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A new weight class and Poincaré inequalities with the Radon measure

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Abstract

We first introduce and study a new family of weights, the $A(\alpha, \beta, \gamma, E)$-class which contains the well-known $A_r(E)$-weight as a proper subset. Then, as applications of the $A(\alpha, \beta, \gamma; E)$-*class*, we prove the local and global Poincaré inequalities with the Radon measure for the solutions of the non-homogeneous A-harmonic equation which belongs to a kind of the nonlinear partial differential equations.

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1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, B be a ball and σB be the ball with the same center and $diam(\sigma B) = \sigma diam(B)$, $\sigma > 0$. We use $|E|$ to denote the Lebesgue measure of the set $E \subset \mathbb{R}^n$. We say w is a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ a.e. In 1972, Muckenhoupt [1] introduced the following $A_r(E)$-weight in order to study the properties of the Hardy-Littlewood maximal operator. We say a weight w satisfies the $A_r(E)$ -condition in a subset $E \subset \mathbb{R}^n$, where $r > 1$, and write $w \in A_r(E)$ when

$$\sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1} < \infty, \tag{1.1}$$

where the supremum is over all balls $B \subset E$. Since then, the weight functions have been well studied and widely used in analysis and PDEs, particularly in areas of the measures and integrals, see [2-11]. In 1998, the following $A_r(\lambda, E)$-weight class was introduced in [12]. We say that a weight w belongs to the $A_r(\lambda, E)$ class, $1 < r < \infty$ and $0 < \lambda < \infty$, or that w is an $A_r(\lambda, E)$-weight, write $w \in A_r(\lambda, E)$, if $sup_B \left(\frac{1}{|B|}\int_B w^\lambda dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1} < \infty$ for all balls $B \subset E$. Notice that if we choose $\lambda = 1$, we find that $A_r(1, E) = A_r(E)$. In 2000, the following class of $A_r^\lambda(E)$-weights was introduced in [13]. We say that the weight $w(x) > 0$ satisfies the

$sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)}dx\right)^{\lambda(r-1)} < \infty$-condition in E, $r > 1$ and $\lambda > 0$, and write $w \in A_r^\lambda(E)$, if $sup_B \left(\frac{1}{|B|}\int_B w dx\right)\left(\frac{1}{|B|}\int_B w^{1/(1-r)}dx\right)^{\lambda(r-1)} < \infty$ for any ball $B \subset E \subset \mathbb{R}^n$. Also, it is easy to see that $A_r^1(E) = A_r(E)$. Both $A_r(\lambda, E)$ and $A_r^\lambda(E)$ have widely been used in the study of the weighted inequalities and integral estimates, see [4-6,12,13] for example.

2. The *A*(α, β, γ; *E*)-class

In this section, we first introduce the $A(\alpha, \beta, \gamma; E)$-class which is an extension of the $A_r(E)$-weight. Then, we study the properties of this class. We will use the following Hölder inequality repeatedly in this article.

Lemma 2.1. *Let* $0 < \alpha < \infty$, $0 < \beta < \infty$ *and* $s^{-1} = \alpha^{-1} + \beta^{-1}$. *If f and g are measurable functions on* $\mathbb{R}^n$, *then* $\| fg \|_{s,E} \leq \| f \|_{\alpha,E} \cdot \| g \|_{\beta,E}$ *for any* $E \subset \mathbb{R}^n$.

We introduce the following class of functions which is an extension of the several existing classes of weights, such as $A_r^\lambda(E)$-weights, $A_r(\lambda, E)$-weights, and $A_r(E)$-weights.

Definition 2.2. We say that a measurable function $g(x)$ defined on a subset $E \subset \mathbb{R}^n$ satisfies the $A(\alpha, \beta, \gamma; E)$-condition for some positive constants α, β, γ, write $g(x) \in A(\alpha, \beta, \gamma; E)$ if $g(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B g^\alpha dx\right)\left(\frac{1}{|B|}\int_B g^{-\beta} dx\right)^{\gamma/\beta} < \infty, \tag{2.1}$$

where the supremum is over all balls $B \subset E$.

We should notice that there are three parameters in the definition of the $A(\alpha, \beta, \gamma; E)$-class. If we choose some special values for these parameters, we may obtain the existing weights. For example, if $\alpha = \lambda$, $\beta = 1/(r - 1)$ and $\gamma = 1$ in above definition, the $A(\alpha, \beta, \gamma; E)$ -class becomes $A_r(\lambda, E)$-weight, that is $A_r(\lambda, E) = A(\lambda, 1/(r - 1), 1; E)$. Similarly, $A_r^\lambda(E) = A(1, 1/(r-1), \lambda; E)$. Also, it is easy to see that the $A(\alpha, \beta, \gamma; E)$-class reduces to the usual $A_r(E)$-weight if $\alpha = \gamma = 1$ and $\beta = 1/(r - 1)$. Moreover, we have the following theorem which establishes the relationship between the $A_r(E)$-weight and the $A(\alpha, \beta, \gamma; E)$-*class*.

Theorem 2.3. *Let* $r > 1$ *be any constant and* $E \subset \mathbb{R}^n$. *Then, (i) There exists a constant* $\alpha_0 > 1$ *such that* $A_r(E) \subset A(\alpha_0, 1/(r-1), \alpha_0; E)$. *(ii) For any* α *with* $0 < \alpha < 1$, $A_r(E) \subset A(\alpha, 1/(r-1), \alpha; E)$.

Proof. For $w(x) \in A_r(E)$, by the reverse Hölder inequality for the $A_r(E)$-weight, there are constants $\alpha_0 > 1$ and $C_1 > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^{\alpha_0} dx\right)^{1/\alpha_0} \leq \frac{C_1}{|B|}\int_B w dx \tag{2.2}$$

for all balls $B \subset E$, i.e.,

$$\frac{1}{|B|}\int_B w^{\alpha_0} dx \leq C_2\left(\frac{1}{|B|}\int_B w dx\right)^{\alpha_0}. \tag{2.3}$$

From (2.3) and (1.1), we obtain

$$\begin{aligned} &\sup_B \left(\frac{1}{|B|} \int_B w^{\alpha_0} dx \right) \left(\frac{1}{|B|} \int_B w^{-\frac{1}{r-1}} dx \right)^{\alpha_0 (r-1)} \\ &\leq C_2 \sup_B \left(\frac{1}{|B|} \int_B w dx \right)^{\alpha_0} \left(\frac{1}{|B|} \int_B w^{-\frac{1}{r-1}} dx \right)^{\alpha_0 (r-1)} \\ &\leq C_2 \left(\sup_B \left(\frac{1}{|B|} \int_B w dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} \right)^{\alpha_0} < \infty, \end{aligned} \tag{2.4}$$

where the supremum is over all balls $B \subset E$. Thus, $w \in A(\alpha_0, 1/(r-1), \alpha_0; E)$. Hence, $A_r(E) \subset A(\alpha_0, 1/(r-1), \alpha_0; E)$. We have completed the proof of the first part of Theorem 2.3. Next, we prove the second part of the theorem. Let $\alpha \in (0,1)$ be any real number. Using the Hölder inequality with $1/\alpha = 1 + (1 - \alpha)/\alpha$, we have

$$\left(\int_B w^{\alpha} dx \right)^{1/\alpha} \leq \left(\int_B w dx \right) \left(\int_B 1^{\frac{\alpha}{1-\alpha}} dx \right)^{(1-\alpha)/\alpha}, \tag{2.5}$$

that is

$$\left(\frac{1}{|B|} \int_B w^{\alpha} dx \right)^{1/\alpha} \leq \frac{1}{|B|} \int_B w dx$$

which can be written as

$$\frac{1}{|B|} \int_B w^{\alpha} dx \leq \left(\frac{1}{|B|} \int_B w dx \right)^{\alpha}. \tag{2.6}$$

Similar to inequality (2.4), using (2.6) and the definitions of the $A_r(E)$-weight and the $A(\alpha, \beta, \gamma; E)$-class, we obtain that $A_r(E) \subset A(\alpha, 1/(r-1), \alpha; E)$ for any α with $0 < \alpha < 1$. The proof of Theorem 2.3 has been completed.

Example 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin and $g(x) = |x|^p$, $x \in \Omega$. We all know that $g(x) = |x|^p \in A_r(\Omega)$ for some $r > 1$ if and only if $-n < p < n(r - 1)$. Now, we consider an example in $\mathbb{R}^2$, that is $n = 2$. Assume that $D \subset \mathbb{R}^2$ is a bounded domain containing the origin and $g(x) = |x|^{-3}$ is a function in D. Since $p = -3 < -2 = -n$, then $g(x) = |x|^{-3} \notin A_r(D)$ for any $r > 1$. However, it is easy to check that $g(x) = |x|^{-3} \in A(\alpha, \beta, \gamma; D)$ for any positive numbers α, β, γ with $0 < \alpha < 2/3$.

Combining Theorem 2.3 and Example 2.4, we find that $A_r(E)$ is a proper subset of $A(\alpha, \beta, \gamma; E)$ for any positive constants α, β, γ and r with $0 < \alpha < 2/3$ and $r > 1$.

Theorem 2.5. *If $g_1(x), g_2(x) \in A(\alpha, \beta, \gamma; E)$ for some $\alpha \geq 1$, $\beta, \gamma > 0$ and a subset $E \subset \mathbb{R}^n$, then $g_1(x) + g_2(x) \in A(\alpha, \beta, \gamma; E)$.*

Proof. Let $g_1(x), g_2(x) \in A(\alpha, \beta, \gamma; E)$. By Minkowski inequality, we find that

$$\left(\int_B |g_1 + g_2|^{\alpha} dx \right)^{\frac{1}{\alpha}} \leq \left(\int_B |g_1|^{\alpha} dx \right)^{\frac{1}{\alpha}} + \left(\int_B |g_2|^{\alpha} dx \right)^{\frac{1}{\alpha}}. \tag{2.7}$$

Since $|a + b|^s \le 2^s(|a|^s + |b|^s)$ for any constants a, b, s with $s > 0$, from (2.7) , we have

$$\begin{aligned} \int_B (g_1 + g_2)^\alpha dx &\le \left(\left(\int_B |g_1|^\alpha dx \right)^{\frac{1}{\alpha}} + \left(\int_B |g_2|^\alpha dx \right)^{\frac{1}{\alpha}} \right)^\alpha \\ &\le 2^\alpha \left(\int_B |g_1|^\alpha dx + \int_B |g_2|^\alpha dx \right). \end{aligned} \tag{2.8}$$

Note that $g_1(x), g_2(x) \in A(\alpha, \beta, \gamma; E)$. Using (2.8) , we obtain

$$\begin{aligned} &\sup_B \left(\frac{1}{|B|} \int_B (g_1 + g_2)^\alpha dx \right) \left(\frac{1}{|B|} \int_B (g_1 + g_2)^{-\beta} dx \right)^{\gamma/\beta} \\ &\le \sup_B 2^\alpha \left(\frac{1}{|B|} \int_B |g_1|^\alpha dx + \frac{1}{|B|} \int_B |g_2|^\alpha dx \right) \left(\frac{1}{|B|} \int_B (g_1 + g_2)^{-\beta} dx \right)^{\gamma/\beta} \\ &\le \sup_B 2^\alpha \left(\frac{1}{|B|} \int_B g_1^\alpha dx \left(\frac{1}{|B|} \int_B g_1^{-\beta} dx \right)^{\gamma/\beta} + \frac{1}{|B|} \int_B g_2^\alpha dx \left(\frac{1}{|B|} \int_B g_2^{-\beta} dx \right)^{\gamma/\beta} \right) \\ &< \infty. \end{aligned}$$

Thus, $g_1(x) + g_2(x) \in A(\alpha, \beta, \gamma; E)$. The proof of Theorem 2.5 has been completed.

Theorem 2.6. *Let* $g_1(x) \in A(\alpha_1, \beta_1, \alpha_1\gamma; E)$ *and* $g_2(x) \in A(\alpha_2, \beta_2, \alpha_2\gamma; E)$ *for some* $\gamma > 0$ *and any subset* $E \subset \mathbb{R}^n$ *, where* $\alpha_i, \beta_i > 0$, $i = 1,2$, *and* $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}, \frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2}$. *Then,* $g_1(x)g_2(x) \in A(\alpha, \beta, \alpha\gamma; E)$.

Proof. Using Lemma 2.1 with $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2}$ and $\frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2}$, respectively, we have

$$\left(\int_B (g_1 g_2)^\alpha dx \right)^{1/\alpha} \le \left(\int_B g_1^{\alpha_1} dx \right)^{1/\alpha_1} \left(\int_B g_2^{\alpha_2} dx \right)^{1/\alpha_2}, \tag{2.9}$$

$$\left(\int_B (g_1 g_2)^{-\beta} dx \right)^{\gamma/\beta} \le \left(\int_B g_1^{-\beta} dx \right)^{\gamma/\beta_1} \left(\int_B g_2^{-\beta_2} dx \right)^{\gamma/\beta_2}. \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\begin{aligned} &\left(\int_B (g_1 g_2)^\alpha dx \right)^{1/\alpha} \left(\int_B (g_1 g_2)^{-\beta} dx \right)^{\gamma/\beta} \\ &\le \left(\int_B g_1^{\alpha_1} dx \right)^{1/\alpha_1} \left(\int_B g_1^{-\beta_1} dx \right)^{\gamma/\beta_1} \left(\int_B g_2^{\alpha_2} dx \right)^{1/\alpha_2} \left(\int_B g_2^{-\beta_2} dx \right)^{\gamma/\beta_2} \end{aligned} \tag{2.11}$$

which is equivalent to

$$\begin{aligned} &\left(\int_B (g_1 g_2)^\alpha dx \left(\int_B (g_1 g_2)^{-\beta} dx \right)^{\alpha\gamma/\beta} \right)^{1/\alpha} \\ &\le \left(\int_B g_1^{\alpha_1} dx \left(\int_B g_1^{-\beta_1} dx \right)^{\alpha_1\gamma/\beta_1} \right)^{1/\alpha_1} \left(\int_B g_2^{\alpha_2} dx \left(\int_B g_2{}^{-\beta_2} dx \right)^{\alpha_2\gamma/\beta_2} \right)^{1/\alpha_2}. \end{aligned} \tag{2.12}$$

Noticing that $g_1(x) \in A(\alpha_1, \beta_1, \alpha_1 \gamma; E)$ and $g_2(x) \in A(\alpha_2, \beta_2, \alpha_2\gamma; E)$, we obtain

$$\begin{aligned}
&\sup_B \left(\frac{1}{|B|} \int_B (g_1 g_2)^{\alpha} dx \right) \left(\frac{1}{|B|} \int_B (g_1 g_2)^{-\beta} dx \right)^{\alpha\gamma/\beta} \\
&\le \left(\sup_B \left(\frac{1}{|B|} \int_B g_1^{\alpha_1} dx \right) \left(\frac{1}{|B|} \int_B g_1^{-\beta_1} dx \right)^{\frac{\alpha_1 \gamma}{\beta_1}} \right)^{\frac{\alpha}{\alpha_1}} \left(\sup_B \left(\frac{1}{|B|} \int_B g_2^{\alpha_2} dx \right) \left(\frac{1}{|B|} \int_B g_2^{-\beta_2} dx \right)^{\frac{\alpha_2 \gamma}{\beta_2}} \right)^{\frac{\alpha}{\alpha_2}} \\
&< \infty.
\end{aligned} \tag{2.13}$$

Thus, $g_1(x)g_2(x) \in A(\alpha, \beta, \alpha\gamma; E)$. The proof of Theorem 2.6 has been completed.

Proposition 2.7. *Let* $0 < p < 1$ *and* $g(x) \in A(\alpha, \beta p, \gamma; E)$. *Then,* $g^p(x) \in A(\alpha, \beta, \gamma; E)$.

Proof. Using Lemma 2.1 with $\frac{1}{\alpha p} = \frac{1}{\alpha} + \frac{1-p}{\alpha p}$ yields

$$\left(\int_B g^{\alpha p} dx \right)^{1/\alpha p} \le |B|^{(1-p)/\alpha p} \left(\int_B g^{\alpha} dx \right)^{1/\alpha},$$

that is

$$\frac{1}{|B|} \int_B (g^p)^{\alpha} dx \le \left(\frac{1}{|B|} \int_B g^{\alpha} dx \right)^p. \tag{2.14}$$

Since $g(x) \in A(\alpha, \beta p, \gamma; E)$, using (2.14) , we find that

$$\begin{aligned}
&\sup_B \left(\frac{1}{|B|} \int_B (g^p)^{\alpha} dx \right) \left(\frac{1}{|B|} \int_B (g^p)^{-\beta} dx \right)^{\gamma/\beta} \\
&\le \sup_B \left(\frac{1}{|B|} \int_B g^{\alpha} dx \right)^p \left(\frac{1}{|B|} \int_B g^{-\beta p} dx \right)^{\gamma/\beta} \\
&\le \sup_B \left(\left(\frac{1}{|B|} \int_B g^{\alpha} dx \right) \left(\frac{1}{|B|} \int_B g^{-\beta p} dx \right)^{\gamma/\beta p} \right)^p \\
&\le \left(\sup_B \left(\frac{1}{|B|} \int_B g^{\alpha} dx \right) \left(\frac{1}{|B|} \int_B g^{-\beta p} dx \right)^{\gamma/\beta p} \right)^p \\
&< \infty.
\end{aligned} \tag{2.15}$$

Therefore, $g^p(x) \in A(\alpha, \beta, \gamma; E)$. The proof of Proposition 2.7 has been completed.

Let $\alpha, \beta, \gamma > 0$ be any constants. It is easy to prove that (i) $\frac{1}{g(x)} \in A(\alpha, \beta, \gamma; E)$ if and only if $g(x) \in A(\beta, \alpha, \alpha\beta/\gamma; E)$. (ii) $g^p(x) \in A(\alpha, \beta, \gamma; E)$ if and only if $g(x) \in A(\alpha p, \beta p, \gamma p; E)$ for any constant $p > 0$. Also, using the Hölder inequality and the definition of the $A(\alpha, \beta, \gamma; E)$-class, we can prove the following monotone properties of the $A(\alpha, \beta, \gamma; E)$-class.

Proposition 2.8. *If* $\alpha_1 < \alpha_2$, *then* $A(\alpha_2, \beta, \gamma; E) \subset A(\alpha_1, \beta, \gamma; E)$ *for any* $\beta, \gamma > 0$. *If* $\beta_1 < \beta_2$, *then* $A(\alpha, \beta_2, \gamma; E) \subset A(\alpha, \beta_1, \gamma; E)$ *for any* $\alpha, \gamma > 0$.

From Theorem 2.3 and Proposition 2.8, we know that for every $r > 1$, there exists a constant $\alpha_0 > 1$ such that $A_r(E) \subset A(\alpha, 1/(r-1), \alpha; E)$ for any α with $0 < \alpha < \alpha_0$.

3. Local Poincaré inequalities

As applications of the $A(\alpha, \beta, \gamma; E)$-class, we prove the local Poincaré inequalities with the Radon measure for the differential forms satisfying the non-homogeneous A-harmonic equation. Differential forms are extensions of functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2,...,x_n)$ is called a 0-form. The 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^{n} u_i(x_1, x_2, ..., x_n)dx_i$. If the coefficient functions $u_i(x_1, x_2,...,x_n)$, $i = 1,2,...,n$, are differentiable, then $u(x)$ is called a differential 1-form. Similarly, a differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}, k = 1, 2, ..., n,$ that is, $u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2,...,i_k)$, $1 \le i_1 < i_2 < ... < i_k \le n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$ and $L^p(\Omega, \Lambda^l)$ be the l-forms $u(x) = \Sigma_I u_I(x) dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1,2,...,n$. We denote the exterior derivative by d and the Hodge star operator by $*$. The Hodge codifferential operator d^* is given by $d^* = (-1)^{nl+1} *d*$, $l = 0,1,..., n - 1$. We consider here the solutions to the nonlinear partial differential equation

$$d^*A(x, du) = B(x, du) \tag{3.1}$$

which is called non-homogeneous A -harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions: $|A(x, \xi)| \le a|\xi|^{p-1}$, $A(x, \xi) \cdot \xi \ge |\xi|^p$ and $|B(x, \xi)| \le b|\xi|^{p-1}$ for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (3.1). A solution to (3.1) is an element of the Sobolev space $W^{1,p}_{loc}(\Omega, \wedge^{l-1})$ such that $\int_\Omega A(x, du) \cdot d\phi + B(x, du) \cdot \phi = 0$ for all $\varphi \in W^{1,p}_{loc}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, the equation (3.1) reduces to

$$\text{div}A(x, \nabla u) = B(x, \nabla u). \tag{3.2}$$

If the operator $B = 0$, Equation (3.1) becomes $d^*A(x, du) = 0$, which is called the (homogeneous) A -harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and $d^*A(x, du) = 0$ becomes the p-harmonic equation $d^*(du|du|^{p-2}) = 0$ for differential forms. See [5,6,9-16] for recent results on the solutions to the different versions of the A-harmonic equation. The operator K_y with the case $y = 0$ was first introduced by Cartan [17]. Then, it was extended to the following version in [18]. Let D be a convex and bounded domain. To each $y \in D$ there corresponds a linear operator K_y: $C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by $(K_y u)(x; \xi_1, ..., \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, ..., \xi_{l-1})dt$. A homotopy operator $T : C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in D$: $Tu = \int_D \phi(y) K_y u dy$, where $\phi \in C_0^\infty(D)$ is normalized so that $\int_D \phi(y)dy = 1$. The l-form is defined by $\omega_D = |D|^{-1} \int_D \omega(y)\, dy$, $l = 0$, and $\omega_D = d(T\omega)$, $l = 1,2,...,n$ for all $\omega \in L^p(D, \wedge^l)$, $1 \le p \le \infty$. For any differential form $u \in L^s_{loc}(D, \wedge^l), l = 1, 2, ..., n, 1 < s < \infty$, we have

$$\|Tu\|_{s,D} \le C|D|\, diam(D)\|u\|_{s,D}. \tag{3.3}$$

Lemma 3.1. [14]*Let u be a differential form satisfying the non-homogeneous A-harmonic equation* (3.1) *in* Ω, $\sigma > 1$ *and* $0 < s, t < \infty$. *Then, there exists a constant C, independent of u, such that* $||du||_{s,B} \le C|B|^{(t-s)/st}||du||_{t,\sigma B}$ *for all balls or cubes B with* $\sigma B \subset \Omega$.

Theorem 3.2. *Let* $u \in L^s_{loc}(\Omega, \wedge^l)$*be a solution of the non-homogeneous A-harmonic equation* (3.1) *in a domain* $\Omega, du \in L^s_{loc}(\Omega, \wedge^{l+1}), l = 0, 1, ..., n-1$*and* $1 < s < \infty$. *Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |u-u_B|^s d\mu\right)^{1/s} \le C\,|B|\,diam(B)\left(\int_{\sigma B} |du|^s d\mu\right)^{1/s} \tag{3.4}$$

for all balls B with $\sigma B \subset \Omega$, *where the Radon measure* μ *is defined by* $d\mu = g(x)dx$ *and* $g \in A(\alpha, \beta, \alpha; \Omega)$, $\alpha > 1$, $\beta > 0$.

Proof. By the decomposition theorem of differential forms, we have $u = d(Tu) + T(du) = u_B + T(du)$, where d is the exterior differential operator and T is the homotopy operator.

From (3.3), we obtain

$$\|u-u_B\|_{t,B} = \left\|T(du)\right\|_{t,B} \le C_1\,|B|\,diam(B)\|du\|_{t,B} \tag{3.5}$$

for any $t > 1$. Now, choose $t = \alpha s/(\alpha - 1)$, then, $t > s$. Using the Hölder inequality and (3.5), we obtain

$$\begin{aligned}\left(\int_B |u-u_B|^s d\mu\right)^{1/s} &= \left(\int_B |u-u_B|^s g(x)dx\right)^{1/s}\\ &= \left(\int_B \left(|u-u_B|\, g^{1/s}(x)\right)^s dx\right)^{1/s}\\ &\le \left(\int_B |u-u_B|^t dx\right)^{1/t}\left(\int_B g^{t/(t-s)}(x)dx\right)^{(t-s)/st}\\ &\le C_2\,|B|\,diam(B)\|du\|_{t,B}\left(\int_B g^{\alpha}(x)dx\right)^{1/\alpha s}.\end{aligned} \tag{3.6}$$

Let $m = \beta s/(1 + \beta)$, then $0 < m < s$. From Lemma 3.1, we have

$$\|du\|_{t,B} \le C_3\,|B|\,\frac{m-t}{mt}\,\|du\|\, m, \sigma_1 B, \tag{3.7}$$

where $\sigma_1 > 1$ is a constant. Using the Hölder inequality again, we find that

$$\begin{aligned}\|du\|_{m,\sigma_1 B} &= \left(\int_{\sigma_1 B}\left(|du|\,(g(x))^{1/s}(g(x))^{-1/s}\right)^m dx\right)^{1/m}\\ &\le \left(\int_{\sigma_1 B}|du|^s g(x)dx\right)^{1/s}\left(\int_{\sigma_1 B}\left(g^{-1/s}(x)\right)^{\frac{ms}{s-m}}dx\right)^{\frac{s-m}{ms}}\\ &\le \left(\int_{\sigma_1 B}|du|^s g(x)dx\right)^{1/s}\left(\int_{\sigma_1 B}(g(x))^{\frac{-m}{s-m}}dx\right)^{\frac{s-m}{ms}}\\ &\le \left(\int_{\sigma_1 B}|du|^s d\mu\right)^{1/s}\left(\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{1/\beta s}.\end{aligned} \tag{3.8}$$

Since $g \in A(\alpha, \beta, \alpha; \Omega)$, it follows that

$$\begin{aligned}
&\left(\int_B g^{\alpha}(x)dx\right)^{1/\alpha s}\left(\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{1/\beta s} \\
&\le \left(\left(\int_{\sigma_1 B} g^{\alpha}(x)dx\right)\left(\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{\alpha/\beta}\right)^{1/\alpha s} \\
&= \left(|\sigma_1 B|^{1+\frac{\alpha}{\beta}}\left(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B} g^{\alpha}(x)dx\right)\left(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{\alpha/\beta}\right)^{1/\alpha s} \\
&\le C_4|B|^{1/\alpha s+1/\beta s}.
\end{aligned} \tag{3.9}$$

Combining (3.6), (3.7), and (3.8) and using (3.9), we have

$$\begin{aligned}
&\left(\int_B |u-uB|^s d\mu\right)^{1/s} \\
&\le C_5\,|B|\,diam(B)|B|^{\frac{m-t}{mt}}\left(\int_{\sigma_1 B}|du|^s d\mu\right)^{1/s}\left(\int_B g^{\alpha}(x)dx\right)^{1/\alpha s}\left(\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{1/\beta s} \\
&\le C_5 diam(B)|B|^{1+\frac{1}{t}-\frac{1}{m}}\left(\int_{\sigma_1 B}|du|^s d\mu\right)^{1/s}\left(\left(\int_B g^{\alpha}(x)dx\right)\left(\int_{\sigma_1 B} g^{-\beta}(x)dx\right)^{\alpha/\beta}\right)^{1/\alpha s} \\
&\le C_6\,|B|\,diam(B)\left(\int_{\sigma_1 B}|du|^s d\mu\right)^{1/s},
\end{aligned}$$

that is

$$\left(\int_B |u-u_B|^s d\mu\right)^{1/s} \le C_6\,|B|\,diam(B)\left(\int_{\sigma_1 B}(du)^s d\mu\right)^{1/s}.$$

We have completed the proof of Theorem 3.2.

Let $g(x) = \frac{1}{|x-x_B|^{\lambda}}$, where x_B be the center of the ball $B \subset \Omega$ and $0 < \lambda < \frac{n}{\alpha}, \alpha > 1$. Then, $g(x) \in A\ (\alpha, \beta, \alpha; \Omega)$. From Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let $u \in L^s_{loc}(\Omega, \wedge^l)$be a solution of the non-homogeneous A-harmonic equation* (3.1) *in a domain $\Omega, du \in L^s_{loc}(\Omega, \wedge^{l+1}), l = 0, 1, \ldots, n-1$and $1 < s < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |u-u_B|^s d\mu\right)^{1/s} \le C\,|B|\,diam(B)\left(\int_{\sigma B}|du|^s d\mu\right)^{1/s} \tag{3.10}$$

for all balls B with $\sigma B \subset \Omega$, where the Radon measure μ is defined by $d\mu = \frac{1}{|x-x_B|^{\lambda}}dx$, x_B is the center of the ball $B \subset \Omega, 0 < \lambda < \frac{n}{\alpha}$and $\alpha > 1$ is a constant.

4. Global Poincaré inequalities

In this section, we will prove the global Poincaré inequalities with the Radon measure for solutions of the nonhomogeneous A-harmonic equation in $L^s\ (\mu)$-averaging domains. In 1989, Staples [19] introduced the following L^s-averaging domains.

Definition 4.1. A proper subdomain $\Omega \subset \mathbb{R}^n$ is called an L^s-averaging domain, $s \ge 1$, if there exists a constant C such that

$$\left(\frac{1}{|\Omega|}\int_{\Omega}|u-u_{\Omega}|^{s}dx\right)^{1/s}\leq C\sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_{B}|u-u_{B}|^{s}dx\right)^{1/s}$$

for all $u\in L^{s}_{loc}(\Omega)$.

Also, in [19], the L^s-averaging domain is characterized in terms of the quasi-hyperbolic metric. Particularly, Staples proved that any John domain is L^s-averaging domain, see [20] for more results on the averaging domains. In [15], the L^s-averaging domains were extended to the following $L^s(\mu)$-averaging domains.

Definition 4.2. We call a proper subdomain $\Omega\subset\mathbb{R}^n$ an $L^s(\mu)$-averaging domain, $s\geq 1$, if there exists a constant C such that

$$\left(\frac{1}{\mu(\Omega)}\int_{\Omega}\left|u-u_{B_0}\right|^{s}d\mu\right)^{1/s}\leq C\sup_{B\subset\Omega}\left(\frac{1}{\mu(B)}\int_{B}|u-u_{B}|^{s}dx\right)^{1/s}$$

for some ball $B_0\subset\Omega$ and all $u\in L^{s}_{loc}(\Omega;\mu)$, where the Radon measure $\mu(x)$ is defined by $d\mu = w(x)dx$ and $w(x)$ is a weight. Here, the supremum is over all balls B with $B\subset\Omega$.

Theorem 4.3. *Let* $u\in L^s(\Omega,\wedge^0)$ *be a solution of the non-homogeneous A -harmonic equation* (3.2) *in a domain* Ω, $du\in L^s(\Omega,\wedge^1)$, $1<s<\infty$. *Then, there exists a constant C, independent of u, such that*

$$\left(\int_{\Omega}\left|u-u_{B_0}\right|^{s}d\mu\right)^{1/s}\leq C\left(\mu(\Omega)\right)^{1+1/n}\left(\int_{\Omega}|du|^{s}d\mu\right)^{1/s} \tag{4.1}$$

for any $L^s(\mu)$*-averaging domain* $\Omega\subset\mathbb{R}^n$ *with* $\mu(\Omega)<\infty$, *where* B_0 *is some ball appearing in Definition 4.2 and the Radon measure* μ *is defined by* $d\mu = g(x)dx$, $g(x)\in A(\alpha,\beta,\alpha;\Omega)$, $\alpha>1$, $\beta>0$.

Proof. We may assume $g(x)\geq 1$ a.e. in Ω. Otherwise, let $\Omega_1=\Omega\cap\{x\in\Omega : 0<g(x)<1\}$ and $\Omega_2=\Omega\cap\{x\in\Omega : g(x)\geq 1\}$. Then, $\Omega=\Omega_1\cup\Omega_2$. We define $G(x)$ by

$$G(x)=\begin{cases}1, & x\in\Omega_1\\ g(x), & x\in\Omega_2.\end{cases}$$

Then, $G(x)\geq g(x)$ and it is easy to check that $g(x)\in A(\alpha,\beta,\alpha;\Omega)$ if and only if $G(x)\in A(\alpha,\beta,\alpha;\Omega)$.

Thus,

$$\begin{aligned}\left(\int_{\Omega}\left|u-u_{B_0}\right|^{s}d\mu\right)^{1/s}&=\left(\int_{\Omega}\left|u-u_{B_0}\right|^{s}g(x)dx\right)^{1/s}\\&\leq\left(\int_{\Omega}\left|u-u_{B_0}\right|^{s}G(x)dx\right)^{1/s}\end{aligned} \tag{4.2}$$

with $G(x)\geq 1$. Hence, we may suppose that $g(x)\geq 1$ a.e. in Ω. Thus, for any $D\subset\Omega$, we have

$$\mu(D)=\int_{D}d\mu=\int_{D}g(x)dx\geq\int_{D}dx=|D|. \tag{4.3}$$

Note that $diam(B) = C_1|B|^{1/n}$. From Theorem 3.2, we obtain

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s d\mu\right)^{1/s} \le C_2|B|^{1+1/n-1/s}\left(\int_{\sigma B}|du|^s d\mu\right)^{1/s}. \tag{4.4}$$

By definition of the L^s (μ) -averaging domain, (4.3) , (4.4) and noticing that $1 + 1/n - 1/s > 0$, we find that

$$\begin{aligned}\left(\frac{1}{\mu(\Omega)}\int_\Omega \left|u-u_{B_0}\right|^s d\mu\right)^{1/s} &\le C_3 \sup_{B\subset\Omega}\left(\frac{1}{\mu(B)}\int_B |u-u_B|^s d\mu\right)^{1/s}\\ &\le C_3 \sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B |u-u_B|^s d\mu\right)^{1/s}\\ &\le C_4 \sup_{B\subset\Omega}|B|^{1+1/n-1/s}\left(\int_{\sigma B}|du|^s d\mu\right)^{1/s}\\ &\le C_4|\Omega|^{1+1/n-1/s}\sup_{B\subset\Omega}\left(\int_{\sigma B}|du|^s d\mu\right)^{1/s}\\ &\le C_4|\Omega|^{1+1/n-1/s}\left(\int_\Omega |du|^s d\mu\right)^{1/s}\\ &\le C_4(\mu(\Omega))^{1+1/n-1/s}\left(\int_\Omega |du|^s d\mu\right)^{1/s},\end{aligned}$$

that is

$$\left(\int_\Omega \left|u-u_{B_0}\right|^s d\mu\right)^{1/s} \le C\big(\mu(\Omega)\big)^{1+1/n}\left(\int_\Omega |du|^s d\mu\right)^{1/s}.$$

The proof of Theorem 4.3 has been completed.

In [15], it has been proved that any John domain is an $L^s(\mu)$-averaging domain. Hence, we have the following corollary.

Corollary 4.4. *Let* $u \in L^s(\Omega, \wedge^0)$ *be a solution of the non-homogeneous A-harmonic equation* (3.2) *in a John domain* Ω *with* $\mu(\Omega) < \infty$, $du \in L^s(\Omega, \wedge^1)$, $1 < s < \infty$. *Then, there exists a constant C, independent of u, such that*

$$\left(\int_\Omega \left|u-u_{B_0}\right|^s d\mu\right)^{1/s} \le C\left(\int_\Omega |du|^s d\mu\right)^{1/s}, \tag{4.5}$$

where B_0 *is some ball appearing in Definition 4.2 and the Radon measure* μ *is defined by* $d\mu = g(x)dx$ *and* $g(x) \in A(\alpha, \beta, \alpha; \Omega)$, $\alpha > 1$, $\beta > 0$.

Example 4.5. Since the usual p-harmonic equation div $(\nabla u|\nabla u|^{p-2}) = 0$ and the A-harmonic equation div $A(x, \nabla u) = 0$ for functions are the special cases of the non-homogeneous A-harmonic equation, all results proved in Sections 3 and 4 are still true for p-harmonic functions and A-harmonic functions.

Remark. (i) Since an L^s-averaging domain is a special L^s (μ)-averaging domain, then the inequality (4.1) still holds in any L^s-averaging domain. (ii) Since $\mu(\Omega) < \infty$, the inequality (4.1) can be written as

$$\left(\int_\Omega \left|u-u_{B_0}\right|^s d\mu\right)^{1/s} \le C\left(\int_\Omega |du|^s d\mu\right)^{1/s},$$

where Ω is an L^s (μ)-averaging domain $\Omega \subset \mathbb{R}^n$ with $\mu(\Omega) < \infty$ and B_0 is some ball appearing in Definition 4.2, and the Radon measure μ is defined by $d\mu = g(x)dx$ and $g(x) \in A(\alpha, \beta, \alpha; \Omega)$, $\alpha > 1$, $\beta > 0$. (iii) The inequalities obtained in this article are extensions of the usual $A_r(E)$-weighted inequalities since the $A_r(E)$ is a proper subset of the $A(\alpha, \beta, \alpha; E)$-class which can be used to extend many results with the $A_r(E)$-weight into the $A(\alpha, \beta, \alpha; E)$-weight.

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References

1. Muckenhoupt, B: Weighted norm inequalities for the Hardy maximal function. Trans Am Math Soc. **165**, 207–226 (1972)
2. Agarwal, RP, O'Regan, D, Shakhmurov, V: Separable anisotropic differential operators in weighted abstract spaces and applications. J Math Anal Appl. **338**, 970–983 (2008). doi:10.1016/j.jmaa.2007.05.078
3. Agarwal, RP, Diagana, T, Hernandez, EM: Weighted pseudo almost periodic solutions to some partial neutral functional differential equations. J Nonlinear Convex Anal. **8**, 397–415 (2007)
4. Bao, G: $A_r(\lambda)$-weighted integral inequalities for *A*-harmonic tensors. J Math Anal Appl. **247**, 466–477 (2000). doi:10.1006/jmaa.2000.6851
5. Liu, B: $A_r^{\lambda}(\Omega)$-weighted imbedding inequalities for A-harmonic tensors. J Math Anal Appl. **273**(2), 667–676 (2002). doi:10.1016/S0022-247X(02)00331-1
6. Liu, B: $A_r(\lambda)$-weighted Caccioppoli-type and Poincare-type inequalities for A-harmonic tensors. Int J Math Math Sci. **31**, 115–122 (2002). doi:10.1155/S0161171202107046
7. Neugebauer, CJ: Inserting A_p-weights. Proc Am Math Soc. **87**, 644–648 (1983)
8. Stein, EM: Harmonic Analysis. Princeton University Press, Princeton (1993)
9. Wang, Y, Wu, Y: Sobolev imbedding theorems and Poincare inequalities for Green's operator on solutions of the nonhomogeneous *A*-harmonic equation. Comput Math Appl. **47**, 1545–1554 (2004). doi:10.1016/j.camwa.2004.06.006
10. Xing, Y: Weighted integral inequalities for solutions of the *A*-harmonic equation. J Math Anal Appl. **279**, 350–363 (2003). doi:10.1016/S0022-247X(03)00036-2
11. Xing, Y: Weighted Poincare-type estimates for conjugate A-harmonic tensors. J Inequal Appl. **1**, 1–6 (2005)
12. Ding, S, Shi, P: Weighted Poincaré-type inequalities for differential forms in $L^s(\mu)$-averaging domains. J Math Anal Appl. **227**, 200–215 (1998). doi:10.1006/jmaa.1998.6096
13. Ding, S: New weighted integral inequalities for differential forms in some domains. Pacific J Math. **149**, 43–56 (2000)
14. Ding, S, Nolder, CA: Weighted Poincaré-type inequalities for solutions to the *A*-harmonic equation. Illinois J Math. **2**, 199–205 (2002)
15. Ding, S, Nolder, CA: L^s (μ)-averaging domains. J Math Anal Appl. **283**, 85–99 (2003). doi:10.1016/S0022-247X(03)00216-6
16. Nolder, CA: Hardy-Littlewood theorems for *A*-harmonic tensors. Illinois J Math. **43**, 613–631 (1999)
17. Cartan, H: Differential Forms. Houghton Mifflin Co., Boston (1970)
18. Iwaniec, T, Lutoborski, A: Integral estimates for null Lagrangians. Arch Ration Mech Anal. **125**, 25–79 (1993). doi:10.1007/BF00411477
19. Staples, SG: L^p-averaging domains and the Poincaré inequality. Ann Acad Sci Fenn, Ser AI Math. **14**, 103–127 (1989)
20. Staples, SG: Averaging domains: from Euclidean spaces to homogeneous spaces. Proc of Conference on Differential & Difference Equations and Applications. pp. 1041–1048.Hindawi Publishing (2006)

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