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Local boundedness results for very weak solutions of double obstacle problems

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Abstract

This article mainly concerns double obstacle problems for second order divergence type elliptic equation div$A(x, u, \nabla u)$ = div$f(x)$. We give local boundedness for very weak solutions of double obstacle problems.

Keywords: double obstacle problems, local boundedness, elliptic equation

1 Introduction

Let Ω be a bounded open set of $\mathbf{R}^n$, $n \geq 2$. We consider the second order divergence type elliptic equation (also called A-harmonic equation or Leray-Lions equation)

$$\operatorname{div} A(x,\ u(x),\ \nabla u(x)) = \operatorname{div} f(x). \tag{1.1}$$

where $A : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is a Carathéodory function satisfying the coercivity and growth conditions: for almost all $x \in \Omega$, all $u \in \mathbf{R}$, and $\xi \in \mathbf{R}^n$,

(i) $\langle A(x, u, \xi), \xi\rangle \geq \alpha\, |\xi|^p$,

(ii) $|A(x, u, \xi)| \leq \beta_1\, |\xi|^{p-1} + \beta_2\, |u|^m + h(x)$,

where $\alpha > 0$, β_1 and β_2 are some nonnegative constants, $1 < p < n$, $p-1 \leq m \leq \frac{n(p-1)}{n-r}$ and $h(x) \in L_{\mathrm{loc}}^{s/(p-1)}(\Omega)$, $f(x) \in \left(L_{\mathrm{loc}}^{s/(p-1)}(\Omega)\right)^n$ for some $s > r$.

Suppose that ψ_1, ψ_2 are any functions in Ω with values in $\mathbf{R} \cup \{\pm\infty\}$, and that $\theta \in W^{1,r}(\Omega)$ with $\max\{1, p - 1\} < r \leq p$. Let

$$K_{\psi_1,\psi_2}^{\theta,r}(\Omega) = \{v \in W^{1,r}(\Omega) : \psi_1 \leq v \leq \psi_2,\ a.e.\ \text{and}\ v - \theta \in W_0^{1,r}(\Omega)\}.$$

The functions ψ_1, ψ_2 are two obstacles and θ determines the boundary values.

For any $u, v \in K_{\psi_1,\psi_2}^{\theta,r}(\Omega)$, we introduce the Hodge decomposition for $|\nabla(v-u)|^{r-p}\nabla(v-u) \in L^{\frac{r}{r-p+1}}$, see [1]:

$$|\nabla(v-u)|^{r-p}\nabla(v-u) = \nabla\phi_{v,u} + h_{v,u} \tag{1.2}$$

where $\phi_{v,u} \in W_0^{1,\frac{r}{r-p+1}}(\Omega)$, $h_{v,u} \in L^{\frac{r}{r-p+1}}(\Omega)$ is a divergence-free vector field and the following estimates hold:

$$\left\|\nabla\phi_{v,u}\right\|_{\frac{r}{r-p+1}} \leq c\left\|\nabla(v-u)\right\|_r^{r-p+1}, \tag{1.3}$$

$$\left\| h_{v,u} \right\|_{\frac{r}{r-p+1}} \leq c(p-r) \left\| \nabla(v-u) \right\|_r^{r-p+1}, \tag{1.4}$$

where $c = c(n, p)$ is a constant depending only on n and p.

Definition 1.1. A very weak solution to the $K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$-double obstacle problem for the A-harmonic Equation (1.1) is a function $u \in K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$ such that

$$\int_\Omega \langle A(x, u, \nabla u), |\nabla(v-u)|^{r-p}\nabla(v-u)-h\rangle dx \geq \int_\Omega \langle f(x), |\nabla(v-u)|^{r-p}\nabla(v-u)-h\rangle dx, \tag{1.5}$$

whenever $v \in K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$.

The obstacle problem has a strong background, and has many applications in physics and engineering. The local boundedness for solutions of obstacle problems plays a central role in many aspects. Based on the local boundedness, we can further study the regularity of the solutions. In [2], Gao et al. first considered the local boundedness for very weak solutions of obstacle problems to the A-harmonic equation in 2010. Precisely, the authors considered the local boundedness for very weak solutions of $K_{\psi,\theta}(\Omega)$-obstacle problems to the A-harmonic equation div $A(x, \nabla u(x)) = 0$ with the obstacle function $\psi \geq 0$, where operator A satisfies conditions $\langle A(x, \xi), \xi\rangle \geq \alpha|\xi|^p$ and $|A(x, \xi)| \leq \beta|\xi|^{p-1}$ with $A(x, 0) = 0$. For the property of weak solutions of nonlinear elliptic equations, we refer the reader to [3-6].

In this article, we continue to consider the local boundedness property. Under some general conditions (i) and (ii) given above on the operator A, we obtain a local boundedness result for very weak solutions of $K^{\theta,r}_{\psi_1,\psi_2}$-double obstacle problems to the A-harmonic Equation (1.1).

Theorem. Let operator A satisfies conditions (i) and (ii). Suppose that $\psi_1, \psi_2 \in W^{1,\infty}_{loc}(\Omega)$. Then a very weak solution u to the $K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$-obstacle problem of (1.1) is locally bounded.

Remark. Since we have assumed that operator A satisfies the conditions (ii), in the proof of the theorem, we have to estimate the integral of some power of $|u|$ by means of $|\nabla u|$. To deal with this difficulty, we will make use of the Sobolev inequality that was used in [4].

2 Preliminary knowledge and lemmas

We give some symbols and preliminary lemmas used in the proof. If $x_0 \in \Omega$ and $t > 0$, then B_t denotes the ball of radius t centered at x_0. For a function $u(x)$ and $k > 0$, let

$$A_k = \{x \in \Omega : |u(x)| > k\}, \ A_k^+ = \{x \in \Omega : u(x) > k\},$$
$$A_{k,t} = A_k \cap B_t, \ A_{k,t}^+ = A_k^+ \cap B_t.$$

Moreover, if $s < n$, s^* is always the real number satisfying $1/s^* = 1/s - 1/n$. Let $t_k(u) = \min\{u, k\}$.

Lemma 2.1. [7] Let $f(\tau)$ be a nonnegative bounded function defined for $0 \leq R_0 \leq t \leq R_1$. Suppose that for $R_0 \leq \tau < t \leq R_1$ one has

$$f(\tau) \leq A(t-\tau)^{-\alpha} + B + \theta f(t),$$

where A, B, α, θ are nonnegative constants and $\theta < 1$. Then there exists a constant $c = c(\alpha, \theta)$, depending only on α and θ, such that for every ρ, R, $R_0 \le \rho < R \le R_1$, one has

$$f(\rho) \le c[A(R - \rho)^{-\alpha} + B].$$

Definition 2.2. [8] A function $u \in W^{1,m}_{loc}(\Omega)$ belongs to the class $\mathbf{B}(\Omega, \gamma, m, k_0)$, if for all $k > k_0$, $k_0 > 0$ and all $B_\rho = B_\rho(x_0)$, $B_{\rho-\rho\sigma} = B_{\rho-\rho\sigma}(x_0)$, $B_R = B_R(x_0)$, one has

$$\int_{A^+_{k,\rho-\rho\sigma}} |\nabla u|^m dx \le \gamma \left\{ \sigma^{-m}\rho^{-m} \int_{A^+_{k,\rho}} (u-k)^m dx + |A^+_{k,\rho}| \right\},$$

for $R/2 \le \rho - \rho\sigma < \rho < R$, $m < n$, where $\left|A^+_{k,\rho}\right|$ is the n-dimensional Lebesgue measure of the set $A^+_{k,\rho}$.

Lemma 2.3. [8] Suppose that $u(x)$ is an arbitrary function belonging to the class $\mathbf{B}(\Omega, \gamma, m, k_0)$ and $B_R \subset\subset \Omega$. Then one has

$$\max_{B_{R/2}} u(x) \le c,$$

in which the constant c is determined only by γ, m, k_0, R, $||\nabla u||_m$.

3 Proof of theorem

Proof. Let u be a very weak solution to the $K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$-obstacle problem for the A-harmonic Equation (1.1). Let $B_{R_1} \subset\subset \Omega$ and $0 < R_1/2 \le \tau < t \le R_1$ be arbitrarily fixed. Fix a cutoff function $\phi \in C^\infty_0(B_{R_1})$, such that

$$\operatorname{supp}\phi \subset B_t,\ 0 \le \phi \le 1,\ \phi \equiv 1 \text{ in } B_\tau,\ |\nabla\phi| \le 2(t-\tau)^{-1}. \tag{3.1}$$

If ψ_2 is an arbitrary function in Ω with values in $\mathbf{R} \cup \{+\infty\}$, consider the function

$$v = u - \phi^r(u - \psi_k), \tag{3.2}$$

where

$$\psi_k = \min\{\max\{\psi_1,\ t_k(u)\},\ \psi_2\},\ \ t_k(u) = \min\{u,\ k\},\ \ k \ge 0.$$

It is easy to see $\psi_1 \le \psi_k \le \psi_2$. Now, $v \in K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$; indeed, since $u \in K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$ and $\phi \in C^\infty_0(\Omega)$, then

$$\begin{aligned} v - \theta &= u - \theta - \phi^r(u - \psi_k) \in W^{1,r}_0(\Omega),\\ v - \psi_1 &= u - \psi_1 - \phi^r(u - \psi_k) \ge (1 - \phi^r)(u - \psi_1) \ge 0 \text{ a.e.in } \Omega,\\ v - \psi_2 &= u - \psi_2 - \phi^r(u - \psi_k) \le (1 - \phi^r)(u - \psi_2) \le 0 \text{ a.e.in } \Omega. \end{aligned} \tag{3.3}$$

For any fixed $k > 0$, let

$$v_0 = \begin{cases} u, & \text{if} u \le k,\\ v, & \text{if} u > k. \end{cases}$$

It is easy to see that $v_0 \in K^{\theta,r}_{\psi_1,\psi_2}(\Omega)$. Then by Definition 1.1 we have

$$\int_\Omega \langle A(x,\ u,\ \nabla u),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u) - \tilde{h}_{v,u}\rangle dx \ge \int_\Omega \langle f(x),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u) - \tilde{h}_{v,u}\rangle dx. \tag{3.4}$$

If $u \leq k$, then $\tilde{h}_{v,u} = 0$, $\nabla\tilde{\phi}_{v,u} = 0$; If $u > k$, then $\tilde{h}_{v,u} = h_{v,u}$, $\tilde{\phi}_{v,u} = \phi_{v,u}$. It's derived from the uniqueness of Hodge decomposition. This means that

$$\begin{aligned}
&\int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ h_{v,u}\rangle dx + \int_{A_{k,t}^{+}} \langle f(x),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u) - \tilde{h}_{v,u}\rangle dx \\
&\leq \int_{\Omega} \langle A(x,\ u,\ \nabla u),\ h_{v,u}\rangle dx + \int_{\Omega} \langle f(x),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u) - \tilde{h}_{v,u}\rangle dx \\
&\leq \int_{\Omega} \langle A(x,\ u,\ \nabla u),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u)\rangle dx \\
&= \left(\int_{\Omega\cap\{u\leq k\}} + \int_{\Omega\cap\{u>k\}}\right) \langle A(x,\ u,\ \nabla u),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u)\rangle dx \\
&= \int_{\Omega\cap\{u>k\}} \langle A(x,\ u,\ \nabla u),\ |\nabla(v_0 - u)|^{r-p}\nabla(v_0 - u)\rangle dx \\
&= \int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ |\nabla(v - u)|^{r-p}\nabla(v - u)\rangle dx.
\end{aligned} \tag{3.5}$$

Let

$$E(v,\ u) = |\phi^r\nabla u|^{r-p}\phi^r\nabla u + |\nabla(v - u)|^{r-p}\nabla(v - u). \tag{3.6}$$

By an elementary inequality [[9], P. 271, (4.1)],

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \leq 2^{\varepsilon}\frac{1+\varepsilon}{1-\varepsilon}|X - Y|^{1-\varepsilon},\ \ X,\ Y \in \mathbf{R}^n,\ \ 0 \leq \varepsilon < 1, \tag{3.7}$$

$$\nabla v = \nabla u - \phi^r(\nabla u - \nabla\psi_k) - r\phi^{r-1}\nabla\phi(u - \psi_k), \tag{3.8}$$

one can derive that

$$|E(v,\ u)| \leq 2^{p-r}\frac{p-r+1}{r-p+1}|\phi^r\nabla\psi_k - r\phi^{r-1}\nabla\phi(u - \psi_k)|^{r-p+1}. \tag{3.9}$$

We get from the definition of $E(v,\ u)$ and (3.5) that

$$\begin{aligned}
&\int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ |\phi^r\nabla u|^{r-p}\phi^r\nabla u\rangle dx \\
&= \int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ E(v,\ u)\rangle dx - \int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ |\nabla(v - u)|^{r-p}\nabla(v - u)\rangle dx \\
&\leq \int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ E(v,\ u)\rangle dx - \int_{A_{k,t}^{+}} \langle A(x,\ u,\ \nabla u),\ h_{v,u}\rangle dx - \int_{A_{k,t}^{+}} \langle f(x),\ E(v,\ u)\rangle dx \\
&\quad + \int_{A_{k,t}^{+}} \langle f(x),\ |\phi^r\nabla u|^{r-p}\phi^r\nabla u\rangle dx + \int_{A_{k,t}^{+}} \langle f(x),\ h_{v,u}\rangle dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{3.10}$$

We now estimate the left-hand side and the right-hand side of (3.10), respectively. Firstly,

$$\begin{aligned}\int_{A_{k,t}^+} \langle A(x,\ u,\ \nabla u),\ |\phi^r \nabla u|^{r-p} \phi^r \nabla u\rangle dx &\geq \int_{A_{k,\tau}^+} \langle A(x,\ u,\ \nabla u),\ |\nabla u|^{r-p} \nabla u\rangle dx \\ &\geq \alpha \int_{A_{k,\tau}^+} |\nabla u|^r dx,\end{aligned} \tag{3.11}$$

here we have used condition (i). Secondly, by condition (ii) and (3.9),

$$\begin{aligned}|I_1| &= \left| \int_{A_{k,t}} \langle A(x,\ u,\ \nabla u),\ E(v,\ u)\rangle dx \right| \\ &\leq \int_{A_{k,t}^+} [\beta_1 |\nabla u|^{p-1} + \beta_2 |u|^m + h_1] |E(v,\ u)| dx \\ &\leq 2^{p-r} \frac{p-r+1}{r-p+1} \int_{A_{k,t}^+} [\beta_1 |\nabla u|^{p-1} + \beta_2 |u|^m + h_1] |\phi^r \nabla \psi_k - r\phi^{r-1} \nabla \phi (u-\psi_k)|^{r-p+1} \\ &= C_1 \beta_1 \int_{A_{k,t}^+} |\nabla u|^{p-1} |\phi^r \nabla \psi_k|^{r-p+1} + C_1 \beta_1 \int_{A_{k,t}^+} |\nabla u|^{p-1} |r\phi^{r-1} \nabla \phi (u-\psi_k)|^{r-p+1} \\ &\quad + C_1 \beta_2 \int_{A_{k,t}^+} |u|^m |\phi^r \nabla \psi_k|^{r-p+1} + C_1 \beta_2 \int_{A_{k,t}^+} |u|^m |r\phi^{r-1} \nabla \phi (u-\psi_k)|^{r-p+1} \\ &\quad + C_1 \int_{A_{k,t}^+} |h| \left|\phi^r \nabla \psi_k\right|^{r-p+1} + C_1 \int_{A_{k,t}^+} |h| \left|r\phi^{r-1} \nabla \phi (u-\psi_k)\right|^{r-p+1} \\ &= I_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16},\end{aligned} \tag{3.12}$$

where $C_1 = 2^{p-r} \frac{p-r+1}{r-p+1}$. By Young's inequality

$ab \leq \varepsilon a^{p'} + C(\varepsilon,\ p) b^p$ valid for $a,\ b \geq 0,\ \varepsilon > 0$ and $p > 1$,

and $\frac{p-1}{r} + \frac{r-p+1}{r} = 1$, we have the estimates

$$|I_{11}| \leq C_1 \beta_1 \left[\varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + C(\varepsilon,\ p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \right], \tag{3.13}$$

$$|I_{12}| \leq C_1 \beta_2 \left[\varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + C(\varepsilon,\ p) \int_{A_{k,t}^+} |r\phi^{r-1} \nabla \phi (u-\psi_k)|^r dx \right], \tag{3.14}$$

$$|I_{13}| \leq C_1 \left[\varepsilon \int_{A_{k,t}^+} |u|^{\frac{mr}{p-1}} dx + C(\varepsilon,\ p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \right], \tag{3.15}$$

where we have used $|\nabla \psi_k| \leq |\nabla \psi_1| + |\nabla \psi_2|$ in $A_{k,t}^+$.

We observe now that, if $w \in W^{1,p}(B_t)$ and $|\text{supp} w| \leq 1/2|B_t|$, then we have the Sobolev inequality (see also [10]),

$$\left(\int_{B_t} |w|^{p*}dx\right)^{p/p*} \leq c_1(n,p)\int_{B_t} |\nabla w|^p dx. \tag{3.16}$$

Set

$$g_k(u) = \begin{cases} u, \text{ if} u & \leq k, \\ 0, \text{ if} u & > k. \end{cases}$$

Since $p - 1 \leq m \leq \frac{n(p-1)}{n-r}$ by assumption, then $r \leq \frac{mr}{p-1} \leq r^*$. (3.16) implies

$$\begin{aligned} \int_{A^+_{k,t}} |u|^{\frac{mr}{p-1}} dx &= \int_{B_t} |u - g_k(u)|^{\frac{mr}{p-1}} dx \\ &\leq ||u - g_k(u)||_{r^*}^{\frac{mr}{p-1} - r} |B_t|^{1 - \frac{mr}{(p-1)r*}} \left(\int_{B_t} |u - g_k(u)|^{r*} dx\right)^{r/r*} \\ &\leq c_1(n, p)||u - g_k(u)||_{r^*}^{\frac{mr}{p-1} - r} |B_t|^{1 - \frac{mr}{(p-1)r*}} \int_{B_t} |\nabla(u - g_k(u))|^r dx \\ &= c_1(n, p)||u - g_k(u)||_{r^*}^{\frac{mr}{p-1} - r} |B_t|^{1 - \frac{mr}{(p-1)r^*}} \int_{A^+_{k,t}} |\nabla u|^r dx, \end{aligned} \tag{3.17}$$

provided that $|\text{supp}(u - g_k(u))|_{B_t}| \leq 1/2\, |B_t|$. Since $\sup \text{p}(u - g_k(u))|_{B_t} \subset A^+_{k,t}$, then $|\text{supp}| \sup \text{p}(u - g_k(u))|_{B_t}| \leq |A^+_{k,t}|$. On the other hand, we have

$$||u||^{r^*}_{r^*,B_t} = \int_{B_t} u^{r^*} dx \geq \int_{A^+_{k,t}} |u|^{r^*} dx \geq k^{r^*} |A^+_{k,t}|.$$

Thus, there exists a constant k_0 >0, such that for all $k \geq k_0$, we have $\left|A^+_{k,t}\right| \leq 1/2|B_t|$. We can also suppose that k_0 such that

$$\int_{A_{k_0,t}} u^{r^*} dx \leq 1.$$

For such values of k we then have inequality

$$\begin{aligned} \int_{A^+_{k,t}} |u|^{m\frac{r}{p-1}} dx &\leq C_2(n,p)||u - g_k(u)||_{r*}^{\frac{mr}{p-1} - r} |B_t|^{1 - \frac{mr}{(p-1)r^*}} \int_{A^+_{k,t}} |\nabla u|^r dx \\ &\leq C_2(n,p)||u - g_k(u)||_{r*}^{\frac{mr}{p-1} - r} |\Omega|^{1 - \frac{mr}{(p-1)r^*}} \int_{A^+_{k,t}} |\nabla u|^r dx \\ &\leq C_2(n,p)|\Omega|^{1 - \frac{mr}{(p-1)r^*}} \int_{A^+_{k,t}} |\nabla u|^r dx \\ &= C_3 \int_{A^+_{k,t}} |\nabla u|^r dx, \end{aligned} \tag{3.18}$$

here $C_3 = C_3(n, m, p, r, k_0, |\Omega|)$. We derive from (3.15) and (3.18) that

$$|I_{13}| \le C_1C_3\varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + C_1C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx. \tag{3.19}$$

$$|I_{14}| \le C_1C_3\varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + C_1C(\varepsilon, p) \int_{A_{k,t}^+} |r\phi^{r-1}\nabla \phi(u - \psi_k)|^r dx. \tag{3.20}$$

I_{15} and I_{16} can be estimated as follows:

$$|I_{15}| \le C_1\varepsilon \int_{A_{k,t}^+} |h|^{\frac{r}{p-1}} dx + C_1C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx. \tag{3.21}$$

$$|I_{16}| \le C_1\varepsilon \int_{A_{k,t}^+} |h|^{\frac{r}{p-1}} dx + C_1C(\varepsilon, p) \int_{A_{k,t}^+} |r\phi^{r-1}\nabla \phi(u - \psi_k)|^r dx. \tag{3.22}$$

In conclusion, we derive from (3.12)-(3.14), (3.19)-(3.22) that

$$\begin{aligned} |I_1| \le\ & (C_1\beta_1\varepsilon + C_1\beta_2\varepsilon + 2C_1C_3\varepsilon) \int_{A_{k,t}^+} |\nabla u|^r dx + 2C_1\varepsilon \int_{A_{k,t}^+} |h|^{\frac{r}{p-1}} dx \\ & + (C_1\beta_1 + 2C_1)C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \\ & + (C_1\beta_2 + 2C_1)C(\varepsilon, p) \int_{A_{k,t}^+} |r\phi^{r-1}\nabla \phi(u - \psi_k)|^r dx. \end{aligned} \tag{3.23}$$

By $|\nabla\varphi| \le 2(t - \tau)^{-1}$ and $|u - \psi_k| \le |u - k|$ a.e. in $A_{k,t}^+$, we have

$$\begin{aligned} |I_1| \le\ & C_4\varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + 2C_1\varepsilon \int_{A_{k,t}^+} |h|^{\frac{r}{p-1}} dx + C_5C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \\ & + C_6C(\varepsilon, p) \frac{2^r r}{(t - \tau)^r} \int_{A_{k,t}^+} |u - k|^r dx. \end{aligned} \tag{3.24}$$

We now estimate $|I_2|$. By condition (ii),

$$\begin{aligned} |I_2| &= \left| \int_{A_{k,t}^+} \langle A(x, u, \nabla u), h_{v,u} \rangle dx \right| \\ &\le \int_{A_{k,t}^+} [\beta_1 |\nabla u|^{p-1} + \beta_2 |u|^m + h] |h_{v,u}| dx \\ &\le \beta_1 \int_{A_{k,t}^+} |\nabla u|^{p-1} |h_{v,u}| dx + \beta_2 \int_{A_{k,t}^+} |u|^m |h_{v,u}| dx + \int_{A_{k,t}^+} |h||h_{v,u}| dx \\ &= I_{21} + I_{22} + I_{23}. \end{aligned} \tag{3.25}$$

By Young's inequality, Hölder's inequality and (1.4), I_{21} and I_{23} can be estimated as

$$\begin{aligned}|I_{21}| &\le \beta_1\left(\int_{A^+_{k,t}} |\nabla u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |h_{v,u}|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}}\\ &\le \beta_1 C_7(p-r)\left(\int_{A^+_{k,t}} |\nabla u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |\nabla(v-u)|^r dx\right)^{\frac{r-p+1}{r}}\\ &\le \beta_1 C_7(p-r)\varepsilon\int_{A^+_{k,t}} |\nabla u|^r dx + \beta_1 C_7(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx.\end{aligned} \tag{3.26}$$

$$\begin{aligned}|I_{23}| &\le \left(\int_{A^+_{k,t}} |h|^{\frac{r}{p-1}} dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |h_{v,u}|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}}\\ &\le C_7(p-r)\left(\int_{A^+_{k,t}} |h|^{\frac{r}{p-1}} dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |\nabla(v-u)|^r dx\right)^{\frac{r-p+1}{r}}\\ &\le C_7(p-r)\varepsilon\int_{A^+_{k,t}} |h|^{\frac{r}{p-1}} dx + C_7(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx.\end{aligned} \tag{3.27}$$

By (3.18), we know that if $k \ge k_0$, then

$$\begin{aligned}|I_{22}| &\le \beta_2\left(\int_{A^+_{k,t}} |u|^{\frac{mr}{p-1}} dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |h_{v,u}|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}}\\ &\le \beta_2 C_4(p-r)\left(\int_{A^+_{k,t}} |u|^{\frac{mr}{p-1}} dx\right)^{\frac{p-1}{r}}\left(\int_{A^+_{k,t}} |\nabla(v-u)|^r dx\right)^{\frac{r-p+1}{r}}\\ &\le \beta_2 C_7(p-r)\varepsilon\int_{A^+_{k,t}} |u|^{\frac{mr}{p-1}} dx + \beta_2 C_7(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx\\ &\le \beta_2 C_7(p-r)\varepsilon C_3\int_{A^+_{k,t}} |\nabla u|^r dx + \beta_2 C_7(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx.\end{aligned} \tag{3.28}$$

Combining (3.25) with (3.26), (3.27), and (3.28), we obtain

$$\begin{aligned}|I_2| &\le (\beta_1 C_4 + \beta_2 C_4 C_3)(p-r)\varepsilon\int_{A^+_{k,t}} |\nabla u|^r dx + C_4(p-r)\varepsilon\int_{A^+_{k,t}} |h|^{\frac{r}{p-1}} dx\\ &\quad + (\beta_1 C_4 + \beta_2 C_4 + C_4)(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx\\ &= C_8\varepsilon\int_{A^+_{k,t}} |\nabla u|^r dx + C_9\int_{A^+_{k,t}} |h|^{\frac{r}{p-1}} dx + C_{10}(p-r)C(\varepsilon,p)\int_{A^+_{k,t}} |\nabla(v-u)|^r dx.\end{aligned} \tag{3.29}$$

We now estimate $|I_3|$, $|I_4|$, and $|I_5|$.

$$
\begin{aligned}
|I_3| &= \left| \int_{A_{k,t}^+} \langle f(x),\ E_{v,u} \rangle dx \right| \\
&\le 2^{p-r} \frac{p-r+1}{r-p+1} \int_{A_{k,t}^+} |f(x)| \left| \phi^r \nabla \psi_k - r\phi^{r-1} \nabla \phi (u - \psi_k) \right|^{r-p+1} dx \\
&\le C_1 \int_{A_{k,t}^+} |f(x)| |\phi^r \nabla \psi_k|^{r-p+1} dx + C_1 \int_{A_{k,t}^+} |f(x)| \left| r\phi^{r-1} \nabla \phi (u - \psi_k) \right|^{r-p+1} dx \\
&\le C_1 \varepsilon \int_{A_{k,t}^+} |f(x)|^{\frac{r}{p-1}} dx + C_1 C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \\
&\quad + C_1 C(\varepsilon, p) \int_{A_{k,t}^+} |r\phi^{r-1} \nabla \phi (u - \psi_k)|^r dx \\
&\le C_1 \varepsilon \int_{A_{k,t}^+} |f(x)|^{\frac{r}{p-1}} dx + C_1 C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla \psi_1|^r + |\nabla \psi_2|^r dx \\
&\quad + C_1 C(\varepsilon, p) \frac{2^r r}{(t-\tau)^r} \int_{A_{k,t}^+} |u-k|^r dx.
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
|I_4| &= \left| \int_{A_{k,t}^+} \langle f(x),\ |\phi^r \nabla u|^{r-p} \phi^r \nabla u \rangle dx \right| \\
&\le \int_{A_{k,t}^+} |f(x)| |\nabla u|^{r-p+1} dx \\
&\le \varepsilon \int_{A_{k,t}^+} |\nabla u|^r dx + C(\varepsilon, p) \int_{A_{k,t}^+} |f(x)|^{\frac{r}{p-1}} dx.
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
|I_5| &= \left| \int_{A_{k,t}^+} \langle f(x),\ h_{v,u} \rangle dx \right| \\
&\le \left(\int_{A_{k,t}^+} |f|^{\frac{r}{p-1}} dx \right)^{\frac{p-1}{r}} \left(\int_{A_{k,t}^+} |h_{v,u}|^{\frac{r}{r-p+1}} dx \right)^{\frac{r-p+1}{r}} \\
&\le C_7 (p-r) \left(\int_{A_{k,t}^+} |f|^{\frac{r}{p-1}} dx \right)^{\frac{p-1}{r}} \left(\int_{A_{k,t}^+} |\nabla (v-u)|^r dx \right)^{\frac{r-p+1}{r}} \\
&\le C_7 (p-r) \varepsilon \int_{A_{k,t}^+} |f|^{\frac{r}{p-1}} dx + C_7 (p-r) C(\varepsilon, p) \int_{A_{k,t}^+} |\nabla (v-u)|^r dx.
\end{aligned}
\tag{3.32}
$$

By (3.8), we have

$$\int_{A_{k,t}^{+}} |\nabla(v-u)|^{r}dx \le \int_{A_{k,t}^{+}} |\nabla u|^{r}dx + \int_{A_{k,t}^{+}} |\nabla\psi_1|^{r} + |\nabla\psi_2|^{r}dx + \frac{2^{r}r}{(t-\tau)^{r}}\int_{A_{k,t}^{+}} |u-k|^{r}dx \tag{3.33}$$

Thus, the inequalities (3.10), (3.11), (3.24), and (3.29)-(3.33) imply that

$$\begin{aligned}
&\int_{A_{k,\tau}^{+}} |\nabla u|^{r}dx \\
&\le \frac{1}{\alpha}\{C_4\varepsilon + C_8\varepsilon + \varepsilon + (C_{10}+C_7)(p-r)C(\varepsilon,p)\}\int_{A_{k,t}^{+}} |\nabla u|^{r}dx \\
&\quad + \frac{1}{\alpha}\{2C_1\varepsilon + C_9\}\int_{A_{k,t}^{+}} |h|^{\frac{r}{p-1}}dx \\
&\quad + \frac{1}{\alpha}\{C_1\varepsilon + C(\varepsilon,\ p) + C_7(p-r)\varepsilon\}\int_{A_{k,t}^{+}} |f|^{\frac{r}{p-1}}dx \\
&\quad + \frac{1}{\alpha}\{C_5C(\varepsilon,\ p) + C_1C(\varepsilon,\ p) + (C_{10}+C_7)(p-r)C(\varepsilon,\ p)\}\int_{A_{k,t}^{+}} |\nabla\psi_1|^{r} + |\nabla\psi_2|^{r}dx \\
&\quad + \frac{1}{\alpha}\{C_6C(\varepsilon,\ p) + C_1C(\varepsilon,\ p) + (C_{10}+C_7)(p-r)C(\varepsilon,\ p)\}\frac{2^{r}r}{(t-\tau)^{r}}\int_{A_{k,t}^{+}} |u-k|^{r}dx.
\end{aligned} \tag{3.34}$$

Choosing ε and p-r small enough such that, the summation θ of the coefficients of the first term in the right-handside of (3.34) is smaller than 1. Let ρ, R be arbitrarily fixed with $R_1/2 \le \rho < R \le R_1$. Thus, from (3.34), we deduce that for every t and τ, such that $R_1/2 \le \tau < t \le R_1$, we have

$$\begin{aligned}
\int_{A_{k,\tau}^{+}} |\nabla u|^{r}dx &\le \frac{C_{11}}{\alpha}\int_{A_{k,R}^{+}} \left(|\nabla\psi_1|^{r} + |\nabla\psi_2|^{r} + |h|^{\frac{r}{p-1}} + |f|^{\frac{r}{p-1}}\right)dx \\
&\quad + \frac{C_{12}}{\alpha(t-\tau)^{r}}\int_{A_{k,R}^{+}} |u-k|^{r}dx + \theta\int_{A_{k,t}^{+}} |\nabla u|^{r}dx,
\end{aligned} \tag{3.35}$$

where C_{11},C_{12} are some constants depending only on n, p, r, m, k_0, $|\Omega|$, α, β_1 and β_2. Applying Lemma 2.1, we conclude that

$$\begin{aligned}
\int_{A_{k,\rho}^{+}} |\nabla u|^{r}dx &\le \frac{cC_{11}}{\alpha(R-\rho)^{r}}\int_{A_{k,R}^{+}} |u-k|^{r}dx \\
&\quad + \frac{cC_{12}}{\alpha}\int_{A_{k,R}^{+}} \left(|\nabla\psi_1|^{r} + |\nabla\psi_2|^{r} + |h_1|^{\frac{r}{p-1}} + |h_2|^{\frac{r}{p-1}}\right)dx \\
&\le \frac{cC_{11}}{\alpha(R-\rho)^{r}}\int_{A_{k,R}^{+}} |u-k|^{r}dx + \frac{cC_{12}C_{13}}{\alpha}|A_{k,R}^{+}|,
\end{aligned} \tag{3.36}$$

where c is the constant given by Lemma 2.1 and $C_{13} = \left\| |\nabla\psi_1|^{r} + |\nabla\psi_2|^{r} + |h|^{\frac{r}{p-1}} + |f|^{\frac{r}{p-1}} \right\|_{L^{\infty}(\Omega)}$. Thus u belongs to the class **B** with $\gamma = \max\{c_2c_7/\alpha,\ c_2c_6c_8/\alpha\}$ and $m = r$. Lemma 2.3 yields

$$\max_{B_{R/2}} u(x) \leq c.$$

If ψ_1 is an arbitrary function in Ω with values in $\mathbf{R} \cup \{-\infty\}$, noticing $-\psi_2 \leq -u \leq -\psi_1$, we only use $-u$ in place of u above.

These results together with the assumptions $\psi_1 \leq u \leq \psi_2$ and $\psi_1, \psi_2 \in W^{1,\infty}_{1\text{oc}}(\Omega)$ yield the desired result.

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Authors' contributions
YT and JL carried out the proof of Theorme in this paper. JG provieded the main idea of this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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