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Ternary γ-homomorphisms and ternary γ-derivations on ternary semigroups

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Abstract

In this paper, we introduce the notions of γ-homomorphism and γ-derivation of a ternary semigroup and investigate γ-homomorphism and γ-derivations on ternary semigroup associated with the following functional in-equality $|f([xyz]) - f(x) - f(y) - f(z)| \leq \phi(x, y, z)$ and $|f([xxx]) - 3f(x)| \leq \phi(x, x, x)$, respectively.

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1 Introduction and preliminaries

Ternary algebraic operations were considered in the 19th century by several mathematicians such as Cayley [1] who introduced the notion of "cubic matrix" which in turn was generalized by Kapranov, Gelfand and Zelevinskii et al. [2]. The simplest example of such non-trivial ternary operation is given by the following composition rule:

$$\{a, b, c\}_{ijk} = \sum_{1 \leq l,m,n \leq N} a_{nil} b_{ljm} c_{mkn} \quad (i, j, k = 1, 2, \ldots\ldots, N).$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their possible applications in physics. Some significant physical applications are described in [3,4].

In 1940, Ulam [5] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homo-morphisms:

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [6] gave a partial solution of Ulams problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [7] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Rassias [7] is called the Hyers-Ulam-Rassias stability. In 1992, a generalization of Rassias theorem was obtained by Găvruta [8].

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During the last decades several stability problems of functional equations have been investigated be many mathematicians. A large list of references concerning the stability of functional equations can be found in [9-15].

In this article, using a sequence of Hyers type, we prove the generalized Hyers-Ulam-Rassias stability of ternary γ-homomorphisms and ternary γ-derivations on commutative ternary semigroups.

In the first section, which have preliminary character, we review some basic definitions and properties related to ternary groups and semigroups (cf. also Rusakov [16]).

Definition 1.1. A nonempty set G with one ternary operation $[\ ]: G \times G \times G \to G$ is called a ternary groupoid and denoted by $(G, [\ ])$.

We say that $(G, [\ ])$ is a ternary semigroup if the operation $[\ ]$ is associative, i.e., if

$$[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$$

hold for all $x, y, z, u, v \in G$ (see [17]). We shall write x^3 instead of $[xxx]$.

Definition 1.2. A ternary semigroup $(G, [\ ])$ is a ternary group if for all $a, b, c \in G$, there are $x, y, z \in G$ such that

$$[xab] = [ayb] = [abz] = c.$$

One can prove (post [18]) that elements x, y, z are uniquely determined. Moreover, according to the suggestion of post [18] one can prove (cf, Dudek et al. [19]) that in the above definition, under the assumption of the associativity, it suffices only to postulate the existence of a solution of $[ayb] = c$, or equivalently, of $[xab] = [abz] = c$.

In a ternary group, the equation $[xxz] = x$ has a unique solution which is denoted by $z = \bar{x}$ and called the skew element to x (cf. Dörnte [20]). As a consequence of results obtained in [20] we have the following theorem:

Theorem 1.3. *In any ternary group* $(G, [\ ])$ *for all* $x, y, z \in G$*, the following identities take place:*

$$[xx\bar{x}] = [x\bar{x}x] = [\bar{x}xx] = x,$$
$$[yx\bar{x}] = [y\bar{x}x] = [x\bar{x}y] = [\bar{x}xy] = y,$$
$$\overline{[xyz]} = [\bar{z}\bar{y}\bar{x}],$$
$$\bar{\bar{x}} = x.$$

Other properties of skew elements are described in [21,22].

Definition 1.4. A ternary groupoid $(G, [\ ])$ is called σ-commutative, if

$$[x_1x_2x_3] = [x_{\sigma_1}x_{\sigma_2}x_{\sigma_3}] \tag{1}$$

holds for all $x_1, x_2, x_3 \in G$ and all $\sigma \in S_3$. If (1) holds for all $\sigma \in S_3$, then $(G, [\ ])$ is a commutative groupoid. If (1) holds only for $\sigma = (13)$, i.e., if $[x_1x_2x_3] = [x_3x_2x_1]$, then $(G, [\ ])$ is called semicommutative.

Definition 1.5. An element $e \in G$ is called a middle identity or a middle neutral element of $(G, [\ ])$, if for all $x \in G$ we have

$$[exe] = x.$$

An element $e \in G$ satisfying the identity

$$[eex] = x$$

is called a left identity or a left neutral element of (*G*, [ ]). Similarly, we define a right identity. An element which is a left, middle, and right identity is called a ternary identity (or simply identity).

A mapping $f : (G, [\ ]) \to (G, [\ ])$ is called a ternary homomorphism if

$$f([xyz]) = [f(x)f(y)f(z)]$$

for all $x, y, z \in G$.

A mapping $f : (G, [\ ]) \to (G, [\ ])$ is called a ternary Jordan homomorphism if

$$f([xxx]) = [f(x)f(x)f(x)]$$

for all $x \in G$.

In Section 2, we define ternary γ-homomorphism on ternary semigroup and investigate their relations.

2 Ternary γ-homomorphisms on ternary semigroups

Definition 2.1. Let *G* be a ternary semigroup. Then the maping $H : G \to G$ is called a ternary γ-homomorphism if there exists a function $\gamma : G \to [0, \infty)$ such that

$$\gamma(H([xyz])) = \gamma([H(x)H(y)H(z)]) = \gamma(H(x)) + \gamma(H(y)) + \gamma(H(z))$$

for all $x, y, z \in G$.

Theorem 2.2. *Let G be a ternary semigroup and* $\phi : G \times G \times G \to [0, \infty)$ *be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that $H : G \to G$ *and* $f : G \to [0, \infty)$ *are functions such that*

$$\left|f([xyz]) - f(x) - f(y) - f(z)\right| \le \varphi(x, y, z) \tag{2}$$

$$\left|f(H([xyz])) - f([H(x)H(y)H(z)])\right| \le \varphi(x, y, z) \tag{3}$$

for all $x, y, z \in G$. *Then there exists a unique function* $\gamma : G \to [0, \infty)$ *such that*

$$\left|f(x) - \gamma(x)\right| \le \tilde{\varphi}(x, x, x)$$

and $\gamma(x^3) = 3\gamma(x)$. *If G is commutative and H is a ternary Jordan homomorphism, then mapping* $H : G \to G$ *is a ternary* γ*-homomorphism.*

Proof. Putting $y = z = x$ in inequality (2), we get

$$\left|f(x^3) - 3f(x)\right| \le \varphi(x, x, x).$$

By induction, one can show that

$$\left|3^{-n}f(x^{3^n}) - f(x)\right| \le \frac{1}{3}\sum_{k=0}^{n-1} 3^{-k}\varphi\left(x^{3^k}, x^{3^k}, x^{3^k}\right), \tag{4}$$

for all $x \in G$ and for all positive integer *n*, and

$$\left|3^{-n}f(3^{3^n}) - 3^{-m}f(x^{3^m})\right| \le \frac{1}{3}\sum_{k=m}^{n-1} 3^{-k}\varphi\left(x^{3^k}, x^{3^k}, x^{3^k}\right)$$

for all $x \in G$ and for all nonnegative integers m, n with $m < n$. Hence, $\{3^{-n}f(x^{3^n})\}$ is a Cauchy sequence in $[0, \infty)$. Due to the completeness of $[0, \infty)$ we conclude that this sequence is convergent. Now, let

$$\gamma(x) = \lim_{n\to\infty} 3^{-n} f(x^{3^n}), \quad x \in G.$$

Hence

$$\gamma(x^3) = \lim_{n\to\infty} 3^{-n} f\left(x^{3^{n+1}}\right) = 3\lim_{n\to\infty} 3^{-(n+1)} f\left(x^{3^{n+1}}\right) = 3\gamma(x)$$

for all $x \in G$. If $n \to \infty$ in inequality (4), we obtain

$$\left|f(x) - \gamma(x)\right| \le \tilde{\varphi}(x, x, x).$$

Next, assume that G is commutative and $H : G \to G$ is a ternary Jordan homomorphism. Replace x by x^{3^n}, y by y^{3^n} and z by z^{3^n} in inequalities (2) and (3) and divide both sides by 3^n to obtain the following:

$$\left|3^{-n} f([xyz]^{3^n}) - 3^{-n} f(x^{3^n}) - 3^{-n} f(y^{3^n}) - 3^{-n} f(z^{3^n})\right| \le 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}),$$

and

$$\left|3^{-n} f((H[xyz])^{3^n}) - 3^{-n} f([H(x)H(y)H(z)]^{3^n})\right| \le 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}).$$

If n tends to infinity. Then

$$\gamma(H[xyz]) = \gamma([H(x)H(y)H(z)]) = \gamma(H(x)) + \gamma(H(y)) + \gamma(H(z)),$$

for all $x, y, z \in G$. If γ' is another mapping with the required properties, then

$$\begin{aligned} \left|\gamma(x) - \gamma'(x)\right| &= \frac{1}{3^n}\left|3^n\gamma(x) - 3^n\gamma'(x)\right| \\ &= \frac{1}{3^n}\left|\gamma(x^{3^n}) - \gamma'(x^{3^n})\right| \\ &\le \frac{1}{3^n}\left(\left|\gamma(x^{3^n}) - f(x^{3^n})\right| + \left|f(x^{3^n}) - \gamma'(x^{3^n})\right|\right) \\ &\le \frac{2}{3^n}\tilde{\varphi}(x^{3^n}, x^{3^n}, x^{3^n}). \end{aligned}$$

Passing to the limit as $n \to \infty$ we get $\gamma(x) = \gamma'(x)$, $x \in G$. So γ is unique. Therefore, the mapping $H : G \to G$ is a unique ternary γ-homomorphism.

Theorem 2.3. *Let G be a commutative ternary semigroup and $\phi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that $H : G \to G$ and $f : G \to [0, \infty)$ are functions satisfying (2) *and* (3). *If there exists a mapping $T : G \to G$ such that T is a ternary Jordan homomorphism and*

$$\left|f(H([xyz])) - f([H(x)H(y)T(z)])\right| \le \varphi(x, y, z) \tag{5}$$

for all $x, y, z \in G$, then the mapping $T : G \to G$ is a ternary γ-homomorphism.

Proof. By Theorem 2.2, there exists a unique mapping $\gamma : G \to [0, \infty)$ such that

$$\gamma(x) = \lim_{n\to\infty} 3^{-n} f(x^{3^n}), \quad x \in G,$$

and $H : G \to G$ is a ternary γ-homomorphism. It follows from (5) that

$$\begin{aligned}&\left|\gamma([H(x)H(y)H(z)]) - \gamma([H(x)H(y)T(z)])\right| \\ &= \left|\gamma(H[xyz]) - \gamma([H(x)H(y)T(z)])\right| \\ &= \lim_{n\to\infty} \frac{1}{3^n}\left|f((H[xyz])^{3^n}) - f([H(x)H(y)T(z)]^{3^n})\right| \\ &\le \lim_{n\to\infty} \frac{1}{3^n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) = 0\end{aligned}$$

for all $x, y, z \in G$. So, $\gamma([H(x)H(y)H(z)]) = \gamma([H(x)H(y)T(z)])$ for all $x, y, z \in G$. By (2), γ is ternary additive. Hence, $\gamma(H(x)) = \gamma(T(x))$ for all $x \in G$. Thus,

$$\begin{aligned}\gamma(T[xyz]) &= \gamma(H[xyz]) = \gamma(H(x)) + \gamma(H(y)) + \gamma(H(z)) \\ &= \gamma(T(x)) + \gamma(T(y)) + \gamma(T(z)) = \gamma([T(x)T(y)T(z)])\end{aligned}$$

for all $x, y, z \in G$. Therefore T is a ternary γ-homomorphism.

Corollary 2.4. Let G be a ternary group with identity element e and $\phi : G^5 \to [0, \infty)$ be a function such that

$$\tilde{\varphi}(x, y, u.v.w) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, u^{3^n}, v^{3^n}, w^{3^n}) < \infty.$$

Suppose that $H : G \to G$ and $f : G \to [0, \infty)$ are functions such that $f(e) = 0$, $H(e) = e$ and

$$\left|f([xyH([uvw])]) - f(x) - f(y) - f([H(u)H(v)H(w)])\right| \tag{6}$$

$$\le \varphi(x, y, H(u), v, w) \tag{7}$$

for all $x, y, u, v, w \in G$. Then there exists a unique function $\gamma : G \to [0, \infty)$ such that

$$\left|f(x) - \gamma(x)\right| \le \tilde{\varphi}(x, x, x, e, e)$$

and $\gamma(x^3) = 3\gamma(x)$. If G is commutative and H is a ternary Jordan homomorphism, then the mapping $H : G \to G$ is a ternary γ-homomorphism.

Proof. Letting $v = w = e$ in (6), we get

$$\left|f([xyH(u)]) - f(x) - f(y) - f(H(u))\right| \le \varphi(x, y, H(u), e, e)$$

and by putting $x = y = e$ in (6) we get

$$\left|f([H([uvw])]) - f([H(u)H(v)H(w)])\right| \le \varphi(e, e, H(u), v, w).$$

The rest of the proof are similar to the proof of Theorem 2.2.

In next section, firstly we define ternary γ-derivation on ternary semigroup and investigate ternary γ-derivations on ternary semigroups with the following functional inequality $|f([xxx]) - 3f(x)| \le \phi(x, x, x)$.

3 Ternary γ-derivations on ternary semigroups

Definition 3.1. Let G be a ternary semigroup. Then the map $D: G \to G$ is called a ternary γ-derivation if there exists a function $\gamma: G \to [0, \infty)$ such that

$$\gamma(D([xyz])) = \gamma([D(x)yz]) + \gamma([xD(y)z]) + \gamma([xyD(z)])$$

for all $x, y, z \in G$.

Theorem 3.2. *Let G be a ternary semigroup and $\phi: G \times G \times G \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that $f: G \to [0, \infty)$ is a function such that

$$\left|f(x^3) - 3f(x)\right| \le \varphi(x, x, x) \tag{8}$$

$$\left|f(D([xyz])) - f([D(x)yz]) - f([xD(y)z]) - f([xyD(z)])\right| \le \varphi(x, y, z) \tag{9}$$

for all $x, y, z \in G$ and mapping $D: G \to G$. Then there exists a unique function $\gamma: G \to [0, \infty)$ such that

$$\left|f(x) - \gamma(x)\right| \le \tilde{\varphi}(x, x, x)$$

and $\gamma(x^3) = 3\gamma(x)$. If G is commutative and D is a ternary Jordan homomorphism, then mapping $D: G \to G$ is a ternary γ-derivation.

Proof. By induction in (8), one can show that

$$\left|3^{-n}f(x^{3^n}) - f(x)\right| \le \frac{1}{3}\sum_{k=0}^{n-1} 3^{-k}\varphi\left(x^{3^k}, x^{3^k}, x^{3^k}\right), \tag{10}$$

for all $x \in G$ and for all positive integer n, and

$$\left|3^{-n}f(3^{3^n}) - 3^{-m}f(x^{3^m})\right| \le \frac{1}{3}\sum_{k=m}^{n-1} 3^{-k}\varphi\left(x^{3^k}, x^{3^k}, x^{3^k}\right)$$

for all $x \in G$ and for all nonnegative integers m, n with $m < n$. Hence, $\{3^{-n}f(x^{3^n})\}$ is a Cauchy sequence in $[0, \infty)$. Due to the completeness of $[0, \infty)$ we conclude that this sequence is convergent. Set now

$$\gamma(x) = \lim_{n\to\infty} 3^{-n}f(x^{3^n}), \quad x \in G.$$

Hence

$$\gamma(x^3) = \lim_{n\to\infty} 3^{-n}f\left(x^{3^{n+1}}\right) = 3\lim_{n\to\infty} 3^{-(n+1)}f\left(x^{3^{n+1}}\right) = 3\gamma(x)$$

for all $x \in G$. If $n \to \infty$ in inequality (10), we obtain

$$\left|f(x) - \gamma(x)\right| \le \tilde{\varphi}(x, x, x).$$

Next, assume that G is commutative and $D: G \to G$ is a ternary Jordan homomorphism. Replace x by x^{3^n}, y by y^{3^n} and z by z^{3^n} in inequality (9) and divide both sides by 3^n, we have

$$\begin{aligned}
&\Big|3^{-n}f(D([xyz])^{3^n}) - 3^{-n}f([D(x)yz]^{3^n}) \\
&-3^{-n}f([xD(y)z]^{3^n}) - 3^{-n}f([xyD(z)]^{3^n})\Big| \\
&\le 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}).
\end{aligned}$$

If n tends to infinity. Then

$$\gamma(D([xyz])) = \gamma([D(x)yz]) + \gamma([xD(y)z]) + \gamma([xyD(z)])$$

for all $x, y, z \in G$. If γ' is another mapping with the required properties, then

$$\begin{aligned}
\left|\gamma(x) - \gamma'(x)\right| &= \frac{1}{3^n}\left|3^n\gamma(x) - 3^n\gamma'(x)\right| \\
&= \frac{1}{3^n}\left|\gamma(x^{3^n}) - \gamma'(x^{3^n})\right| \\
&\le \frac{1}{3^n}\left(\left|\gamma(x^{3^n}) - f(x^{3^n})\right| + \left|f(x^{3^n}) - \gamma'(x^{3^n})\right|\right) \\
&\le \frac{2}{3^n}\tilde{\varphi}(x^{3^n}, x^{3^n}, x^{3^n}).
\end{aligned}$$

Passing to the limit as $n \to \infty$ we get $\gamma(x) = \gamma'(x)$, $x \in G$. This proves the uniqueness of γ. Thus, the mapping $D : G \to G$ is a unique ternary γ-derivation.

Corollary 3.3. Let G be a ternary semigroup, and $\epsilon > 0$. Suppose that $f : G \to [0, \infty)$ is a function such that

$$\left|f(x^3) - 3f(x)\right| \le \varepsilon,$$

$$\left|f(D([xyz])) - f([D(x)yz]) - f([xD(y)z]) - f([xyD(z)])\right| \le \varepsilon$$

for all $x, y, z \in G$ and mapping $D : G \to G$. Then there exists a unique function $\gamma : G \to [0, \infty)$ such that

$$\left|f(x) - \gamma(x)\right| \le \frac{1}{2}\varepsilon$$

and $\gamma(x^3) = 3\gamma(x)$. If G is commutative and D is a ternary Jordan homomorphism, then mapping $D : G \to G$ is a ternary γ-derivation.

Theorem 3.4. *Let G be a commutative ternary semigroup and $\phi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that $D : G \to G$ is a ternary Jordan homomorphism and $f : G \to [0, \infty)$ is a function such that

$$f(x^{3^n}) = 3^n f(x)$$

$$\left|f(D([xyz])) - f([D(x)yz]) - f([xD(y)z]) - f([xyD(z)])\right| \le \varphi(x, y, z) \tag{11}$$

for all $x, y, z \in G$ and for all positive integer n. Then the mapping $D : G \to G$ is a ternary f-derivation.

Proof. Since G is commutative and $D : G \to G$ is ternary Jordan homomorphism. Replace x by x^{3^n}, y by y^{3^n} and z by z^{3^n} in inequality (11) and divide both sides by 3^n to obtain the following:

$$\begin{aligned}&\Big|3^{-n}f(D([xyz])^{3^n}) - 3^{-n}f([D(x)yz]^{3^n})\\&-3^{-n}f([xD(y)z]^{3^n}) - 3^{-n}f([xyD(z)]^{3^n})\Big|\\&\leq 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}).\end{aligned}$$

If n tends to infinity. Then

$$f(D([xyz])) = f([D(x)yz]) + f([xD(y)z]) + f([xyD(z)])$$

for all $x, y, z \in G$. Thus, the mapping $D : G \to G$ is a ternary f-derivation.

4 Ternary (γ, h)-derivations on ternary semigroups

In this section, we introduce concept ternary (γ, h)-derivations on ternary semigroups and investigate ternary (γ, h)-derivations on ternary semigroups with the following functional inequality $|f([xxx]) - 3f(x)| < \phi(x, x, x)$.

Definition 4.1. Let G be a ternary semigroup. Then the maping $D : G \to G$ is called ternary (γ, h)-derivation if there exists mappings $h : G \to G$ and $\gamma : G \to [0, \infty)$ such that

$$\gamma(D([xyz])) = \gamma([D(x)h(y)h(z)]) + \gamma([h(x)D(y)h(z)]) + \gamma([h(x)h(y)D(z)])$$

for all $x, y, z \in G$.

Theorem 4.2. *Let G be a ternary semigroup, and let $\phi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that D, $h : G \to G$ and $f : G \to [0, \infty)$ are functions such that

$$\left|f(x^3) - 3f(x)\right| \leq \varphi(x, x, x) \tag{12}$$

$$\Big|f(D([xyz])) - f([D(x)h(y)h(z)]) - f([h(x)D(y)h(z)]) \tag{13}$$

$$-f([h(x)h(y)D(z)])\Big| \leq \varphi(x, y, z) \tag{14}$$

for all $x, y, z \in G$. Then there exist a unique function $\gamma : G \to [0, \infty)$ such that

$$\left|f(x) - \gamma(x)\right| \leq \tilde{\varphi}(x, x, x)$$

and $\gamma(x^3) = 3\gamma(x)$. If G is commutative and D, h are ternary homomorphisms, then mapping $D : G \to G$ is a ternary (γ, h)-derivation.

Proof. By a similar method to the proof of Theorem 3.2 we obtain

$$\gamma(x) = \lim_{n\to\infty} 3^{-n}f(x^{3^n}), \quad x \in G.$$

Such that

$$\left|f(x) - \gamma(x)\right| \leq \tilde{\varphi}(x, x, x)$$

and

$$\gamma(x^3) = 3\gamma(x)$$

for all $x \in G$.

Now suppose that G is commutative and D, $h : G \to G$ are ternary homomorphism. Replace x by x^{3^n}, y by y^{3^n} and z by z^{3^n} in inequality (13) and divide both sides by 3^n to obtain the following:

$$\begin{aligned}&\Big|3^{-n}f(D([xyz])^{3^n}) - 3^{-n}f([D(x)h(y)h(z)]^{3^n})\\ &-3^{-n}f([h(x)D(y)h(z)]^{3^n}) - 3^{-n}f([h(x)h(y)D(z)]^{3^n})\Big|\\ &\le 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}).\end{aligned}$$

Let n tend to infinity. Then

$$\gamma(D([xyz])) = \gamma([D(x)h(y)h(z)]) + \gamma([h(x)D(y)h(z)]) + \gamma([h(x)h(y)D(z)])$$

for all x, y, $z \in G$.

If in Theorem 4.2 replace inequality 12 by equation $f(x^{3^n}) = 3^n f(x)$ to obtain the following Theorem.

Theorem 4.3. *Let G be a commutative ternary semigroup and $\phi : G \times G \times G \to [0, \infty)$ be a function such that*

$$\tilde{\varphi}(x, y, z) := \frac{1}{3}\sum_{n=0}^{\infty} 3^{-n}\varphi(x^{3^n}, y^{3^n}, z^{3^n}) < \infty.$$

Suppose that D, $h : G \to G$ are ternary Jordan homomorphism and $f : G \to [0, \infty)$ is a function such that

$$f(x^{3^n}) = 3^n f(x)$$

$$\begin{aligned}&|f(D([xyz])) - f([D(x)h(y)h(z)]) - f([h(x)D(y)h(z)])\\ &\qquad -f([h(x)h(y)D(z)])| \le \varphi(x, y, z)\end{aligned}$$

for all x, y, $z \in G$ and for all positive integer n. Then the mapping $D : G \to G$ is a ternary (f, h)-derivation.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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