Zhang et al. Journal of Inequalities and Applications 2012, **2012**:42
http://www.journalofinequalitiesandapplications.com/content/2012/1/42

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Boundedness of (k + 1)-linear fractional integral with a multiple variable kernel

Huihui Zhang[1], Jianmiao Ruan[2] and Xiao Yu[1*]

* Correspondence: yx2000s@163.com
[1]Department of Mathematics, Shangrao Normal University, Shangrao, 334001, P.R. China
Full list of author information is available at the end of the article

Abstract
In this article, we discuss the boundedness of $(k + 1)$-linear fractional integrals with variable kernels on product L^p spaces. Our results improved some known results.
2000 Mathematics Subject Classification: 42B20; 42B25.

Keywords: multilinear, variable kernel, fractional integral.

1. Introduction

It is well known that multilinear theory plays an important role in harmonic analysis and mathematicians pay much attention to it, see [1,2] for more details. In 1992, Grafakos [1] first proved that the multilinear fractional operator $I_\beta^m(\vec{f})(x)$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ spaces to L^r space with $1/r + \alpha/n = 1/s$, where $1/s = 1/p_1 + \cdots + 1/p_m$ and s satisfies $n/(n + \alpha) \leq s < n/\alpha$ with multilinear fractional $I_\beta^m(\vec{f})(x)$ defined as following:

$$I_\beta^m(\vec{f})(x) = \int\limits_{\mathbb{R}^n} \frac{1}{|y|^{n-\beta}} \prod_{i=1}^m f_i(x - \theta_i y)dy, \tag{1.1}$$

for fixed nonzero real numbers $\theta_i (i = 1, ..., m)$ and $0 < \beta < n$.

Later, Ding and Lu [3] improved Grafakos's results to the case when $I_\beta^m(\vec{f})(x)$ has a rough kernel $\Omega_0(x)$ with $\Omega_0(x) \in L^r(S^{n-1})$ and

$$I_{\beta,\Omega_0}^m(\vec{f})(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega_0(y)}{|y|^{n-\beta}} \prod_{i=1}^m f_i(x - \theta_i y)dy, \tag{1.2}$$

Ding and Lu proved that $I_{\beta,\Omega_0}^m(\vec{f})(x)$ is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ spaces to L^q spaces with $1/p_1 + \ldots\ldots + 1/p_k - 1/q = \beta/$n. Obviously, Ding and Lu's results improved the main results in [1].

In 1999, Kenig and Stein [2] studied a new kind of multilinear fractional integral associated with the bilinear fractional integrals operators, they defined the $(k + 1)$-linear fractional integrals as following,

$$I_{\alpha,A}\left(f_1,\ldots,f_{k+1}\right)(x) = \int\limits_{(R^n)^k} f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)$$
$$\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\frac{dy_1,\ldots,dy_k}{\left|\left(y_1,\ldots,y_k\right)\right|^{nk-\alpha}}, 0<\alpha<kn,$$

where for a fixed $k \in N$ and $1 \le i, j \le k + 1$, a linear mapping $\ell_j : R^{n(k+1)} \to R^n$, $1 \le j \le k + 1$ is defined by

$$\ell_j\left(x_1\ldots,x_k,x\right) = A_{1j}x_1 + \cdots + A_{kj}x_k + A_{k+1,j}x. \tag{1.3}$$

Here, A_{ij} is an $n \times n$ matrix and a $(k + 1)n \times (k + 1)n$ matrix $A = (A_{ij})$ $(i = 1, \ldots, k + 1, j = 1, \ldots, k + 1,)$ satisfies the following assumptions:

(I) For each $1 \le j \le k + 1$, $A_{k+1,i}$ is an invertible $n \times n$ matrix.
(II) A is an invertible $(k + 1)n \times (k + 1)n$ matrix.
(III) For each j_0, $1 \le j_0 \le k + 1$, consider the $kn \times kn$ matrix $A_{j_0} = \left(A_{j_0}\right)_{\ell m}$, where

$$\left(A_{j_0}\right)_{\ell m} = \begin{cases} A_{\ell,m} & 1\le\ell\le k, 1\le m\le k, m<j_0 \\ A_{\ell,m+1} & 1\le\ell\le k, 1\le m\le k, m<j_0. \end{cases}$$

Obviously, when $k = 1$ and $A_{11} = I$, $A_{21} = I$, $A_{12} = -I$, $A_{22} = I$, $I_{\alpha,A}$ $(f_1, f_2)(x)$ becomes the classical bilinear fractional integral, that is

$$I_{\alpha,A}\left(f_1,f_2\right)(x) = B_\alpha\left(f_1,f_2\right)(x) = \int\limits_{R^n} f_1(x+t)f_2(x-t)\frac{dt}{|t|^{n-\alpha}}. \tag{1.4}$$

In [2], Kenig and Stein proved that $B_\alpha(f_1, f_2)(x)$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q with $1/p_1 + 1/p_2$-$1/q = \alpha/n$ for $1 \le p_1, p_2 \le \infty$. Later, Ding and Lin [4] considered the following bilinear fractional integral with a rough kernel,

$$B_{\alpha,\Omega_0}\left(f_1,f_2\right)(x) = \int\limits_{R^n} f_1(x+t)f_2(x-t)\Omega_0(t)\frac{dt}{|t|^{n-\alpha}},$$

where $\Omega_0(y')$ is a rough kernel belongs to $L^s(S^{n-1})(s > 1)$ without any smoothness on the unit sphere.

Ding and Lin proved the following theorem,

Theorem A ([4])

Assume that $0 < \alpha < n, 1 < s' < \frac{n}{\alpha}$, $1/p_1 + 1/p_2$ - α/n, $1/q = 1/p_1 + 1/p_2$ - α/n, and that $s < \min\{p_1, p_2\}$, then for $1 \le p_1, p_2 \le \infty$, we have

$$\left\|B_{\alpha,\Omega_0}\left(f_1,f_2\right)\right\|_{L^q} \le C\left\|f_1\right\|_{L^{p_1}}\left\|f_2\right\|_{L^{p_2}}.$$

For the research of partial differential equation, mathematicians pay much attention to the singular integral (or fractional integral) with a variable kernel $\Omega(x, y)$, see [5,6] for more details. A function $\Omega(x, y)$ is said to be belonged to $L^\infty(R^n) \times L^q(S^{n-1})$ if the function $\Omega(x, y)$ satisfies the following conditions:

(i) $\Omega(x, \lambda z) = \Omega(x, z)$ for any $x, z \in R^n$ and $\lambda > 0$.
(ii) $\|\Omega\|_{L^\infty(R^n)\times L^q(S^{n-1})} = \sup\limits_{x\in R^n}\left(\int_{S^{n-1}}\left|\Omega(x,z')\right|^q d\sigma\left(z'\right)\right)^{1/q} < \infty.$

Recently, Chen and Fan [7] considered the following bilinear fractional integral with a variable kernel,

$$B_{\alpha,\Omega}\left(f_1, f_2\right)(x) = \int_{R^n} f_1(x+t) f_2(x-t) \frac{\Omega(x,t)}{|t|^{n-\alpha}} dt, \tag{1.5}$$

they proved the following result,

Theorem B([7])

Let $1/p = 1/p_1 + 1/p_2 - \alpha/n$ and $\Omega(x, y) \in L^\infty(R^n) \times L^s(S^{n-1})$ with $s' < \min\{p_1, p_2\}$ and $s > \frac{n}{n-\alpha}$, then

$$\left\|B_{\alpha,\Omega}\left(f_1, f_2\right)\right\|_{L^p} \leq C \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

Obviously, Chen and Fan's result improved the main results in [4] and the method they used is different from [4].

In this article, we will consider the $(k + 1)$-linear fractional integral with a multiple variable kernel $\Omega\left(x, \vec{y}\right)$. Before state the main results in this article, we first introduce a multiple variable function $\Omega\left(x, \vec{y}\right) \in L^\infty(R^n) \times L^r\left(S^{nk-1}\right)$ satisfying the following conditions:

(i) $\Omega\left(x, \lambda\vec{y}\right) = \Omega\left(x, \vec{y}\right)$ for any $\lambda > 0$.

(ii) $\|\Omega\|_{L^\infty(R^n)\times L^r(S^{nk-1})} = \sup_{x\in R^n}\left(\int_{S^{nk-1}} \left|\Omega(x, \vec{y}')\right|^r d\sigma\left(\vec{y}'\right)\right) < \infty.$

Now, we define the $(k + 1)$-linear fractional integral with a multiple variable kernel $\Omega(x, \vec{y}) \in L^\infty(R^n) \times L^r\left(S^{nk-1}\right)$ as following:

$$I_{\alpha,A}^{\Omega}\left(f_1, \ldots, f_{k+1}\right)(x) = \int_{(R^n)^k} f_1\left(\ell_1\left(y_1, \ldots, y_k, x\right)\right) \cdots f_{k+1}\left(\ell_{k+1}\left(y_1, \ldots, y_k, x\right)\right) \frac{\Omega(x, \vec{y})}{\left|\left(y_1, \ldots, y_k\right)\right|^{nk-\alpha}}, dy_1, \ldots, dy_k,$$

where the linear mapping ℓ_j is defined as in (1.3) and the corresponding matrix A satisfies the assumptions (I), (II) and (III). What's more, we assume that for each $1 \leq j_0 \leq k + 1$, A_{j_0} is an invertible $kn \times kn$ matrix.

Our main results are as following,

Theorem 1.1.

Assume that (I), (II) and (III) hold, if $\Omega(x, \vec{y}) \in L^\infty(R^n) \times L^r\left(S^{nk-1}\right)$ for $r > \frac{nk}{nk-\alpha}$ and $0 < \alpha < kn$, then

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1, \ldots, f_{k+1}\right)\right\|_{L^{p,\infty}} \leq C \prod_{i=1}^{k+1} \|f_i\|_{L^{r'}}$$

with $1/p = (k + 1)/r' - \alpha/n$.

Theorem 1.2.

Assume that (I), (II) and (III) hold, if $\Omega(x, \vec{y}) \in L^\infty(R^n) \times L^r\left(S^{nk-1}\right)$ for $r' < \min\{p_1, \ldots, p_{k+1}\}$, $r > \frac{nk}{nk-\alpha}$ and $0 < \alpha < kn$, then

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^q}\leq C\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{p_i}}$$

with $1/q = 1/p_1 + \ldots\ldots + 1/p_{k+1} - \alpha/n$.

Remark 1.3.

As far as we know, our results are also new even in the case that if we replace $\Omega(x,\vec{y})$ by $\Omega_0(\vec{y})\in L^r\left(S^{nk-1}\right)$.

Remark 1.4.

Obviously, our results improved the main results in [2,4,7].

2. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. First we introduce some definitions and lemmas that will be used throughout this article.

Denote

$$M_{A,s}\left(f_1,\ldots,f_{k+1}\right)(x)=\int_{2^{-s}\leq|(y_1,\ldots,y_k)|\leq 2^{-s+1}}\Omega\left(x,\vec{y}\right)f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\ldots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)d\vec{y},$$

thus we have the following conclusion.

Lemma 2.1.

Let $\Omega(x,\vec{y})$ be as in Theorem 1.1 and assume (I), (II) and (III) hold, then

$$\left\|M_{A,s}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{\frac{r'}{k+1}}}\leq C\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}2^{-nks}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}
M_{A,s}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)(x)&\leq C\left(\int_{2^{-s}\leq|(y_1,\ldots,y_k)|\leq 2^{-s+1}}\left|\Omega(x,\vec{y})\right|^r d\vec{y}\right)^{1/r}\\
&\times\left(\int_{2^{-s}\leq|(y_1,\ldots,y_k)|\leq 2^{-s+1}}\left|f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\right|^{r'}d\vec{y}\right)^{1/r'}\\
&\leq C2^{\frac{-nks}{r}}\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}\\
&\times\left(\int_{2^{-s}\leq|(y_1,\ldots,y_k)|\leq 2^{-s+1}}\left|f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\right|^{r'}d\vec{y}\right)^{1/r'}
\end{aligned}$$

Then by the estimate in page 8 of [2], we have

$$\begin{aligned}
\left\|M_{A,s}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{\frac{r'}{k+1}}}&\leq C2^{\frac{-nks}{r}}\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}\\
&\times\left(\int_{R^n}\left|\int_{2^{-s}\leq|(y_1,\ldots,y_k)|\leq 2^{-s+1}}\left|f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\right|^{r'}dy\right|^{\frac{1}{k+1}}dx\right)^{\frac{k+1}{r'}}\\
&\leq C\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}2^{\frac{-nks}{r'}}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}\\
&\leq C\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}2^{-kns}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}.
\end{aligned}$$

So far, the proof of Lemma 2.1 has been finished.

Lemma 2.2.

Under the same conditions as in Theorem 1.1, for

$$I_{\alpha,\Omega}\left(\vec{f}\right)(x) = \int\limits_{R^{nk}} \frac{\Omega(x,\vec{y})}{\left|(y_1,\ldots,y_k)\right|^{kn-\alpha}} f_1(x-y_1)\cdots f_k(x-y_k)d\vec{y},$$

with $\Omega(x,\vec{y}) \in L^\infty(R^n) \times L^r\left(S^{nk-1}\right)$ for $1 < r' < \frac{kn}{\alpha}$ and $0 < \alpha < kn$.

Let $1/s = 1/r_1 + \ldots\ldots 1/r_k - \alpha/n > 0$ with $1 \le r_i \le \infty$, then,

(i) if each $r_i > r'$, then there exists a constant C such that

$$\left\| I_{\alpha,\Omega}\left(\vec{f}\right)\right\|_{L^s} \le C \prod_{i=1}^{k} \|f_i\|_{L^{r_i}},$$

(ii) if $r_i = r'$ for some i, then there exists a constant C such that

$$\left\| I_{\alpha,\Omega}\left(\vec{f}\right)\right\|_{L^{s,\infty}} \le C \prod_{i=1}^{k} \|f_i\|_{L^{r_i}}.$$

Proof. In [8], Lemma 2.2 was proved in the case $\Omega\left(x,\vec{y}\right) = \Omega_0\left(\vec{y}\right) \in L^r\left(S^{nk-1}\right)$. When consider the case if the multiple kernel function is a multiple variable kernel, by the similar argument as in [3,8], we can prove Lemma 2.2. Here we state the main steps to prove Lemma 2.2 for the completeness of this article.

First, we introduce the multilinear fractional maximal function $\mathcal{M}_\alpha\left(\vec{f}\right)(x)$ and multilinear fractional maximal function with a multiple variable kernel $\mathcal{M}_{\Omega,\alpha}\left(\vec{f}\right)(x)$, respectively.

$$\mathcal{M}_\alpha\left(\vec{f}\right)(x) = \sup_{r>0} \frac{1}{r^{kn-\alpha}} \int\limits_{|\vec{y}|<r} \prod_{i=1}^{k} \left|f_i(x-y_i)\right| d\vec{y}.$$

$$\mathcal{M}_{\Omega,\alpha}\left(\vec{f}\right)(x) = \sup_{r>0} \frac{1}{r^{kn-\alpha}} \int\limits_{|\vec{y}|<r} \Omega(x,\vec{y}) \prod_{i=1}^{k} \left|f_i(x-y_i)\right| d\vec{y}.$$

By Hölder's inequality, we can easily get the boundedness of $M_\alpha\left(\vec{f}\right)(x)$ on product L^p spaces and the following fact is also obvious by a simple computation,

$$M_{\Omega,\alpha}\left(\vec{f}\right)(x) \le C\|\Omega\|_{L^\infty(R^n)\times L^r(S^{nk-1})}\left[M_{\alpha s'}\left(|f_1|^{s'}, |f_2|^{s'}, \ldots, |f_k|^{s'}\right)(x)\right]^{1/s'}$$

which implies the boundedness of $M_{\Omega,\alpha}\left(\vec{f}\right)(x)$ on product L^p spaces.

Then by a classical augment as in [3,8], we have the following point estimate for

$$\left|I_{\alpha,\Omega}\left(\vec{f}\right)(x)\right| \le C\left[M_{\Omega,\alpha+\varepsilon}\left(\vec{f}\right)(x)\right]^{\frac{1}{2}}\left[M_{\Omega,\alpha-\varepsilon}\left(\vec{f}\right)(x)\right]^{\frac{1}{2}},$$

$$\left|I_{\alpha,\Omega}\left(\vec{f}\right)(x)\right| \le C\left[M_{\Omega,\alpha+\varepsilon}\left(\vec{f}\right)(x)\right]^{\frac{1}{2}}\left[M_{\Omega,\alpha-\varepsilon}\left(\vec{f}\right)(x)\right]^{\frac{1}{2}}, \tag{2.1}$$

So, by inequality (2.1) and the boundedness of $M_{\Omega,\alpha}\left(\vec{f}\right)(x)$ on product L^p spaces, we get Lemma 2.2 easily.

To finish the proof of Theorem 1.1, we define

$$F_{A,s}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)(x) = \int\limits_{2^{-s}\le|(y_1,\ldots,y_k)|\le 2^{-s+1}} \frac{\Omega(x,\vec{y})}{|(y_1,\ldots,y_k)|^{nk}} f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)d\vec{y}.$$

Then, we have

$$I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)(x) \le H(x) + G(x)$$

with $H(x) = \sum_{s\ge s_0} 2^{-s\alpha}F_{A,s}^{\Omega}(x)$ and

$$G(x) = \int\limits_{|(y_1,\ldots,y_k)|\ge 2^{-s_0}} \frac{\Omega(x,\vec{y})}{|(y_1,\ldots,y_k)|^{nk-\alpha}} f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)d\vec{y}.$$

For $r > \frac{kn}{kn-\alpha}$, we have

$$\begin{aligned} G(x) &= \int\limits_{|(y_1,\ldots,y_k)|\ge 2^{-s_0}} \frac{\Omega(x,\vec{y})}{|(y_1,\ldots,y_k)|^{nk-\alpha}} f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)d\vec{y} \\ &\le C\left(\int\limits_{|(y_1,\ldots,y_k)|\ge 2^{-s_0}} \frac{|\Omega(x,\vec{y})|^r}{|(y_1,\ldots,y_k)|^{(nk-\alpha)r}}d\vec{y}\right)^{1/r} \\ &\times\left(\int\limits_{R^{nk}} \left|f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\right|^{r'}d\vec{y}\right)^{1/r'} \\ &\le 2^{s_0\left[(kn-\alpha)-\frac{kn}{r}\right]}\left(\int\limits_{R^{nk}} \left|f_1\left(\ell_1\left(y_1,\ldots,y_k,x\right)\right)\cdots f_{k+1}\left(\ell_{k+1}\left(y_1,\ldots,y_k,x\right)\right)\right|^{r'}d\vec{y}\right)^{1/r'} \end{aligned}$$

Now using the linear change of variables as in page 14 of [2], that is for each $1 \le j \le k+1$, we define $f_j\left(\ell_j\left(y_1,\ldots,y_k,x\right)\right) = f_j'\left(A_{k+1,j}^{-1}\ell_j(x)\right) = f_j'\left(x - \sum_{i=1}^{k} A'_{ij}x_i\right) = f_j'(x-y_j)$ with $A'_{ij} = -A_{k+1,j}^{-1}A_{ij}$ and $y_j = \sum_{i=1}^{k} A'_{ij}x_i$ we have

$$\|G\|_{L^{r'}} \le 2^{s_0\left[(kn-\alpha)-\frac{kn}{r}\right]}\prod_{i=1}^{k+1}\|f_i\|_{L^{r'}}$$

For the estimate of $H(x)$, first by Lemma 2.1, we have

$$\left\| F_{A,s}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{\frac{r'}{k+1}}} \leq C\|\Omega\|_{L^{\infty}(R^n)\times L^r(S^{nk-1})}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}.$$

So when $\frac{r'}{k+1}\leq 1$, we get

$$\begin{aligned}\|H\|_{L^{\frac{r'}{k+1}}}^{\frac{r'}{k+1}} &\leq C\sum_{s\geq s_0}2^{-\frac{r'}{k+1}s\alpha}\left\|F_{A,s}^{\Omega}\right\|_{L^{\frac{r'}{k+1}}}^{\frac{r'}{k+1}}\\ &\leq C2^{-\frac{r'}{k+1}s_0\alpha}\prod_{i=1}^{m}\left\|f_i\right\|_{L^{r'}}^{\frac{r'}{k+1}}\end{aligned}$$

When $\frac{r'}{k+1}>1$, we can easily get

$$\begin{aligned}\|H\|_{L^{\frac{r'}{k+1}}} &\leq C\sum_{s\geq s_0}2^{-s\alpha}\left\|F_{A,s}^{\Omega}\right\|_{L^{\frac{r'}{k+1}}}\\ &\leq C2^{-s_0\alpha}\prod_{i=1}^{m}\left\|f_i\right\|_{L^{r'}}.\end{aligned}$$

Combine the estimate above can we easily get

$$\|H\|_{L^{\frac{r'}{k+1}}}\leq 2^{-s_0\alpha}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}.$$

By the above estimates, we have

$$\begin{aligned}&\left|\left\{I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)(x)>\lambda\right\}\right|\\ &\leq\left|\left\{x\in R^n: H(x)>\frac{\lambda}{2}\right\}\right|+\left|\left\{x\in R^n: G(x)>\frac{\lambda}{2}\right\}\right|\\ &\leq\frac{\|G\|_{L^{r'}}^{r'}}{\lambda^{r'}}+\frac{\|H\|_{L^{\frac{r'}{k+1}}}^{\frac{r'}{k+1}}}{\lambda^{\frac{r'}{k+1}}}\\ &\leq\frac{2^{s_0\left(kn-\alpha-\frac{kn}{r}\right)r'}}{\lambda^{r'}}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}^{r'}+\frac{2^{-s_0\alpha\frac{r'}{k+1}}}{\lambda^{\frac{r'}{k+1}}}\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{r'}}^{\frac{r'}{k+1}}.\end{aligned}$$

Now we may assume that $\left\|f_i\right\|_{L^{r'}}=1$ for $i=1,\ldots,k+1$, and choose $s_0=\frac{\frac{k}{k+1}log_2\lambda}{\frac{kn}{r'}-\frac{\alpha k}{k+1}}$,, we get

$$\left|\left\{x\in R^n: I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)(x)>\lambda\right\}\right|\leq\frac{C}{\lambda^p},$$

with $1/p=\frac{k+1}{r'}-\frac{\alpha}{n}$.

So far, the proof of Theorem 1.1 has been finished.

3. Proof of Theorem 1.2

For any p_1 that is larger than and sufficiently close to r', by the proof of Theorem 1.1, we get

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q_{1,\infty}}} \le C\left\|f_1\right\|_{L^{p_1}}\cdots\left\|f_k\right\|_{L^{p_1}}\left\|f_{k+1}\right\|_{L^{p_1}}$$

with $1/q_1 = (k + 1)/p_1 - \alpha/n$. On the other hand, by Lemma 2.2 and the same linear change of variables as in Section 2, we have

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q_2}} \le C\left\|f_1\right\|_{L^{p_1}}\cdots\left\|f_k\right\|_{L^{p_1}}\left\|f_{k+1}\right\|_{L^{\infty}}$$

with $1/q_2 = k/p_1 - \alpha/n$.

Then by interpolation, we have

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q_{3,\infty}}} \le C\left\|f_1\right\|_{L^{p_1}}\cdots\left\|f_k\right\|_{L^{p_1}}\left\|f_{k+1}\right\|_{L^{p_{k+1}}}$$

with $1/q_3 = k/p_1 + 1/p_{k+1} - \alpha/n$ and $p_1 \le p_{k+1}$.

Again, by Lemma 2.2 and the same linear change of variables as in Section 2, we have

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q_4}} \le C\left\|f_1\right\|_{L^{p_1}}\cdots\left\|f_{k-1}\right\|_{L^{p_1}}\left\|f_k\right\|_{L^{\infty}}\left\|f_{k+1}\right\|_{L^{p_{k+1}}}$$

with $1/q_4 = (k - 1)/p_1 + 1/p_{k+1} - \alpha/n$

Then by interpolation, we have,

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q_{5,\infty}}} \le C\left\|f_1\right\|_{L^{p_1}}\cdots\left\|f_{k-1}\right\|_{L^{p_1}}\left\|f_k\right\|_{L^{p_k}}\left\|f_{k+1}\right\|_{L^{p_{k+1}}}$$

with $1/q_5 = (k- 1)/p_1 + 1/p_k + 1/p_{k+1} - \alpha/n$ and $p_1 \le \min\{p_k, p_{k+1}\}$.

Again using the above methods can we easily get

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q,\infty}} \le C\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{p_i}}$$

for any $p_1 \le \min\{p_2,\ldots ,p_{k+1}\}$ with $1/q = 1/p_1 + \cdots 1/p_{k+1} - \alpha/n$.

Similarly, for any $p_i(1 \le i \le k + 1)$ that is larger than and sufficiently close to r', we can also get

$$\left\|I_{\alpha,A}^{\Omega}\left(f_1,\ldots,f_{k+1}\right)\right\|_{L^{q\infty}} \le C\prod_{i=1}^{k+1}\left\|f_i\right\|_{L^{p_i}},$$

for any $p_i \le \min\{p_1,\ldots, p_{i-1}, p_{i+1},\ldots p_{k+1}\}$ with $1/q = 1/p_1+\cdots 1/p_{k+1} -\alpha/n$.

Now, we obtain Theorem 1.2 by multilinear interpolation from [2,9].

Acknowledgements
Xiao Yu was partially supported by the NSFC under grant \# 10871173 and NFS of Jiangxi Province under grant \#2010GZC185 and \#20114BAB211007.

Author details
[1]Department of Mathematics, Shangrao Normal University, Shangrao, 334001, P.R. China [2]Department of Mathematics, Zhejiang International Studies University, Hangzhou, 310012, P.R. China

Authors' contributions
HZ discovered the problem of this paper and participated in the proof of Theorem 1.1. JR contributed a lot in the revised version of this manuscript and pointed out several mistakes of this paper. XY participated in the proof of Theorem 1.2 and checked the proof of the whole paper carefully. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 10 July 2011 Accepted: 24 February 2012 Published: 24 February 2012

References

1. Grafakos, L: On multilinear fractional integrals. Studia Math. **102**(1):49–56 (1992)
2. Kenig, CE, Stein, EM: Multilinear estimates and fractional integration. Math Res Lett. **6**, 1–15 (1999)
3. Ding, Y, Lu, SZ: The $L^{p_1} \times \cdots \times L^{p_k}$ boundedness for some multilinear operators. J Math Anal Appl. **203**(1):166–186 (1996). doi:10.1006/jmaa.1996.0373
4. Ding, Y, Lin, CC: Rough bilinear fractional integrals. Math Nachr. **246-247**, 47–52 (2002). doi:10.1002/1522-2616(200212)246:13.0.CO;2-7
5. Calderön, AP, Zygmund, A: On a problem of Mihlin. Trans Am Math Soc. **78**, 209–224 (1955)
6. Chen, JC, Ding, Y, Fan, DS: On a Hyper Hilbert Transform. Chinese Annals Math Ser B. **24**, 475–484 (2003). doi:10.1142/S0252959903000475
7. Chen, JC, Fan, DS: Rough Bilinear Fractional Integrals with Variable Kernels. Frontier Math China. **5**(3):369–378 (2010). doi:10.1007/s11464-010-0061-1
8. Shi, YL: Related Estimates for Some Multilinear Operators and Commutators. Master's Thesis. Ningbo University, P.R. China (2009)
9. Janson, S: On interpolation of multilinear operators. In Springer Lecture Notes in Math, vol. 1302, pp. 290–302.Springer-Verlag, Berlin-New York (1988). doi:10.1007/BFb0078880

doi:10.1186/1029-242X-2012-42
Cite this article as: Zhang *et al.*: **Boundedness of (*k* + 1)-linear fractional integral with a multiple variable kernel.** *Journal of Inequalities and Applications* 2012 **2012**:42.