H_{2n}|^{2+\delta} \leq c\lambda_{5n}^{2+\delta}, \ P\left(\left|H_{2n}\right| > \lambda_{5n}^{(2+\delta)/(3+\delta)}\right) \leq \lambda_{5n}^{(2+\delta)/(3+\delta)}, \ |H_{3n}| \leq c\lambda_{5n}.$$

Lemma 4.3. Under the conditions of Theorem 2.2, set $s_{n2}^2 = \sum_{w=1}^{k} \text{Var}(\gamma_{2nw})$, then

$$\left|s_{n2}^2 - 1\right| \leq C\left(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + \lambda_{3n}^{1/2} + u(q)\right).$$

Lemma 4.4. Suppose that the conditions in Theorem 2.2 are satisfied, then

$$\sup_u \left|P\left(T_{n2}/B_{n2} \leq u\right) - \Phi(u)\right| \leq c\lambda_{2n}^{\rho}.$$

Lemma 4.5. Suppose that the conditions in Theorem 2.2 are satisfied, then

$$\sup_u \left|P\left(H'_{11n} \leq u\right) - P\left(T_{n2} \leq u\right)\right| \leq C\left\{\lambda_{2n}^{\rho} + \gamma_{4n}^{1/4}\right\}.$$

Proof of Lemma 4.2. Similar to the proof of (A.8) in [6], we first verify that

$$\lim_{n\to\infty} S_n/n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i^2 = \sum, \qquad \text{where } 0 < \sum < \infty. \tag{4.3}$$

From (1.2), we write

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \tilde{x}_i^2 &= \frac{1}{n}\sum_{i=1}^{n} u_i^2 + \frac{1}{n}\sum_{i=1}^{n} \tilde{h}_i^2 + \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} u_j \int_{A_j} E_m(t_i, s)\,ds\right)^2 + \frac{2}{n}\sum_{i=1}^{n} u_i\tilde{h}_i \\
&\quad - \frac{2}{n}\sum_{i=1}^{n} u_i\left(\sum_{j=1}^{n} u_j \int_{A_j} E_m(t_i, s)\,ds\right) - \frac{2}{n}\sum_{i=1}^{n} \tilde{h}_i\left(\sum_{j=1}^{n} u_j \int_{A_j} E_m(t_i, s)\,ds\right) \\
&= L_{1n} + L_{2n} + L_{3n} + 2L_{4n} - 2L_{5n} - 2L_{6n}.
\end{aligned} \tag{4.4}$$

By assumption (A2)(i) and Remark 2.1, we have

$$L_{1n} \to \sum, \ L_{2n} \leq \max_{1\leq i\leq n} \tilde{h}_i^2 = O\left(n^{-1} + 2^{-m}\right) \to 0, \tag{4.5}$$

and by assumption (A2)(iii), Lemmas A.4 and A.5, we get that

$$
\begin{aligned}
L_{3n} &\leq \frac{c}{n} \max_{1 \leq i,j \leq n} \int_{A_j} |E_m(t_i, s)| \, ds \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\int_{A_j} E_m(t_i, s)| \, ds \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| \right)^2 \\
&= O\left(\frac{2^m \log^2 n}{n} \right) \to 0, \\
|L_{4n}| &\leq \frac{c}{n} \max_{1 \leq i \leq n} |\tilde{h}_i| \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| = O\left(\frac{\log n}{2^m \sqrt{n}} \right) \to 0, \\
|L_{5n}| &\leq \frac{c}{n} \max_{1 \leq i,j \leq n} \int_{A_j} |E_m(t_i, s)| \, ds \cdot \max_{1 \leq l \leq n} \left| \sum_{i'=1}^{l} u_{j_{i'}} \right| \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| \\
&= O\left(\frac{2^m \log^2 n}{n} \right) \to 0, \\
|L_{6n}| &\leq \frac{c}{n} \max_{1 \leq i \leq n} |\tilde{h}_i| \cdot \max_{1 \leq j \leq n} \sum_{i=1}^{n} \int_{A_j} |E_m(t_i, s)| \, ds \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| = O\left(\frac{\log n}{2^m \sqrt{n}} \right) \to 0.
\end{aligned}
\tag{4.6}
$$

Therefore, from (4.4) to (4.6), we complete the proof of (4.3).

Recalling the fact that if $\xi_n \Rightarrow \xi \sim N(0,1)$ then $E|\xi_n| \to E|\xi| = \sqrt{2/\pi}$ and $E|\xi_n|^{2+\delta} \to E|\xi|^{2+\delta} < \infty$, by Theorem 2.1, Lemma 3.1 and relation (4.3), we deduce that

$$
\left| \beta - E\hat{\beta}_n \right| \leq E\left| \beta - \hat{\beta}_n \right| \leq O\left(\sigma_{n1}/S_n^2 \right) = O\left(n^{-1/2} \right),
$$

and

$$
E\left| \hat{\beta}_n - \beta \right|^{2+\delta} \leq O\left(\left(\sigma_{n1}/S_n^2 \right)^{2+\delta} \right) = O\left(n^{-(1+\delta/2)} \right).
$$

Therefore, applying the Abel Inequality, by (A2)(iii) and (A4)(i), we get that

$$
\begin{aligned}
|H_{3n}| &= \sigma_{n2}^{-1} \left| \beta - E\hat{\beta}_n \right| \cdot \left| \sum_{i=1}^{n} x_i \int_{A_i} E_m(t, s) \, ds \right| \\
&\leq \sigma_{n2}^{-1} n^{-1/2} \left(\sup_{0 \leq t \leq 1} |h(t)| + \max_{1 \leq i \leq n} \int_{A_i} |E_m(t, s)| \, ds \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| \right) \\
&\leq c\left(2^{-m/2} + \sqrt{2^m/n} \log n \right) = c\lambda_{5n},
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
E|H_{2n}|^{2+\delta} &= \sigma_{n2}^{-(2+\delta)} E\left| \beta - \hat{\beta}_n \right|^{2+\delta} \cdot \left| \sum_{i=1}^{n} x_i \int_{A_i} E_m(t, s) \, ds \right|^{2+\delta} \\
&\leq c\sigma_{n2}^{-(2+\delta)} n^{-(2+\delta)/2} \left(\sup_{0 \leq t \leq 1} |\tilde{h}(t)| + \max_{1 \leq i \leq n} \int_{A_i} |E_m(t, s)| \, ds \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} u_{j_i} \right| \right)^{2+\delta} \\
&\leq c\lambda_{5n}^{2+\delta}.
\end{aligned}
\tag{4.8}
$$

By (4.8), it follows

$$
P\left(|H_{2n}| > \lambda_{5n}^{(2+\delta)/(3+\delta)} \right) \leq \lambda_{5n}^{(2+\delta)/(3+\delta)}.
\tag{4.9}
$$

Therefore, the proof is completed by (4.7)-(4.9).

5 Proofs of main results

Proof of Theorem 2.1. Note that

$$
\sup_u \left| P\left(S'_{n111} \le u\right) - \Phi(u) \right| \le \sup_u \left| P\left(S'_{n111} \le u\right) - P(T_n \le u) \right| + \sup_u \left| P(T_n \le u) - \Phi(u/s_n) \right|
$$
$$
+ \sup_u \left| \Phi(u/s_n) - \Phi(u) \right| =: J_{1n} + J_{2n} + J_{3n}. \tag{5.1}
$$

By Lemmas 3.6, 3.5, and 3.4, we obtain, respectively

$$
J_{1n} \le C \left\{ \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} \right\}, \tag{5.2}
$$

$$
J_{2n} = \sup_u \left| P\left(T_n/s_n \le u/s_n\right) - \Phi(u/s_n) \right| = \sup_u \left| P\left(T_n/s_n \le u\right) - \Phi(u) \right| \le C\gamma_{4n}^{\rho}, \tag{5.3}
$$

and

$$
J_{3n} \le C \left| s_n^2 - 1 \right| \le C \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} + u(q) \right). \tag{5.4}
$$

Hence, from (5.1) to (5.4), we get that

$$
\sup_u \left| P\left(S'_{n111} \le u\right) - \Phi(u) \right| \le C \left\{ \sum_{i=1}^{3} \gamma_{in}^{1/2} + u(q) + \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} \right\}. \tag{5.5}
$$

Therefore, according to Lemma A.3, relations (3.8), (3.12), and (5.5), we obtain

$$
\sup_u \left| P\left(S_{n\beta} \le u\right) - \Phi(u) \right|
$$
$$
= \sup_u \left| P\left(S'_{n111} + S''_{n111} + S'''_{n111} + S_{n112} + S_{n12} + S_{n13} + S_{n2} + S_{n3} \le u\right) - \Phi(u) \right|
$$
$$
\le C \left\{ \sup_u \left| P\left(S'_{n111} \le u\right) - \Phi(u) \right| \right.
$$
$$
\left. + \sum_{i=1}^{3} \gamma_{in}^{1/3} + P\left(\left|S''_{n111}\right| \ge \gamma_{1n}^{1/3}\right) + P\left(\left|S'''_{n111}\right| \ge \gamma_{2n}^{1/3}\right) + P\left(\left|S_{n112}\right| \ge \gamma_{3n}^{1/3}\right) \right\}
$$
$$
+ \left(n^{-1} + 2^{-m}\right)^{2/3} + \left(2^m n^{-1} \log^2 n\right)^{1/3} + 2^{-m} \log n + n^{1/2} 2^{-2m}
$$
$$
+ P\left(\left|S_{n12}\right| \ge \left(n^{-1} + 2^{-m}\right)^{2/3}\right) + P\left(\left|S_{n13}\right| \ge \left(2^m n^{-1} \log^2 n\right)^{1/3}\right)
$$
$$
+ P\left(\left|S_{n21}\right| \ge \left(2^m n^{-1} \log^2 n\right)^{1/3}\right) + P\left(\left|S_{n22}\right| \ge \left(n^{-1} + 2^{-m}\right)^{2/3}\right)
$$
$$
+ P\left(\left|S_{n23}\right| \ge \left(2^m n^{-1} \log^2 n\right)^{1/3}\right)
$$
$$
\le c \left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{3n}^{1/2} + u(q) + \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} + \sum_{i=1}^{3} \gamma_{in}^{1/3} \right)
$$
$$
+ c \left(2^{-\frac{2m}{3}} + \left(2^m n^{-1} \log^2 n\right)^{1/3} + 2^{-m} \log n + n^{1/2} 2^{-2m} \right)
$$
$$
\le c \left(\sum_{i=1}^{3} \gamma_{in}^{1/3} + u(q) + \gamma_{2n}^{\rho} + \gamma_{4n}^{1/4} \right) + c \left(2^{-2m/3} + \left(2^m n^{-1} \log^2 n\right)^{1/3} + 2^{-m} \log n + n^{1/2} 2^{-2m} \right)
$$
$$
= O\left(\mu_n(\rho, p)\right) + O\left(\upsilon_n(m)\right).
$$

This completes the proof of the theorem. ♣

Proof of Corollary 2.1. Let $p = [n^{\tau}]$, $q = \left[n^{2\tau-1}\right]$, where $\tau = \frac{3\lambda+7}{6\lambda+7}$, then

$$
\gamma_{1n}^{1/3} = \gamma_{2n}^{1/3} = O\left(n^{(\tau-1)/3}\right) = O\left(n^{-\frac{\lambda}{6\lambda+7}}\right), \quad \gamma_{4n}^{1/4} = O\left(n^{-\frac{\lambda}{6\lambda+7}}\right),
$$

$$
\lambda_{3n}^{1/3} = n^{-1/4} (\log n)^{3/4} \left(\sup_{n \ge 1} n^{7/8} (\log n)^{-9/8} \sum_{|j|>n} |a_j| \right)^{2/3} = O\left(n^{-1/4} (\log n)^{3/4}\right).
$$

By $\delta > 1/3, \lambda \geq \max\left\{\frac{2+\delta}{\delta}, \frac{7\delta+14}{6\delta-2}\right\}$, we get

$$u(q) = O\left(q^{-\lambda\delta/(2+\delta)+1}\right) = O\left(n^{-\frac{7}{6\lambda+7}\left(\frac{\lambda\delta}{2+\delta}-1\right)}\right) \leq O\left(n^{-\frac{\lambda}{6\lambda+7}}\right).$$

Thus, we have that

$$\mu_n(\rho, p) = O\left(n^{-\frac{\lambda}{6\lambda+7}}\right), \quad \upsilon_n(m) = O\left(n^{-\frac{1}{5}}\right).$$

Therefore, (2.2) follows from Theorem 2.1 directly. ♣

Proof of Theorem 2.2. Analogous to the Proof of Theorem 2.1, we write

$$\sup_u |P(H'_{11n} \leq u) - \Phi(u)|$$
$$\leq \sup_u |P(H'_{11n} \leq u) - P(T_{n2} \leq u)| + \sup_u |P(T_{n2} \leq u) - \Phi(u/s_{n2})| \qquad (5.6)$$
$$+ \sup_u |\Phi(u/s_{n2}) - \Phi(u)| =: J'_{1n} + J'_{2n} + J'_{3n}.$$

By Lemmas 4.5, 4.4, and 4.3, we obtain, respectively

$$J'_{1n} \leq C\left\{\lambda_{2n}^{\rho} + \gamma_{4n}^{1/4}\right\}, \quad J'_{2n} = \sup_u |P\left(T_{n2}/s_{n2} \leq u\right) - \Phi(u)| \leq C\lambda_{2n}^{\rho}, \qquad (5.7)$$

$$J'_{3n} \leq C|s_{n2}^2 - 1| \leq C\left(\lambda_{1n}^{1/2} + \lambda_{2n}^{1/2} + \lambda_{3n}^{1/2} + u(q)\right). \qquad (5.8)$$

Therefore, from (5.6) to (5.8), we have

$$\sup_n |P\left(H'_{11n} \leq u\right) - \Phi(u)| \leq C\left\{\sum_{i=1}^{3} \lambda_{in}^{1/2} + u(q) + \lambda_{2n}^{\rho} + \lambda_{4n}^{1/4}\right\}.$$

Thus, according to Lemmas A.3, 4.1, and 4.2, we obtain

$$\sup_u |P(S_{ng} \leq u) - \Phi(u)|$$
$$= \sup_u |P(H'_{11n} + H''_{11n} + H'''_{11n} + H_{12n} + H_{n2} + H_{n3} \leq u) - \Phi(u)|$$
$$\leq C\left\{\sup_u |P(H'_{11n} \leq u) - \Phi(u)| + \sum_{i=1}^{3} \lambda_{in}^{1/3} + P\left(|H''_{11n}| \geq \lambda_{1n}^{1/3}\right) + P\left(|H'''_{11n}| \geq \lambda_{2n}^{1/3}\right)\right.$$
$$\left. + P\left(|H_{12n}| \geq \lambda_{3n}^{1/3}\right) + \lambda_{5n} + \lambda_{5n}^{(2+\delta)/(3+\delta)} + P\left(|H_{12n}| > \lambda_{5n}^{(2+\delta)/(3+\delta)}\right)\right\}$$
$$\leq c\left(\sum_{i=1}^{3} \lambda_{in}^{1/2} + u(q) + \lambda_{2n}^{\rho} + \lambda_{4n}^{1/4} + \sum_{i=1}^{3} \lambda_{in}^{1/3} + \lambda_{5n}^{(2+\delta)/(3+\delta)}\right)$$
$$\leq c\left(\sum_{i=1}^{3} \lambda_{in}^{1/3} + u(q) + \lambda_{2n}^{\rho} + \lambda_{4n}^{1/4} + \lambda_{5n}^{(2+\delta)/(3+\delta)}\right).$$

Now we complete the Proof of Theorem 2.2. ♣

Proof of Corollary 2.2. Let $p = [n^{\tau}]$, $q = [n^{2\tau-1}]$, where $\tau = \frac{1}{2} + \frac{8\theta-1}{2(6\lambda+7)}$. We can get $\tau < \theta$ by $\frac{\lambda+1}{2\lambda+1} < \theta$. By this, we can obtain

$$\lambda_{1n}^{1/3} = \lambda_{2n}^{1/3} = O\left(n^{-(\theta-\tau)/3}\right) = O\left(n^{-\left(\frac{\lambda(2\theta-1)+(\theta-1)}{6\lambda+7}\right)}\right), \quad \lambda_{4n}^{1/4} = o\left(n^{-\left(\frac{\lambda(2\theta-1)+(\theta-1)}{6\lambda+7}\right)}\right),$$

$$\lambda_{3n}^{1/3} = n^{-1/4}(\log n)^{3/4}\left(\sup_{n\geq 1} n^{7/8}(\log n)^{-9/8}\sum_{|j|>n}|a_j|\right)^{2/3} = O\left(n^{-1/4}(\log n)^{3/4}\right).$$

And by $\delta > 2/3$, and $\theta \leq 3/4$, we have

$$\lambda_{5n}^{(2+\delta)/(3+\delta)} = O\left(n^{-\frac{\theta}{2}\cdot\frac{2+\delta}{3+\delta}}\right) \leq O\left(n^{-\frac{4}{11}\theta}\right) \leq O\left(n^{-\frac{4\lambda+4}{22\lambda+11}}\right).$$

Since $\lambda > \frac{(2+\delta)(9\theta-2)}{2\theta(3\delta-2)+2}$, we can obtain

$$\frac{(8\theta-1)(\lambda\delta-\delta-2)}{(7+6\lambda)(2+\delta)} > \frac{\lambda(2\theta-1)+(\theta-1)}{6\lambda+7}.$$

By this we have

$$u(q) = O\left(q^{-\lambda\delta/(2+\delta)+1}\right) = O\left(n^{-\frac{(8\theta-1)(\lambda\delta-\delta-2)}{(7+6\lambda)(2+\delta)}}\right) = O\left(n^{-\frac{\lambda(2\theta-1)+(\theta-1)}{6\lambda+7}}\right).$$

Therefore, (2.3) follows from Theorem 2.2 directly. ♣

Appendix

Lemma A.1. [8,9] Let $\{b_j: j \geq 1\}$ be a real sequence.

(i) Let $\{X_j : j \geq 1\}$ be an α-mixing sequence of random variables with $E|X_i|^{2+\delta} < \infty, \delta > 0$, then $E\left(\sum_{j=1}^n b_j X_j\right)^2 \leq \left(1 + 20\sum_{m=1}^n \alpha^{\delta/(2+\delta)}(m)\right)\sum_{j=1}^n b_j^2 \|X_j\|_{2+\delta}^2, \forall n \geq 1$.

(ii) Let $r > 2$, $\delta > 0$, and $\{X_i, i \geq 1\}$ be an α-mixing sequence of random variables with $EX_i = 0$ and $E|X_i|^{r+\delta} < \infty$. Suppose that $\theta > r(r+\delta)/(2\delta)$ and $\alpha(n) \leq Cn^{-\theta}$ for some $C > 0$. Then, for any $\varepsilon > 0$, there exists a positive constants $K = K(\varepsilon, r, \delta, \theta, C) < \infty$ such that

$$E\max_{1\leq j\leq n}\left|\sum_{i=1}^j X_i\right|^r \leq K\left\{n^\varepsilon\sum_{i=1}^n E|X_i|^r + \left(\sum_{i=1}^n \|X_i\|_{r+\delta}^2\right)^{r/2}\right\}.$$

Lemma A.2. [10] Let $\{X_j : j \geq 1\}$ be an α-mixing sequence of random variables, p and, q are two positive integers. Let $\eta_l := \sum_{j=(l-1)(p+q)+1}^{(l-1)(p+q)+p} X_j$ for $1 \leq l \leq k$. If $r > 0$, $s > 0$ and $\frac{1}{r} + \frac{1}{s} = 1$, then

$$\left|E\exp\left(it\sum_{l=1}^k \eta_l\right) - \prod_{l=1}^k E\exp(it\eta_l)\right| \leq C|t|\alpha^{1/s}(q)\sum_{l=1}^k \|\eta_l\|_r.$$

Lemma A.3. [14] Suppose that $\{\zeta_n : n \geq 1\}$, $\{\eta_n : n \geq 1\}$ and $\{\xi_n : n \geq 1\}$ are random sequences, $\{\gamma_n : n \geq 1\}$ is a positive constant sequence with $\gamma_n \to 0$. If $\sup_u|F_{\zeta_n}(u) - \Phi(u)| \leq C\gamma_n$, then for any $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$

$$\sup_u |F_{\zeta_n+\eta_n+\xi_n}(u) - \Phi(u)| \le C\{\gamma_n + \varepsilon_1 + \varepsilon_2 + P(|\eta_n| \ge \varepsilon_1) + P(|\xi_n| \ge \varepsilon_2)\}.$$

Lemma A.4. [19] Under assumptions (A5)-(A6), we have

(i) $|\int_{A_i} E_m(t,s)ds| = O\left(\frac{2^m}{n}\right)$, $\quad i = 1, 2, ..., n$; (ii) $\sum_{i=1}^{n}\left(\int_{A_i} E_m(t,s)ds\right)^2 = O\left(\frac{2^m}{n}\right)$;

(iii) $\sup_m \int_0^1 |E_m(t,s)ds| \le C$; (iv) $\sum_{i=1}^{n} |\int_{A_i} E_m(t,s)ds| \le C$.

Lemma A.5. [20] Under assumptions (A5) and (A7), we have

(i) $\max_{1 \le i \le n} \sum_{j=1}^{n} \int_{A_j} |E_m(t_i,s)|ds \le c$; (ii) $\max_{1 \le i \le n} \sum_{j=1}^{n} \int_{A_i} |E_m(t_j,s)|ds \le c$.

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Authors' contributions

The two authors contributed equally to this work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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