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Results regarding the argument of certain *p*-valent analytic functions defined by a generalized integral operator

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Abstract

The integral operator $J_p^m(\lambda,\ell)(\lambda > 0; \ell \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1,2,...\})$ for functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic and *p*-valent in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ was introduced by El-Ashwah and Aouf. The object of the present article is to drive interesting argument results of *p*-valent analytic functions defined by this integral operator.

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1 Introduction

Let $A(p)$ denotes the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$. We note that $A(1) = A$, the class of univalent functions.

In [1], Catas defined the linear operator $J_p^m(\lambda,\ell)f(z)$ as follows:

$$I_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell+\lambda(k-p)}{p+\ell}\right)^m a_k z^k \\ (\lambda \geq 0; \ell \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0). \tag{1.2}$$

Also, El-Ashwah and Aouf [2] defined the integral operator $J_p^m(\lambda,\ell)f(z)$ as follows:

$$J_p^m(\lambda,\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell}{p+\ell+\lambda(k-p)}\right)^m a_k z^k \\ (\lambda \geq 0; \ell \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0). \tag{1.3}$$

The operator $J_p^m(\lambda,\ell)f(z)$ was studied by Srivastava et al. [3] and Aouf et al. [4].

From (1.2) and (1.3), we observe that $J_p^{-m}(\lambda,\ell)f(z) = I_p^m(\lambda,\ell)f(z)(m > 0)$, so the operator $J_p^m(\lambda,\ell)f(z)$ is well-defined for $\lambda \geq 0$, $\ell \geq 0$, $p \in \mathbb{N}$ and $m \in \mathbb{Z} = \{..., -2, -1,0,1,2,...\}$.

From (1.3), it is easy to verify that (see, [2])

$$\lambda z(J_p^{m+1}(\lambda,\ell)f(z))' = (\ell+p)J_p^m(\lambda,\ell)f(z) - [\ell+p(1-\lambda)]J_p^{m+1}(\lambda,\ell)f(z) \\ (\lambda > 0; \ell \geq 0; p \in \mathbb{N}; m \in \mathbb{N}_0). \tag{1.4}$$

We note that:

(i) $J_1^m(\lambda,0)f(z) = I_\lambda^{-m}f(z)(m \geq 0)$ (see Patel [5]);

(ii) $J_p^\alpha(1,1)f(z) = I_p^\alpha f(z)(\alpha > 0)$ (see Shams et al. [6]);

(iii) $J_p^m(1,1)f(z) = D^m f(z)$ (see Patel and Sahoo [7]);

(iv) $J_1^m(\lambda,0)f(z) = I_\lambda^m f(z)$ (see Al-Oboudi and Al-Qahtani [8]);

(v) $J_1^\alpha(1,1)f(z) = I^\alpha f(z)(\alpha > 0)$ (see Jung et al. [9]);

(vi) $J_1^m(1,1)f(z) = I^m f(z)$ (see Flett [10]);

(vii) $J_1^m(1,0)f(z) = \mathcal{L}^m f(z)$ (see, Salagean [11]).

Also we note that:

(i) $J_p^m(1,0)f(z) = J_p^m f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p}{k}\right)^m a_k z^k (p \in \mathbb{N}; m \in \mathbb{N}_0; z \in U)$;

(ii) $J_p^m(1,\ell)f(z) = J_p^m(\ell)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell}{k+\ell}\right)^m a_k z^k$

$(p \in \mathbb{N}; m \in \mathbb{N}_0; \ell \geq 0; z \in U)$.

In this article, we drive interesting argument results of p-valent analytic functions defined by the integral operator $J_p^m(\lambda,\ell)f(z)$.

2 Main results

In order to prove our main results, we recall the following lemma.

Lemma 1 [12]. *Let* $h(z)$ be analytic in U with $h(0) \neq 0$ $(z \in U)$. Further suppose that $\alpha, \beta \in \mathbb{R}^+ = (0, \infty)$ and

$$\left|\arg(h(z) + \beta z h'(z))\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan(\beta\alpha)\right) \quad (\alpha, \beta > 0), \tag{2.1}$$

then

$$\left|\arg(h(z))\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.2}$$

Unless otherwise mentioned we shall assume throughout the article that $\alpha, \gamma, \delta \in \mathbb{R}^+$, $\lambda > 0$, $\ell \geq 0$, $p \in \mathbb{N}$, $m \in \mathbb{Z}$ and the powers are understood as principle values.

Theorem 1. Let $g(z) \in A(p)$. Suppose $f(z) \in A(p)$ satisfies the following condition

$$\left|\arg\left(\left\{\frac{J_p^m(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)g(z)}\right\}^\gamma \left\{1 + \frac{\delta}{\lambda}\left(\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)f(z)} - \frac{J_p^{m-1}(\lambda,\ell)g(z)}{J_p^m(\lambda,\ell)g(z)}\right)\right\}\right)\right| \\ < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{\gamma(\ell+p)}\alpha\right]\right) \tag{2.3}$$

then

$$\left|\arg\left\{\frac{J_p^m(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)g(z)}\right\}^\gamma\right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.4}$$

Proof. Define a function

$$h(z) = \left\{ \frac{J_p^m(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)g(z)} \right\}^{\gamma}, \gamma \neq 0 \tag{2.5}$$

then $h(z) = 1 + c_1 z + ...,$ is analytic in U with $h(0) = 1$ and $h'(0) \neq 0$.

Differentiating (2.5) logarithmically with respect to z and multiplying by z, we have

$$\frac{1}{\gamma}\frac{zh'(z)}{h(z)} = \left\{ \frac{z(J_p^m(\lambda,\ell)f(z))'}{J_p^m(\lambda,\ell)f(z)} - \frac{z(J_p^m(\lambda,\ell)g(z))'}{J_p^m(\lambda,\ell)g(z)} \right\}. \tag{2.6}$$

Using (1.4) in (2.6), we obtain

$$h(z) + \frac{\delta}{\gamma(\ell+p)} zh'(z) = \left\{ \frac{J_p^m(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)g(z)} \right\}^{\gamma} \left\{ 1 + \frac{\delta}{\lambda}\left(\frac{J_p^{m-1}(\lambda,\ell)f(z)}{J_p^m(\lambda,\ell)f(z)} - \frac{J_p^{m-1}(\lambda,\ell)g(z)}{J_p^m(\lambda,\ell)g(z)} \right) \right\}. \tag{2.7}$$

By using Lemma 1, the proof of Theorem 1 is completed.

Remark 1. Putting $\lambda = \delta = p = 1$, $\ell = m = 0$, and $g(z) = z$, in Theorem 1, we obtain the result obtained by Lashin [12, Theorem 2.2].

Putting $\gamma = 1$ and $g(z) = z^p$ in Theorem 1, we obtain the following corollary:

Corollary 1. If $f(z) \in A(p)$ satisfies the following condition

$$\left| \arg\left\{ \frac{\delta}{\lambda}\frac{J_p^{m-1}(\lambda,\ell)f(z)}{z^p} + (1-\frac{\delta}{\lambda})\frac{J_p^m(\lambda,\ell)f(z)}{z^p} \right\} \right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{(\ell+p)}\alpha \right] \right) \tag{2.8}$$

then

$$\left| \arg\left(\frac{J_p^m(\lambda,\ell)f(z)}{z^p} \right) \right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.9}$$

Next, putting $p = 1$ in Corollary 1, we obtain the following corollary:

Corollary 2. If $f(z) \in A$ satisfies the following condition

$$\left| \arg\left\{ \frac{\delta}{\lambda}\frac{J_1^{m-1}(\lambda,\ell)f(z)}{z} + \left(1-\frac{\delta}{\lambda}\right)\frac{J_1^m(\lambda,\ell)f(z)}{z} \right\} \right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{(\ell+1)}\alpha \right] \right) \tag{2.10}$$

then

$$\left| \arg\left(\frac{J_1^m(\lambda,\ell)f(z)}{z} \right) \right| < \frac{\pi}{2}\alpha \quad (z \in U). \tag{2.11}$$

Remark 2. Putting $\lambda = 1$ and $\ell = m = 0$ in Corollary 2 we obtain the result obtained by Lashin [12, Example 2.2].

Finally, putting $\gamma = 1$ and $f(z) = z^p$ in Theorem 1, we obtain the following corollary:

Corollary 3. Let $\frac{z^p}{J_p^m(\lambda,\ell)g(z)} \neq 0, g(z) \in A(p)$ and $\delta \geq 0$. Suppose that

$$\begin{aligned} &\left| \arg\left\{ \left(1+\frac{\delta}{\lambda}\right)\frac{z^p}{J_p^m(\lambda,\ell)g(z)} - \frac{\delta}{\lambda}\frac{J_p^{m-1}(\lambda,\ell)g(z)}{J_p^m(\lambda,\ell)g(z)}\left(\frac{z^p}{J_p^m(\lambda,\ell)g(z)} \right) \right\} \right| \\ &\quad < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{(\ell+p)}\alpha \right] \right) \end{aligned} \tag{2.12}$$

then

$$\left|\arg\left(\frac{z^p}{J_p^m(\lambda,\ell)g(z)}\right)\right| < \frac{\pi}{2}\alpha. \tag{2.13}$$

Theorem 2. Let $0 < \delta \leq 1$. Suppose $f(z) \in A(p)$ satisfies the following condition

$$\left|\arg\left(\frac{J_p^m(\lambda,\ell)f(z)}{z^p}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta}{\gamma(\ell+p)}\alpha\right]\right) \quad (z \in U) \tag{2.14}$$

then we have

$$\left|\arg\left(\frac{\gamma(\ell+p)}{\delta}z^{-\frac{\gamma(\ell+p)}{\delta}}\int_0^z t^{\frac{\gamma(\ell+p)-\delta(p+1)}{\delta}} J_p^m(\lambda,\ell)f(t)dt\right)\right| < \frac{\pi}{2}\alpha. \tag{2.15}$$

Proof. Consider the function

$$h(z) = \frac{\gamma(\ell+p)}{\delta}z^{-\frac{\gamma(\ell+p)}{\delta}}\int_0^z t^{\frac{\gamma(\ell+p)-\delta(p+1)}{\delta}} J_p^m(\lambda,\ell)f(t)dt \tag{2.16}$$

then $h(z) = 1 + c_1 z + \ldots$, is analytic in U with $h(0) = 1$ and $h'(0) \neq 0$.
Differentiating (2.16) with respect to z, we have

$$h(z) + \frac{\delta}{\gamma(\ell+p)}zh'(z) = \frac{J_p^m(\lambda,\ell)f(z)}{z^p}. \tag{2.17}$$

By using Lemma 1, the proof of Theorem 2 is completed.
Putting $p = \delta = \gamma = 1$ and $m = 0$ in Theorem 2 we obtain the following corollary:
Corollary 4. Let $f(z) \in A$ satisfies the following condition

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\alpha}{\ell+1}\right]\right) \tag{2.18}$$

then

$$\left|\arg\left(\frac{\ell+1}{z^{(\ell+1)}}\int_0^z t^{(\ell-1)}f(t)dt\right)\right| < \frac{\pi}{2}\alpha(z \in U). \tag{2.19}$$

Remark 3. (i) Putting $\ell = 0$ in Corollary 4 we obtain the result obtained by Goyal and Goswami [13, Corollary 3.6].

(ii) By specifying the parameters p, λ, ℓ, and m we obtain various results for different operators reminded in the introduction.

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Competing interests
The authors declare that they have no competing interests.

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