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A general iterative method for quasi-nonexpansive mappings in Hilbert space

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Abstract

Iterative algorithms have been extensively studied over the class of nonexpansive mappings in Hilbert spaces. Recall that nonexpansive mappings belong to quasi-nonexpansive mappings. The aim of this article is expanding the general approximation method proposed by Marino and Xu to quasi-nonexpansive mappings in Hilbert spaces.

Keywords: quasi-nonexpansive mapping, iterative analysis, variational inequality, fixed point, viscosity method

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and induced norm $||\cdot||$. A mapping T: $H \to H$ is called nonexpansive if $||Tx - Ty|| \le ||x - y||$ for all $x, y \in H$. The set of the fixed points of T is denoted by $Fix(T)$: $= \{x \in H: Tx = x\}$.

Iterative theory and methods for nonlinear mappings and variational inequalities have recently been applied to solve convex minimization problems, zero point problems and many others; see, e.g., [1-9] and references therein.

The viscosity approximation method was first introduced by Moudafi [10]. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ generated by:

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} Tx_n, \ \forall n \ge 0, \tag{1.1}$$

where f is a contraction with a coefficient $\alpha \in [0,1)$ on H, i.e., $||f(x) - f(y)|| \le \alpha||x - y||$ for all $x, y \in H$, and $\{\varepsilon_n\}$ is a sequence in (0,1) satisfying the following given conditions:

(1) $\lim_{n\to\infty} \varepsilon_n = 0$;

(2) $\sum_{n=0}^{\infty} \varepsilon_n = \infty$;

(3) $\lim_{n\to\infty}\left(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}\right) = 0$.

It is proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution $x^* \in C(C: = Fix(T))$ of the variational inequality:

$$\langle (I-f)x^*, x-x^*\rangle \ge 0, \quad \forall x \in Fix(T).$$

In [1], Xu proved that the sequence $\{x_n\}$ defined by the below process started with an arbitrary initial $x_0 \in H$:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.2}$$

converges strongly to the unique solution of the minimization problem (1.3) provided the the sequence $\{\alpha_n\}$ satisfies certain conditions:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where C is the set of fixed points set of T on H and b is a given point in H.

In [2], Marino and Xu combined the iterative method (1.2) with the viscosity approximation method (1.1) and considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \ \forall n \geq 0. \tag{1.4}$$

It is proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality:

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0, \ \forall x \in C, \tag{1.5}$$

or equivalently $\tilde{x} = P_{Fix(T)}(I - A + \gamma f)\tilde{x}$, where C is the fixed point set of a nonexpansive mapping T.

In [11], Maingé considered the viscosity approximation method (1.1), and expanded the strong convergence to quasi-nonexpansive mappings in Hilbert space. Motivated by Marino and Xu [2] and Maingé [11], we consider the following iterative process:

$$\begin{cases} x_0 = x \in H \quad \textit{arbitrarily chosen}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T_\omega x_n, \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where $T_\omega = (1 - \omega)I + \omega T$, and T is a quasi-nonexpansive mapping. Under some appropriate conditions on ω and $\{\alpha_n\}$, we obtain strong convergence over the class of quasi-nonexpansive mappings in Hilbert spaces. Our result is more general than Maingé's [11] conclusion, and also extends the iterative method (1.4) to quasi-nonexpansive mappings.

2. Preliminaries

Throughout this article, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that the sequence $\{x_n\}$ converges strongly to x. The following lemmas are useful for our article.

The following identities are valid in a Hilbert space H: for each $x,y \in H$, $t \in [0, 1]$

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$;

(ii) $||(1 - t)x + ty||^2 = (1 - t)||x||^2 + t||y||^2 - (1 - t)\, t||x - y||^2$;

(iii) $\langle x, y\rangle = -\frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$.

Lemma 2.1. [2]*Let H be a Hilbert space H. Given* $x \in H$, *C is a closed convex subset of H,* $f: H \to H$ *is a contraction with coefficient* $0 < \alpha < 1$, *and A is a strongly positive linear bounded operator with coefficient* $\bar{\gamma}$. *Then for* $0 < \gamma < \bar{\gamma}/\alpha$,

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $A - \gamma f$ *is strongly monotone with coefficient* $\bar{\gamma} - \gamma\alpha$.

Lemma 2.2. [2]*Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\bar{\gamma} > 0$*and* $0 < \rho \le ||A||^{-1}$. *Then* $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.

Lemma 2.3. [11]*Let* $T_\omega := (1 - \omega)I + \omega T$, *with T being a quasi-nonexpansive mapping on H,* $Fix(T) \neq \emptyset$, *and* $\omega \in (0, 1]$. *Then the following statements are reached:*

(a1) $Fix(T) = Fix(T_\omega)$;
(a2) T_ω *is quasi-nonexpansive;*
(a3) $||T_\omega x - q||^2 \le ||x - q||^2 - \omega(1 - \omega)||Tx - x||^2$ *for all* $x \in H$ *and* $q \in Fix(T)$;
(a4) $\langle x - T_\omega x, x - q\rangle \ge \frac{\omega}{2}\|x - Tx\|^2$*for all* $x \in H$ *and* $q \in Fix(T)$.

Remark 2.4. (a4) was revised by Wongchan and Saejung [12] (Proposition 2).

Lemma 2.5. [13]*Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exist a subsequence* $\{\Gamma_{n_j}\}_{j\ge 0}$*of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_j} < \Gamma_{n_j+1}$*for all* $j \ge 0$. *Also consider the sequence of integers* $\{\tau(n)\}_{n\ge n_0}$*defined by*

$$\tau(n) = \max\{k \le n | \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$*is a nondecreasing sequence verifying* $\lim_{n\to\infty} \tau(n) = \infty$ *and for all* $n \ge n_0$, *it holds that* $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ *and we have*

$$\Gamma_n \le \Gamma_{\tau(n)+1}.$$

Recall the metric projection P_K form a Hilbert space H to a closed convex subset K of H is defined: for each $x \in H$, there exists a unique element $P_K x \in K$ such that

$$\|x - P_K x\| := \inf\{\|x - y\| : y \in K\}.$$

Lemma 2.6. *Let K be a closed convex subset of H. Given* $x \in H$, *and* $z \in K$, $z = P_K x$, *if and only if there holds the inequality:*

$$\langle x - z, y - z\rangle \le 0, \ \forall y \in K.$$

Lemma 2.7. *If* x^* *is the solution of the variational inequality (1.5) with demi-closedness of T and* $\{y_n\} \in H$ *is a bounded sequence such that* $||Ty_n - y_n|| \to 0$, *then*

$$\liminf_{n\to\infty}\langle (A - \gamma f)x^*, y_n - x^*\rangle \ge 0. \quad (2.1)$$

Proof. We assume that there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \rightharpoonup \tilde{y}$. From the given conditions $||Ty_n - y_n|| \to 0$ and $T: H \to H$ demi-closed, we have that any weak cluster point of $\{y_n\}$ belongs to the fixed point set $Fix(T)$. Hence, we conclude that $\tilde{y} \in Fix(T)$, and also have that

$$\liminf_{n\to\infty}\langle (A - \gamma f)x^*, y_n - x^*\rangle = \lim_{j\to\infty}\langle (A - \gamma f)x^*, y_{n_j} - x^*\rangle.$$

Recalling the (1.5), we immediately obtain

$$\liminf_{n\to\infty}\langle (A - \gamma f)x^*, y_n - x^*\rangle = \langle (A - \gamma f)x^*, \tilde{y} - x^*\rangle \ge 0.$$

This completes the proof.

3. Main results

Let H be a real Hilbert space, let A be a bounded linear operator on H, and let T be a quasi-nonexpansive mapping on H, and f is a contraction with coefficient α; that is $||f$

(x) - $f(y)|| \leq \alpha||x$ - $y||$ for all x, $y \in H$. Assume the set $Fix(T)$ of fixed points of T is nonempty and we note that $Fix(T)$ is closed and convex (see [14] for more general results).

Throughout this article, we assume that A is strongly positive; that is, there exist a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$, for all $x \in H$. Let $0 < \gamma < \bar{\gamma}/\alpha$.

Theorem 3.1. *Starting with an arbitrary chosen* $x_0 \in H$, *let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T_\omega x_n, \tag{3.1}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Also* $\omega \in \left(0, \frac{1}{2}\right)$, T_ω: = $(1 - \omega)I + \omega T$ *with two conditions on T:*

(C1) $||Tx - q|| \leq ||x - q||$ *for any* $x \in H$, *and* $q \in Fix(T)$; *this means that T is a quasi-nonexpansive mapping;*
(C2) *T is demiclosed on H; that is: if* $\{y_k\} \in H$, $y_k \rightharpoonup z$, *and* $(I - T)y_k \to 0$, *then* $z \in Fix(T)$.

Then $\{x_n\}$ *converges strongly to the* $x^* \in Fix(T)$ *which is the unique solution of the VIP:*

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0, \ \forall x \in Fix(T). \tag{3.2}$$

Remark 3.2. Equivalently, from the VIP (3.2), we have

$$x^* = P_{Fix(T)} \circ (I - A + \gamma f)x^*. \tag{3.3}$$

Proof. First we show that $\{x_n\}$ is bounded.

Take any $p \in Fix(T)$, from Lemma 2.3 (a3), we have

$$\begin{aligned} \left\|x_{n+1} - p\right\| &= \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n A)T_\omega x_n - p\right\| \\ &= \left\|\alpha_n\gamma\left(f(x_n) - f(p)\right) + \alpha_n(\gamma f(p) - Ap) + (I - \alpha_n A)(T_\omega x_n - p)\right\| \\ &\leq \alpha_n\gamma\alpha\left\|f(x_n) - f(p)\right\| + \alpha_n\left\|\gamma f(p) - Ap\right\| + (1 - \alpha_n\bar{\gamma})\left\|x_n - p\right\| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\left\|x_n - p\right\| + \alpha_n\left\|\gamma f(p) - Ap\right\|. \end{aligned} \tag{3.4}$$

By induction

$$\left\|x_n - p\right\| \leq \max\left\{\left\|x_0 - p\right\|, \frac{\left\|\gamma f(p) - Ap\right\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad \forall n \geq 0.$$

Hence $\{x_n\}$ is bounded, so are the $\{f(x_n)\}$ and $\{A(x_n)\}$.

Let $x^* = P_{Fix(T)}$ o$(I - A + \gamma f)x^*$ From (3.1), we have

$$x_{n+1} - x_n + \alpha_n(Ax_n - \gamma f(x_n)) = (I - \alpha_n A)(T_\omega x_n - x_n). \tag{3.5}$$

Since $x^* \in Fix(T)$, from (a4), and together with (3.5), we obtain

$$\begin{aligned} &\langle x_{n+1} - x_n + \alpha_n(Ax_n - \gamma f(x_n)), x_n - x^*\rangle \\ &\quad = \langle(I - \alpha_n A)(T_\omega x_n - x_n), x_n - x^*\rangle \\ &\quad = (1 - \alpha_n)\langle T_\omega x_n - x_n, x_n - x^*\rangle + \alpha_n\langle(I - A)(T_\omega x_n - x_n), x_n - x^*\rangle \\ &\quad \leq -\frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \omega\alpha_n\langle(I - A)(T - I)x_n, x_n - x^*\rangle, \end{aligned}$$

it follows from the previous inequality that

$$-\langle x_n - x_{n+1}, x_n - x^* \rangle \le -\alpha_n \langle (A - \gamma f)x_n, x_n - x^* \rangle - \frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \omega\alpha_n \langle (I - A)(T - I)x_n, x_n - x^* \rangle. \quad (3.6)$$

From (iii), we obviously have

$$\langle x_n - x_{n+1}, x_n - x^* \rangle = -\frac{1}{2}\left\|x_{n+1} - x^*\right\|^2 + \frac{1}{2}\left\|x_n - x^*\right\|^2 + \frac{1}{2}\|x_{n+1} - x_n\|^2. \quad (3.7)$$

Set $\Gamma_n := \frac{1}{2}\|x_n - x^*\|^2$, and combine with (3.6), it follows that

$$\Gamma_{n+1} - \Gamma_n - \frac{1}{2}\|x_{n+1} - x_n\|^2 \le -\alpha_n \langle (A - \gamma f)x_n, x_n - x^* \rangle - \frac{\omega}{2}(1 - \alpha_n)\|x_n - Tx_n\|^2 + \omega\alpha_n \langle (I - A)(T - I)x_n, x_n - x^* \rangle. \quad (3.8)$$

Now, we calculate $||x_{n+1} - x_n||$.

From the given condition: $T_\omega := (1 - \omega)I + \omega T$, it is easy to deduce that $||T_\omega x_n - x_n|| = \omega||x_n - Tx_n||$. Thus, it follows from (3.5) that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \left\|\alpha_n(\gamma f(x_n) - Ax_n) + (I - \alpha_n A)(T_\omega x_n - x_n)\right\|^2 \\ &\le 2\alpha_n^2\left\|\gamma f(x_n) - Ax_n\right\|^2 + 2(1 - \alpha_n\bar{\gamma})^2\|T_\omega x_n - x_n\|^2 \\ &\le 2\alpha_n^2\left\|\gamma f(x_n) - Ax_n\right\|^2 + 2(1 - \alpha_n\bar{\gamma})\|T_\omega x_n - x_n\|^2 \\ &\le 2\alpha_n^2\left\|\gamma f(x_n) - Ax_n\right\|^2 + 2\omega^2(1 - \alpha_n\bar{\gamma})\|Tx_n - x_n\|^2.\end{aligned} \quad (3.9)$$

Then from (3.8) and (3.9), we have

$$\begin{aligned}&\Gamma_{n+1} - \Gamma_n + \left[\frac{\omega}{2}(1 - \alpha_n) - \omega^2(1 - \alpha_n\bar{\gamma})\right]\|x_n - Tx_n\|^2 \\ &\quad\le \alpha_n\left[\alpha_n\left\|\gamma f(x_n) - Ax_n\right\|^2 - \langle (A - \gamma f)x_n, x_n - x^* \rangle \right. \\ &\qquad\left. + \omega\langle (I - A)(T - I)x_n, x_n - x^* \rangle\right].\end{aligned} \quad (3.10)$$

Finally, we prove $x_n \to x^*$. To this end, we consider two cases.

Case 1: Suppose that there exists n_0 such that $\{\Gamma_n\}_{n \ge n_0}$ is nonincreasing, it is equal to $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge n_0$. It follows that $\lim_{n\to\infty} \Gamma_n$ exists, so we conclude that

$$\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0. \quad (3.11)$$

It follows from (3.10), (3.11) and the fact that $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Again, from (3.10), we have

$$-\alpha_n[\alpha_n\left\|\gamma f(x_n) - Ax_n\right\|^2 - \langle (A - \gamma f)x_n, x_n - x^* \rangle + \omega\langle (I - A)(T - I)x_n, x_n - x^* \rangle] \le \Gamma_n - \Gamma_{n+1}. \quad (3.12)$$

Then, by $\sum_{n=0}^{\infty} \alpha_n = \infty$, we conclude that

$$\liminf_{n\to\infty} -[\alpha_n\left\|(\gamma f - A)x_n\right\|^2 - \langle (A - \gamma f)x_n, x_n - x^* \rangle + \omega\langle (I - A)(T - I)x_n, x_n - x^* \rangle] \le 0. \quad (3.13)$$

Since $\{f(x_n)\}$ and $\{x_n\}$ are both bounded, as well as $\alpha_n \to 0$, and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, it follows from (3.13) that

$$\liminf_{n\to\infty}\langle (A - \gamma f)x_n, x_n - x^* \rangle \le 0. \quad (3.14)$$

From Lemma 2.1, it is obvious that

$$\langle (A-\gamma f)x_n, x_n - x^*\rangle \geq \langle (A-\gamma f)x^*, x_n - x^*\rangle + 2(\bar{\gamma} - \gamma\alpha)\Gamma_n. \tag{3.15}$$

Thus, from (3.14), (3.15) and the fact that $\lim_{n\to\infty}\Gamma_n$ exists, we immediately obtain

$$\begin{aligned}&\liminf_{n\to\infty}[\langle (A-\gamma f)x^*, x_n - x^*\rangle + 2(\bar{\gamma} - \gamma\alpha)\Gamma_n]\\ &\quad = 2(\bar{\gamma} - \gamma\alpha)\lim_{n\to\infty}\Gamma_n + \liminf_{n\to\infty}\langle (A-\gamma f)x^*, x_n - x^*\rangle \leq 0,\end{aligned} \tag{3.16}$$

or equivalently

$$2(\bar{\gamma} - \gamma\alpha)\lim_{n\to\infty}\Gamma_n \leq -\liminf_{n\to\infty}\langle (A-\gamma f)x^*, x_n - x^*\rangle. \tag{3.17}$$

Finally, by Lemma 2.7, we have

$$2(\bar{\gamma} - \gamma\alpha)\lim_{n\to\infty}\Gamma_n \leq 0, \tag{3.18}$$

so we conclude that $\lim_{n\to\infty}\Gamma_n = 0$, which equivalently means that $\{x_n\}$ converges strongly to x^*.

Case 2: Assume that there exists a subsequence$\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.5 that there exists a subsequence $\{\Gamma_{\tau(n)}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, and $\{\tau(n)\}$ is defined as in Lemma 2.5.

Invoking the (3.10) again, it follows that

$$\begin{aligned}&\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} + \left[\frac{\omega}{2}(1-\alpha_{\tau(n)}) - \omega^2(1-\alpha_{\tau(n)}\bar{\gamma})\right]\left\|x_{\tau(n)} - Tx_{\tau(n)}\right\|^2\\ &\quad \leq \alpha_{\tau(n)}[\alpha_{\tau(n)}\left\|\gamma f(x_{\tau(n)}) - Ax_{\tau(n)}\right\|^2 - \langle (A-\gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle\\ &\qquad + \omega\langle (I-A)(T-I)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle].\end{aligned}$$

Recalling the fact that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, we have

$$\begin{aligned}&\left[\frac{\omega}{2}(1-\alpha_{\tau(n)}) - \omega^2(1-\alpha_{\tau(n)}\bar{\gamma})\right]\left\|x_{\tau(n)} - Tx_{\tau(n)}\right\|^2\\ &\quad \leq \alpha_{\tau(n)}[\alpha_{\tau(n)}\left\|\gamma f(x_{\tau(n)}) - Ax_{\tau(n)}\right\|^2 - \langle (A-\gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle\\ &\qquad + \omega\langle (I-A)(T-I)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle].\end{aligned} \tag{3.19}$$

From the preceding results, we get the boundedness of $\{x_n\}$ and $\alpha_n \to 0$, which obviously lead to

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - Tx_{\tau(n)}\right\| = 0. \tag{3.20}$$

Hence, combining (3.19) with (3.20), we immediately deduce that

$$\begin{aligned}\langle (A-\gamma f)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle &\leq \alpha_{\tau(n)}\left\|\gamma f(x_{\tau(n)}) - Ax_{\tau(n)}\right\|^2\\ &\quad + \omega\langle (I-A)(T-I)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle.\end{aligned} \tag{3.21}$$

Again, (3.15) and (3.21) yield

$$\begin{aligned}\langle (A-\gamma f)x^*, x_{\tau(n)} - x^*\rangle + 2(\bar{\gamma} - \gamma\alpha)\Gamma_{\tau(n)} &\leq \alpha_{\tau(n)}\left\|\gamma f(x_{\tau(n)}) - Ax_{\tau(n)}\right\|^2\\ &\quad + \omega\langle (I-A)(T-I)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle.\end{aligned} \tag{3.22}$$

Recall that $\lim_{n\to\infty} \alpha_{\tau(n)} = 0$ and (3.20), we immediately have

$$2(\bar{\gamma} - \gamma\alpha) \limsup_{n\to\infty} \Gamma_{\tau(n)} \leq -\liminf_{n\to\infty} \langle (A - \gamma f)x^*, x_{\tau(n)} - x^* \rangle \tag{3.23}$$

By Lemma 2.7, we have

$$\liminf_{n\to\infty} \langle (A - \gamma f)x^*, x_{\tau(n)} - x^* \rangle \geq 0. \tag{3.24}$$

Consider (3.23) again, we conclude that

$$\limsup_{n\to\infty} \Gamma_{\tau(n)} = 0, \tag{3.25}$$

which means that $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. By Lemma 2.5, it follows that $\Gamma_n \leq \Gamma_{\tau(n)}$, thus, we get $\lim_{n\to\infty} \Gamma_n = 0$, which is equivalent to $x_n \to x^*$.

Corollary 3.3. [11]*Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_\omega x_n, \tag{3.26}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Also* $\omega \in (0, \frac{1}{2})$, *and* $T_\omega := (1 - \omega)I + \omega T$ *with two conditions on T:*

(C1) $||Tx - q|| \leq ||x - q||$ *for any* $x \in H$, *and* $q \in Fix(T)$; *this means that T is a quasi-nonexpansive mapping;*
(C2) *T is demiclosed on H; that is: if* $\{y_k\} \in H$, $y_k \rightharpoonup z$, *and* $(I - T)y_k \to 0$, $z \in Fix(T)$.

Then $\{x_n\}$ *converges strongly to the* $x^* \in Fix(T)$ *which is the unique solution of the VIP* (3.27):

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \ \forall x \in Fix(T). \tag{3.27}$$

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
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