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A new system of generalized quasi-variational-like inclusions with noncompact valued mappings

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Abstract

In this article, we introduce and study a new system of generalized quasi-variational-like inclusions with noncompact valued mappings. By using the η-proximal mapping technique, we prove the existence of solutions and the convergence of some new N-step iterative algorithms for this system of generalized quasi-variational-like inclusions. Our results extend and improve some known results in the literature.

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1 Introduction

Let H be a real Hilbert space, and $CB(H)$ be the family of all nonempty bounded closed subsets of H. We will consider the following problem:

For i, j = 1, 2, . . . , N, let A_{ij}: $H \to CB(H)$, η_i: $H \times H \to H$, g_i: $H \to H$, $T_i : \underbrace{H \times H \times \cdots \times H}_{N} \to H$ be nonlinear mappings, and let ϕ_i: $H \to R \cup \{+\infty\}$ be real function.

Find $x_1^*, x_2^*, \ldots, x_N^* \in H$, $u_{11}^* \in A_{11}x_1^*, u_{12}^* \in A_{12}x_2^*, \ldots, u_{1N}^* \in A_{1N}x_N^*, \ldots, u_{N1}^* \in A_{N1}x_1^*, u_{N2}^* \in A_{N2}x_2^*, \ldots, u_{NN}^* \in A_{NN}x_N^*$ such that

$$\langle T_i(u_{i1}^*, u_{i2}^*, \ldots, u_{iN}^*), \eta_i(x, g_i(x_i^*))\rangle \geq \varphi_i(g_i(x_i^*)) - \varphi_i(x), \quad \forall x \in H, \quad i = 1, 2, \ldots N. \tag{1.1}$$

Problem (1.1) is called the set-valued nonlinear generalized quasi-variational-like inclusions.

Various special cases of the problem (1.1) had been studied by many authors before. Here, we mention some of them as follows:

(1) If $N = 2$, $A_{11} = A_{12} = A$, $A_{21} = A_{22} = B$, $T_1 = T$, $T_1(A(\cdot), B(\cdot)) : H \to CB(H)$, then the problem (1.1) reduces to find $x^* \in H$, $u^* \in Ax^*$, $v^* \in Bx^*$ such that

$$\langle T(u^*, v^*), \eta(x, g(x^*))\rangle \geq \varphi(g(x^*)) - \varphi(x), \quad \forall x \in H. \tag{1.2}$$

Problem (1.2) was introduced and studied by Ding [1] in 2001.

(2) If $N = 2$, $A_{11} = A_{12} = A_{21} = A_{22} = I$, $g_i = I$ (identical operator), $\eta(x, y) = x-y$, $\phi_1 = \phi_2 = \phi$, $T : H \times H \to H$, $T_1(A_{11}x, A_{12}y) = \rho_1 T(A_{12}y, A_{11}x) + A_{11}x - A_{12}y$, $T_2(A_{21}x, A_{22}y) = \rho_2 T(A_{21}x, A_{22}y) + A_{22}y - A_{21}x$, then the problem (1.1) reduces to find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho_1 T(y^*,\ x^*) + x^* - y^*,\ x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, & \forall x \in H,\ \rho_1 > 0; \\ \langle \rho_2 T(x^*,\ y^*) + y^* - x^*,\ x - y^* \rangle + \varphi(x) - \varphi(y^*) \geq 0, & \forall x \in H,\ \rho_2 > 0. \end{cases} \tag{1.3}$$

Problem (1.3) was studied by He and Gu [2] in 2009.

(3) Let $K \subset H$ be a closed convex subset, $\phi(x) = I_K(x)$, the problem (1.3) reduces to find x^*, $y^* \in K$ such that

$$\begin{cases} \langle \rho_1 T(y^*,\ x^*) + x^* - y^*,\ x - x^* \rangle \geq 0, & \forall x \in K,\ \rho_1 > 0; \\ \langle \rho_2 T(x^*,\ y^*) + y^* - x^*,\ x - y^* \rangle \geq 0, & \forall x \in K,\ \rho_2 > 0. \end{cases} \tag{1.4}$$

Problem (1.4) was inspected and studied by Chang [3], Verma [4,5] and Huang [6].

(4) If $N = 2$, $A_{11} = A_{12} = A_{21} = A_{22} = I$, $g : H \to H$, $T_1(x, y) = \rho_1 Ty + g(x) - g(y)$, $T_2(x, y) = \rho_2 Tx + g(y) - g(x)$, $(\rho_1, \rho_2 > 0)$, $\eta(x, y) = g(x) - g(y)$, then the problem (1.1) reduces to find x^*, $y^* \in H$ such that

$$\begin{cases} \langle \rho_1 Ty^* + g(x^*) - g(y^*), g(x) - g(x^*) \rangle \geq 0, & \forall x \in H; \\ \langle \rho_2 Tx^* + g(y^*) - g(x^*), g(x) - g(y^*) \rangle \geq 0, & \forall x \in H. \end{cases} \tag{1.5}$$

Problem (1.5) was introduced and studied by Hajjafar and Verma [7].

(5) If $N = 3$, $\eta(x, y) = x - y$, then the problem (1.1) reduces to find $x_1^*,\ x_2^*,\ x_3^* \in H,\ u_{i1}^* \in A_{i1}x_1^*,\ u_{i2}^* \in A_{i2}x_2^*,\ u_{i3}^* \in A_{i3}x_3^* (i = 1, 2, 3)$ such that

$$\begin{cases} \langle T_1(u_{11}^*,\ u_{12}^*,\ u_{13}^*), x - g_1(x_1^*) \rangle \geq \varphi_1(g_1(x_1^*)) - \varphi_1(x), & \forall x \in H; \\ \langle T_2(u_{21}^*,\ u_{22}^*,\ u_{23}^*), x - g_2(x_2^*) \rangle \geq \varphi_2(g_2(x_2^*)) - \varphi_2(x), & \forall x \in H; \\ \langle T_3(u_{31}^*,\ u_{32}^*,\ u_{33}^*), x - g_3(x_3^*) \rangle \geq \varphi_3(g_3(x_3^*)) - \varphi_3(x), & \forall x \in H. \end{cases} \tag{1.6}$$

Problem (1.6) was studied by Kazmi et al. [8].

For more special cases, please refer to [1-9] and the references therein.

Remark 1.1. Yang [10] pointed out a fact for the problem (1.4) discussed in reference [5], namely, if the problem (1.4) has a solution (x^*, y^*), then $x^* = y^*$. Therefore, actually, the problem(1.4) is a single variational inequality:

$$\langle T(x^*,\ x^*),\ x - x^* \rangle \geq 0, \quad \forall x \in K.$$

In this article, we study the problem (1.1). By using the η proximal mapping technique, we prove the existence of solutions and approximate the solutions by some new N -step iterative algorithms. Our results extend and improve some known results in the references [1-9].

2 Preliminaries

In this article, we need the following concepts and lemmas.

Definition 2.1 [1] A mapping $g : H \to H$ is said to be

(i) ξ-strongly monotone if there exists a constant $\xi > 0$ such that

$$\langle g(x) - g(y),\ x - y \rangle \geq \xi \left\| x - y \right\|^2, \quad \forall x,\ y \in H.$$

(ii) ζ-Lipschitz continuous if there exists a constant $\zeta > 0$ such that

$$\left\| g(x) - g(y) \right\| \leq \zeta \left\| x - y \right\|, \quad \forall x,\ y \in H.$$

Definition 2.2 [1] A mapping $\eta : H \times H \to H$ is said to be

(i) σ - strongly monotone if there exists a constant σ >0 such that

$$\langle x-y,\ \eta(x,\ y)\rangle \geq \sigma\left\|x-y\right\|^2, \quad \forall x,\ y\in H;$$

(ii) τ - Lipschitz continuous if there exists a constant τ >0 such that

$$\left\|\eta(x,\ y)\right\| \ \leq\ \tau\left\|x-y\right\|, \quad \forall x,\ y\in H.$$

Definition 2.3 [1,11] Let $A : H \to CB(H)$ be a set-valued mapping, $T:\underbrace{H\times H\times\cdots\times H}_{N}\to H$ is said to be

(i) α - (A, g)-strongly monotone in the ith argument if α >0 such that

$$\langle T(\ldots,\ u_i,\ \ldots)-T(\ldots,\ v_i,\ \ldots),\ g(x)-g(y)\rangle \geq \alpha\left\|x-y\right\|^2, \quad \forall x,\ y\in H, \quad u_i\in Ax, \quad v_i\in Ay.$$

(ii) $(s_1, s_2, \ldots, s_N)$-Lipschitz continuous if there exist constants $s_1, s_2, \ldots, s_N$ >0 such that for all $x_i, y_i \in H$, $i = 1, 2, \ldots, N$,

$$\left\|T(x_1,\ x_2,\ \ldots,\ x_N)-T(y_1,\ y_2,\ \ldots,\ y_N)\right\| \ \leq\ s_1\left\|x_1-y_1\right\|+s_2\left\|x_2-y_2\right\|+\cdots+s_N\left\|x_N-y_N\right\|.$$

(iii) A set-valued A is said to be δ - H - Lipschitz continuous if there exists a constant δ >0 such that

$$H(Ax,\ Ay)\leq\delta\left\|x-y\right\|, \quad \forall x,\ y\in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

Definition 2.4 [1] A functional $f : H \times H \to R\cup\{+\infty\}$ is said to be 0-diagonally quasi-concave (in short,0-DQCV) in x, if for any finite set $\{x_1, \ldots, x_N\} \subset H$ and for any $y = \sum_{i=1}^{n}\lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n}\lambda_i = 1$,

$$\min_{1\leq i\leq n} f(x_i,\ y)\leq 0.$$

Definition 2.5 [1] Let $\eta : H \times H \to H$ be a single-valued mapping. A proper functional $\phi : H \to R\cup\{+\infty\}$ is said to be η-subdifferentiable at a point $x \in H$, if there exists a point $f^* \in H$ such that

$$\langle f^*,\ \eta(y,\ x)\rangle \leq \varphi(y)-\varphi(x), \quad \forall y\in H,$$

where f^* is called a η- subgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial_\eta\phi(x)$. We have

$$\partial_\eta\varphi(x) = \{f^*\in H,\ \langle f^*,\ \eta(y,\ x)\rangle \leq \varphi(y)-\varphi(x), \quad \forall y\in H.\} \tag{2.1}$$

Definition 2.6 [1] Let η, ϕ be according to Definition 2.5, if for each $x \in H$ and ρ >0,there exists a unique point $u \in H$ such that

$$\langle u-x,\ \eta(y,\ u)\rangle \geq \rho\varphi(u)-\rho\varphi(y), \quad \forall y\in H, \tag{2.2}$$

then the mapping $x \mapsto u$ denoted by J_φ^ρ, is said to be η- proximal mapping of ϕ. By (2.1) and the definition of J_φ^ρ, we have $x - u \in \rho\partial_\eta\phi(x)$, it follows that

$$J_\varphi^\rho(x) = (I\ +\rho\partial_\eta\varphi)^{-1}(x).$$

Lemma 2.1 [1] Let $\eta : H \times H \to H$ be continuous and σ-strongly monotone such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in H$. And for any given $x \in H$, the function $h(y, u) = \langle x - u, \eta(y, u)\rangle$ is 0-DQCV in y. Let $\phi : H \to R \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional on H, then for any given $\rho > 0$ and $x \in H$ there exists a unique $u \in H$ such that

$$\langle u - x,\ \eta(y,\ u)\rangle \geq \rho\varphi(u) - \rho\varphi(y), \quad \forall y \in H.$$

i.e., $u = J_{\varphi}^{\rho}(x)$.

Lemma 2.2 Let $\eta : H \times H \to H$ be σ- strongly monotone and τ -Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$. Let $h(y, u)$, ϕ, ρ be according to Lemma 2.1, then the η-proximal mapping $J_{\varphi}^{\rho}(x)$ of ϕ is $\frac{\tau}{\sigma}$-Lipschitz continuous.

3 Main results

Theorem 3.1 $(x_1^*,\ x_2^*,\ \ldots,\ x_N^*; u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*,\ i = 1, 2,\ \ldots,\ N.)$ is a solution of problem (1.1) if and only if $(x_1^*,\ x_2^*,\ \ldots,\ x_N^*; u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*,\ i = 1, 2,\ \ldots,\ N.)$ satisfies the following relation: For every $i = 1, 2, \ldots N$,

$$g_i(x_i^*) = J_{\varphi_i}^{\rho_i}(g_i(x_i^*) - \rho_i T_i(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*)), \tag{3.1}$$

where $J_{\varphi^i}^{\rho_i} = (I + \rho_i \partial_\eta \varphi_i)^{-1}, \quad \rho_i > 0.$

Proof. Assume the $(x_1^*,\ x_2^*,\ \ldots,\ x_N^*; u_{11}^*,\ u_{12}^*,\ \ldots,\ u_{1N}^*,\ \ldots,\ u_{N1}^*,\ u_{N2}^*,\ \ldots,\ u_{NN}^*)$ satisfies relation (3.1). Since $J_{\varphi_i}^{\rho_i} = (I + \rho_i \partial_{\eta_i} \varphi_i)^{-1}$, we have

$$g_i(x_i^*) + \rho_i \partial_{\eta_i}\varphi_i(g_i(x_i^*)) \in g_i(x_i^*) - \rho_i T_i(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*).$$

i.e.,

$$-T_1(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*) \in \partial_{\eta_i}\varphi_i(g_i(x_i^*)).$$

By the Definition 2.5 of η_i-subdifferential, the above relation holds if and only if

$$-\langle T_i(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*),\ \eta_i(x,\ g_i(x_i^*))\rangle \leq \varphi_i(x) - \varphi_i(g_i(x_i^*)), \quad \forall x \in H,$$

and hence

$$\langle T_i(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*),\ \eta_i(x,\ g_i(x_i^*))\rangle \geq \varphi_i(g_i(x_i^*)) - \varphi_i(x), \quad \forall x \in H, i = 1, 2, \ldots, N.$$

i.e., $(x_1^*,\ x_2^*,\ \ldots,\ x_N^*; u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*,\ i = 1, 2,\ \ldots,\ N.)$ is a solution of the problem (1.1). □

Now, we give iterative algorithms of problem (1.1).

Algorithm(I) For given $x_1^0,\ x_2^0,\ \ldots,\ x_N^0 \in H,\ u_{i1}^0 \in A_{i1}x_1^0,\ u_{i2}^0 \in A_{i2}x_2^0,\ \ldots,\ u_{iN}^0 \in A_{iN}x_N^0$, let

$$\begin{aligned}
x_1^1 &= x_1^0 - g_1(x_1^0) + J_{\varphi_1}^{\rho_1}(g_1(x_1^0) - \rho_1 T_1(u_{11}^0,\ u_{12}^0,\ \ldots,\ u_{1N}^0));\\
x_2^1 &= x_2^0 - g_2(x_2^0) + J_{\varphi_2}^{\rho_2}(g_2(x_2^0) - \rho_2 T_2(u_{21}^0,\ u_{22}^0,\ \ldots,\ u_{2N}^0));\\
&\vdots\\
x_N^1 &= x_N^0 - g_N(x_N^0) + J_{\varphi_N}^{\rho_N}(g_N(x_N^0) - \rho_N T_N(u_{N1}^0,\ u_{N2}^0,\ \ldots,\ u_{NN}^0)).
\end{aligned}$$

By Nadler [12], for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, there exists $u_{ij}^1 \in A_{ij}x_j^1$ such that

$$\left\| u_{ij}^0 - u_{ij}^1 \right\| \le (1+1)H(A_{ij}x_j^0,\ A_{ij}x_j^1), \quad j = 1, 2, \ldots, N.$$

Let

$$x_i^2 = x_i^1 - g_i(x_i^1) + J_{\varphi_i}^{\rho_i}(g_i(x_i^1) - \rho_i T_i(u_{i1}^1,\ u_{i2}^1,\ \ldots,\ u_{iN}^1)), \quad i = 1, 2, \ldots, N.$$

By induction, we can define sequences $\{x_i^n\}$, $\{u_{ij}^n\}$ satisfying

$$x_i^{n+1} = x_i^n - g_i(x_i^n) + J_{\varphi_i}^{\rho_i}(g_i(x_i^n) - \rho_i T_i(u_{i1}^n,\ u_{i2}^n,\ \ldots,\ u_{iN}^n)),$$

where for any $i, j = 1, 2, \ldots, N$; $n = 0, 1, 2, \ldots,$

$$u_{ij}^n \in A_{ij}^n x_j^n, \quad u_{ij}^n - u_{ij}^{n+1} \le \left(1 + \frac{1}{n+1}\right) H\left(A_{ij}\left(x_j^n\right),\ A_{ij}\left(x_j^{n+1}\right)\right).$$

Theorem 3.2 Let H be a real Hilbert space. For $i, j = 1, 2, \ldots, N$, let set-valued mapping $A_{ij}: H \to CB(H)$ be δ_{ij} - H - *Lipschitz* continuous. Let mapping $\eta_i: H \times H \to H$ be σ_i- strongly monotone and τ_i-Lipschitz continuous such that $\eta_i(x, y) = -\eta_i(y, x)$ for all $x, y \in H$ and for any given $x \in H$, the function $h_i(y, u) = \langle x - g_i(u), \eta_i(y, u)\rangle$ is 0-DQCU in y. Let mapping $g_i: H \to H$ be ξ_i-strongly monotone and ζ_i-Lipschitz continuous, and $T_i : \underbrace{H \times H \times \cdots \times H}_{N} \to H$ be $(s_1, \ldots, s_N)$-Lipschitz continuous and α_i - (A_{ij}, g_i)-strongly monotone in the ith argument. Let $\phi_i: H \to R \cup \{+\infty\}$ be a lower semi-continuous η_i-subdifferentiable proper functional. If there exist $\rho_1, \ldots, \rho_N > 0$ such that for all $i = 1, 2, \ldots, N$

$$\left(1 - 2\xi_i + \zeta_i^2\right)^{\frac{1}{2}} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\right)^{\frac{1}{2}} + \sum_{k=1, k\neq i}^{N} \frac{\tau_k}{\sigma_k}\rho_k s_i \delta_{ki} < 1; \tag{3.2}$$

then the iterative sequences $\{x_1^n\},\ \ldots,\ \{x_N^n\},\ \{u_{11}^n\},\ \ldots,\ \{u_{1N}^n\},\ \ldots,\ \{u_{N1}^n\},\ \ldots,\ \{u_{NN}^n\}$, generated by algorithm (I) converge strongly to $x_1^*,\ \ldots,\ x_N^*,\ u_{11}^*,\ \ldots,\ u_{1N}^*,\ \ldots,\ u_{N1}^*,\ \ldots,\ u_{NN}^*$, respectively, and $\left(x_1^*,\ x_2^*,\ \ldots,\ x_N^*,\ u_{11}^*,\ u_{12}^*,\ \ldots,\ u_{1N}^*,\ \ldots,\ u_{N1}^*,\ u_{N2}^*,\ \ldots,\ u_{NN}^*\right)$ is a solution of the problem (1.1).

Proof. For $i = 1, 2, \ldots, N$, by algorithm (I) and Lemma 2.2, we have

$$\begin{aligned}
\|x_i^{n+1} - x_i^n\| &= \|x_i^n - g_i\left(x_i^n\right) + J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^n\right) - \rho_i T_i\left(u_{i1}^n,\ u_{i2}^n,\ \ldots,\ u_{iN}^n\right)\right) \\
&\quad - x_i^{n-1} + g_i\left(x_i^{n-1}\right) - J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^{n-1}\right) - \rho_i T_i\left(u_{i1}^{n-1},\ u_{i2}^{n-1},\ \ldots,\ u_{iN}^{n-1}\right)\right)\| \\
&\le \|x_i^n - x_i^{n-1} - g_i\left(x_i^n\right) + g_i\left(x_i^{n-1}\right)\| \\
&\quad + \|J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^n\right) - \rho_i T_i\left(u_{i1}^n,\ u_{i2}^n,\ \ldots,\ u_{iN}^n\right)\right) \\
&\quad - J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^{n-1}\right) - \rho_i T_i\left(u_{i1}^{n-1},\ u_{i2}^{n-1},\ \ldots,\ u_{iN}^{n-1}\right)\right)\| \\
&\le \|x_i^n - x_i^{n-1} - g_i\left(x_i^n\right) + g_i\left(x_i^{n-1}\right)\| \\
&\quad + \frac{\tau_i}{\sigma_i}\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right) - \rho_i T_i\left(u_{i1}^n,\ \ldots,\ u_{iN}^n\right) + \rho_i T_i\left(u_{i1}^{n-1},\ \ldots,\ u_{iN}^{n-1}\right)\|.
\end{aligned} \tag{3.3}$$

Since $g_i$is ξ_i-strongly monotone and ζ_i-Lipschitz continuous, we obtain

$$\left\|x_i^n - x_i^{n-1} - \left(g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right)\right)\right\| \le \sqrt{1 - 2\xi_i + \zeta_i^2}\left\|x_i^n - x_i^{n-1}\right\|. \tag{3.4}$$

Notice that,

$$\begin{aligned}
&\left\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right) - \rho_i\left(T_i\left(u_{i1}^n, \ldots, u_{iN}^n\right) - T_i\left(u_{i1}^{n-1}, \ldots, u_{iN}^{n-1}\right)\right)\right\| \\
&\le \left\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right) - \rho_i\left(T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^n, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right.\right. \\
&\quad \left.\left. - T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right)\right\| \\
&+ \rho_i\left\|T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right) - T_i\left(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{iN}^{n-1}\right)\right\|.
\end{aligned} \tag{3.5}$$

Since T_i is $(s_1, \ldots, s_N)$-Lipschitz continuous and α_i - (A_{ij}, g_i)- strongly monotone in the ith argument, we get

$$\begin{aligned}
&\left\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right) - \rho_i\left(T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^n, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right.\right. \\
&\quad \left.\left. - T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right)\right\|^2 \\
&= \left\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right)\right\|^2 - 2\rho_i\left\langle g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right), T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^n, u_{i,i+1}^n, \ldots, u_{i,N}^n\right)\right. \\
&\quad \left. - T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right\rangle \\
&+ \rho_i^2\left\|T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^n, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right. \\
&\quad \left. - T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{i,i}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right\|^2 \\
&\le \xi_i^2\left\|x_i^n - x_i^{n-1}\right\|^2 - 2\rho_i\,\alpha_i\left\|x_i^n - x_i^{n-1}\right\|^2 + \rho_i^2\, s_i^2\left\|u_{ii}^n - u_{ii}^{n-1}\right\|^2 \\
&\le \left(\xi_i^2 - 2\rho_i\alpha_i\right)\left\|x_i^n - x_i^{n-1}\right\|^2 + \rho_i^2 s_i^2\left(1 + \frac{1}{n}\right)^2\left(H\left(A_{ii}x_i^n, A_{ii}x_i^{n-1}\right)\right)^2 \\
&\le \left[\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2\delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right]\left\|x_i^n - x_i^{n-1}\right\|^2.
\end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned}
&\left\|g_i\left(x_i^n\right) - g_i\left(x_i^{n-1}\right) - \rho_i\left(T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^n, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right.\right. \\
&\quad \left.\left. - T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^{n-1}, u_{i,i+1}^n, \ldots, u_N^n\right)\right)\right\| \\
&\quad \le \left[\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2\delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right]^{\frac{1}{2}}\left\|x_i^n - x_i^{n-1}\right\|.
\end{aligned} \tag{3.7}$$

And

$$\begin{aligned}
&\rho_i\left\|T_i\left(u_{i1}^n, u_{i2}^n, \ldots, u_{i,i-1}^n, u_{ii}^{n-1}, u_{i,i+1}^n, \ldots, u_{iN}^n\right)\right. \\
&\quad \left. - T_i\left(u_{i1}^{n-1}, u_{i2}^{n-1}, \ldots, u_{i,i-1}^{n-1}, u_{ii}^{n-1}, u_{i,i+1}^{n-1}, \ldots, u_{iN}^{n-1}\right)\right\| \\
&\le \rho_i\left(s_1\left\|u_{i1}^n - u_{i1}^{n-1}\right\| + s_2\left\|u_{i2}^n - u_{i2}^{n-1}\right\| + \cdots\right. \\
&\quad \left. + s_{i-1}\left\|u_{i,i-1}^n - u_{i,i-1}^{n-1}\right\| + s_{i+1}\left\|u_{i,i+1}^n - u_{i,i+1}^{n-1}\right\| + \cdots + s_N\left\|u_{iN}^n - u_{iN}^{n-1}\right\|\right) \\
&\le \rho_i\left[s_1\left(1 + \frac{1}{n}\right)H\left(A_{i1}x_1^n, A_{i1}x_1^{n-1}\right)\right] + s_2\left(1 + \frac{1}{n}\right)H\left(A_{i2}x_2^n, A_{i2}x_2^{n-1}\right) + \cdots \\
&\quad + s_{i-1}\left(1 + \frac{1}{n}\right)H\left(A_{i,i-1}x_{i-1}^n, A_{i,i-1}x_{i-1}^{n-1}\right) \\
&\quad + s_{i+1}\left(1 + \frac{1}{n}\right)H\left(A_{i,i+1}x_{i+1}^n, A_{i,i+1}x_{i+1}^{n-1}\right) + \cdots + s_N\left(1 + \frac{1}{n}\right)H\left(A_{iN}x_N^n, A_{iN}x_N^{n-1}\right)\Big] \\
&\le \rho_i\left(1 + \frac{1}{n}\right)\left[s_1\delta_{i1}\left\|x_1^n - x_1^{n-1}\right\| + s_2\delta_{i2}\left\|x_2^n - x_2^{n-1}\right\| + \cdots\right. \\
&\quad \left. + s_{i-1}\delta_{i,i-1}\left\|x_{i-1}^n - x_{i-1}^{n-1}\right\| + s_{i+1}\delta_{i,i+1}\left\|x_{i+1}^n - x_{i+1}^{n-1}\right\| + \cdots + s_N\delta_{iN}\left\|x_N^n - x_N^{n-1}\right\|\right].
\end{aligned} \tag{3.8}$$

It follows from (3.3)-(3.8) that for every $i = 1, 2, \ldots, N$,

$$\begin{aligned}\left\|x_i^{n+1} - x_i^n\right\| &\le \left[\sqrt{1 - 2\xi_i + \zeta_i^2} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right)^{\frac{1}{2}}\right]\left\|x_i^n - x_i^{n-1}\right\| \\ &\quad + \rho_i\frac{\tau_i}{\sigma_i}\left(1 + \frac{1}{n}\right)\sum_{k=1,k\neq i}^{N} s_k\delta_{ik}\left\|x_k^n - x_k^{n-1}\right\|.\end{aligned} \tag{3.9}$$

So,

$$\begin{aligned}&\left\|x_1^{n+1} - x_1^n\right\| + \left\|x_2^{n+1} - x_2^n\right\| + \cdots + \left\|x_N^{n+1} - x_N^n\right\| \\ &\le \sum_{i=1}^{N}\left\{\left[\sqrt{1 - 2\xi_i + \zeta_i^2} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right)^{\frac{1}{2}}\right]\left\|x_i^n - x_i^{n-1}\right\|\right. \\ &\qquad \left. + \rho_i\frac{\tau_i}{\sigma_i}\sum_{k=1,k\neq i}^{N} s_k\delta_{ik}\left(1 + \frac{1}{n}\right)\left\|x_k^n - x_k^{n-1}\right\|\right\} \\ &= \sum_{i=1}^{N}\left\{\sqrt{1 - 2\xi_i + \zeta_i^2} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right)^{\frac{1}{2}}\right. \\ &\qquad \left. + \sum_{k=1,k\neq i}^{N}\rho_k\frac{\tau_k}{\sigma_k}s_i\delta_{ki}\left(1 + \frac{1}{n}\right)\right\}\left\|x_i^n - x_i^{n-1}\right\| \\ &\le \theta_n\left(\left\|x_1^n - x_1^{n-1}\right\| + \left\|x_2^n - x_2^{n-1}\right\| + \cdots + \left\|x_N^n - x_N^{n-1}\right\|\right),\end{aligned} \tag{3.10}$$

where

$$\theta_n = \max_{i=1,2,\ldots N}\left\{\sqrt{1 - 2\xi_i + \zeta_i^2} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\left(1 + \frac{1}{n}\right)^2\right)^{\frac{1}{2}} + \sum_{k=1,k\neq i}^{N}\rho_k\frac{\tau_k}{\sigma_k}s_i\delta_{ki}\left(1 + \frac{1}{n}\right)\right\}$$

Letting

$$\theta = \max_{i=1,2,\ldots N}\left\{\sqrt{1 - 2\xi_i + \zeta_i^2} + \frac{\tau_i}{\sigma_i}\left(\xi_i^2 - 2\rho_i\alpha_i + \rho_i^2 s_i^2 \delta_{ii}^2\right)^{\frac{1}{2}} + \sum_{k=1,k\neq i}^{N}\rho_k\frac{\tau_k}{\sigma_k}s_i\delta_{ki}\right\},$$

from (3.2) we have $0 < \theta < 1$, and hence $\{x_1^n\}\ldots\{x_N^n\}$ are also Cauchy sequences. Thus there exist $x_1^*, \ldots, x_N^* \in H$ such that $x_i^n \to x_i^*(n \to \infty), i = 1, 2, \ldots, N$.

Now we prove $u_{ij}^n \to u_{ij}^*(n \to \infty)$, for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. By $\left\|u_{ij}^n - u_{ij}^{n-1}\right\| \le \left(1 + \frac{1}{n}\right)H\left(A_{ij}x_j^n, A_{ij}x_j^{n-1}\right) \le \left(1 + \frac{1}{n}\right)\delta_{ij}\left\|x_j^n - x_j^{n-1}\right\|$.

It follows that $\{u_{ij}^n\}$ are also Cauchy sequence. Therefore, there exist $u_{ij}^* \in H$ such that $u_{ij}^n \to u_{ij}^*(n \to \infty)$.

Note that

$$d\left(u_{ij}^*,\ A_{ij}x_j^*\right) \le \left\|u_{ij}^* - u_{ij}^n\right\| d\left(u_{ij}^n,\ A_{ij}x_j^*\right)$$
$$\le \left\|u_{ij}^* - u_{ij}^n\right\| + H\left(A_{ij}x_j^n,\ A_{ij}x_j^*\right) \le \left\|u_{ij}^* - u_{ij}^n\right\| + \delta_{ij}\left\|x_j^n - x_j^*\right\| \to 0(n \to \infty).$$

We have $d\left(u_{ij}^*,\ A_{ij}x_j^*\right) = 0$. Since $A_{ij}x_j^*$ is closed, $u_{ij}^* \in A_{ij}x_j^*$, for each i = 1,2, . . . ,N, j = 1, 2, . . . ,N. By

$$x_i^{n+1} = x_i^n - g_i\left(x_i^n\right) + J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^n\right) - \rho_i T_i\left(u_{i1}^n,\ u_{i2}^n,\ \ldots,\ u_{iN}^n\right)\right), \quad i = 1, 2,\ \ldots,\ N,$$

and the continuity of g_i, $J_{\varphi^i}^{\rho_i}$, T_i, let $n \to \infty$, we have that

$$0 = -g_i\left(x_i^*\right) + J_{\varphi_i}^{\rho_i}\left(g_i\left(x_i^*\right) - \rho_i T_i\left(u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*\right)\right),$$

and

$$u_{i1}^* \in A_{i1}x_1^*,\ u_{i2}^* \in A_{i2}x_2^*,\ \ldots,\ u_{iN}^* \in A_{iN}x_N^*.$$

0By Theorem 3.1, $\left(x_1^*,\ x_2^*,\ \ldots,\ x_N^*,\ u_{i1}^*,\ u_{i2}^*,\ \ldots,\ u_{iN}^*,\ i = 1, 2,\ \ldots,\ N\right)$ is a solution of the problem (1.1). This completes the proof.□

Remark 3.1 For a suitable choice of T_i, A_{ij}, η_i, $g_i$and ϕ_i, Theorem 3.2 includes many known results of generalized quasi-variational-like inclusions as special cases (see [1-8]),where $\phi_i$is nonconvex and $A_{ij}$is noncompact.

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Competing interests
The authors declare that they have no competing interests.

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