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Some generalized nonlinear retarded integral inequalities with applications

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Abstract

In this article we discuss some new generalized nonlinear Gronwall-Bellman-Type integral inequalities with two variables, which include a non-constant term outside the integrals. We use our result to deal with the estimate on the solutions of partial differential equations with the initial and boundary conditions.

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1 Introduction

Various generalizations of Gronwall inequality [1,2] are fundamental tools in the study of existence, uniqueness, boundedness, stability and other qualitative properties of solutions of differential equations, integral equations, and differential-integral equations. There are a lot of articles investigating its generalizations such as [3-23]. Recently, Pachpatte [19] provided the explicit estimations of following integral inequalities:

$$u^p(t) \le c + p\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}[a_i(s)u^p(s) + b_i(s)u(s)]ds,$$

$$u^p(t) \le c + p\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)}[a_i(s)u(s)w(u(s)) + b_i(s)u(s)]ds,$$

and

$$u^p(x,y) \le c + p\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[a_i(s,t)u^p(s,t) + b_i(s,t)u(s,t)]dtds,$$

$$u^p(x,y) \le c + p\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[a_i(s,t)u(s,t)w(u(s,t)) + b_i(s,t)u(s,t)]dtds,$$

where c is a constant. Cheung [7] investigated the inequality

$$u^p(x,y) \le a + \frac{p}{p-q}\int_{b_1(x_0)}^{b_1(x)}\int_{c_1(y_0)}^{c_1(y)} g_1(s,t)u^q(s,t)dtds + \frac{p}{p-q}\int_{b_2(x_0)}^{b_2(x)}\int_{c_2(y_0)}^{c_2(y)} g_2(s,t)u^q(s,t)\psi(u(s,t))dtds.$$

Agarwal et al. [3] obtained the explicit bounds to the solutions of the following retarded integral inequalities:

$$\varphi(u(t)) \le c + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi(u(s)) + g_i(s)]ds,$$

$$\varphi(u(t)) \le c + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)\varphi_1(u(s)) + g_i(s)\varphi_2(\log(u(s)))]ds,$$

$$\varphi(u(t)) \le c + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s)[f_i(s)L(s,u(s)) + g_i(s)u(s)]ds,$$

where c is a constant. Chen et al. [6] discussed the following inequalities:

$$\psi(u(x,y)) \le c + \int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} f(s,t)\varphi(u(s,t))dtds,$$

$$\psi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)u(u,s)dtds + \int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} f(s,t)u(s,t)\varphi(u(s,t))dtds,$$

$$\psi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)w(u(s,t))dtds + \int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} f(s,t)w(u(s,t))\varphi(u(s,t))dtds,$$

where c is a constant.

In this article, motivated mainly by the works of Agarwal et al. [3] and Chen et al. [6], Cheung [7], Pachpatte [19], we discuss more general forms of following integral inequalities:

$$\psi(u(x,y)) \le a(x,y) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))[f_i(s,t)\varphi(u(s,t)) + g_i(s,t)]dtds, \tag{1.1}$$

$$\psi(u(x,y)) \le a(x,y) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))[f_i(s,t)\varphi_1(u(s,t)) + g_i(s,t)\varphi_2(\log(u(s,t)))]dtds, \tag{1.2}$$

$$\psi(u(x,y)) \le a(x,y) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)]dtds, \tag{1.3}$$

for $(x, y) \in [x_0, x_1) \times [y_0, y_1)$, where $a(x, y)$, $b(x, y)$ are nonnegative and nondecreasing functions in each variable. In inequalities (1.1)-(1.3), we generalized the constant c in [1,5] to the function $a(x,y)$, the function $u(x)$ in [1] to the $u(x,y)$ with two variables.

2 Main result

Throughout this article, x_0, x_1, y_0, $y_1 \in \mathbb{R}$ are given numbers. $I := [x_0,x_1)$, $J := [y_0,y_1)$, $\Delta := [x_0,x_1) \times [y_0,y_1)$, $\mathbb{R}_+ := [0,\infty)$. Consider (1.1)-(1.3), and suppose that

(H_1) $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a strictly increasing function with $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$;

(H_2) a, b: $\Delta \to (0, \infty)$ are nondecreasing in each variable;

(H_3) w, φ, φ_1, $\varphi_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$ are nondecreasing with $w(0) > 0$, $\varphi(r) > 0$, $\varphi_1(r) > 0$ and $\varphi_2(r) > 0$ for $r > 0$;

(H_4) $\alpha_i \in C^1(I,I)$ and $\beta_i \in C^1(J,J)$ are nondecreasing such that $\alpha_i(x) \le x$, $\alpha_i(x_0) = x_0$, $\beta_i(y) \le y$ and $\beta_i(y_0) = y_0$, $i = 1, 2,..., n$;

(H_5) f_i, $g_i \in C(\Delta,\mathbb{R}_+)$, $i = 1,2,...,n$.

Theorem 1. *Suppose that* $(H_1$-$H_5)$ *hold and* $u(x,y)$ *is a nonnegative and continuous function on* Δ *satisfying* (1.1). *Then we have*

$$u(x,y) \le \psi^{-1}(W^{-1}(\Phi^{-1}(B(x,y)))), \tag{2.1}$$

for all $(x,y) \in [x_0,X_1) \times [y_0,Y_1)$, *where*

$$W(r) := \int_1^r \frac{ds}{w(\psi^{-1}(s))}, \quad r > 0, \quad W(0) := \lim_{r\to 0^+} W(r), \tag{2.2}$$

$$\Phi(r) := \int_1^r \frac{ds}{\varphi(\psi^{-1}(W^{-1}(s)))}, \quad r > 0, \quad \Phi(0) := \lim_{r\to 0^+} \Phi(r), \tag{2.3}$$

$$B(x,y) := \Phi(A(x,y)) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds, \tag{2.4}$$

$$A(x,y) := W(a(x,y)) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds, \tag{2.5}$$

ψ^{-1}, W^{-1} and Φ^{-1} denote the inverse function of ψ, W and Φ, respectively, and $(X_1,Y_1) \in \Delta$ is arbitrarily given on the boundary of the planar region

$$\mathcal{R} := \{(x,y) \in \Delta : B(x,y) \in \mathrm{Dom}(\Phi^{-1}), \Phi^{-1}(B(x,y)) \in \mathrm{Dom}(W^{-1})\}. \tag{2.6}$$

Proof. From assumption H_2 and the inequality (1.1), we have

$$\psi(u(x,y)) \le a(X,y) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))[f_i(s,t)\varphi(u(s,t)) + g_i(s,t)]dtds, \tag{2.7}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$, where $x_0 \le X \le X_1$ is chosen arbitrarily. Define a function $\eta(x,y)$ by the right-hand side of (2.7). Clearly, $\eta(x,y)$ is a positive and nondecreasing function in each variable, $\eta(x_0,y) = a(X,y) > 0$. Then, (2.7) is equivalent to

$$u(x,y) \le \psi^{-1}(\eta(x,y)), \tag{2.8}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$. By the fact that $\alpha_i(x) \le x$ for $x \in [x_0,x_1)$, $\beta_i(y) \le y$ for $y \in [y_0,y_1)$, $i = 1,2,\ldots,n$, and the monotonicity of w,ψ^{-1},η, we have for all $(x,y) \in [x_0,X] \times [y_0,y_1)$,

$$\begin{aligned}\eta_x(x,y) &= b(X,y)\sum_{i=1}^{n}\alpha'_i(x)\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(\alpha_i(x),t))[f_i(\alpha_i(x),t)\varphi(u(\alpha_i(x),t)) + g_i(\alpha_i(x),t)]dt\\ &\le w(\psi^{-1}(\eta(x,y)))b(X,y)\sum_{i=1}^{n}\alpha'_i(x)\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)\varphi(\psi^{-1}(\eta(\alpha_i(x),t)))]\\ &\quad + g_i(\alpha_i(x),t)]dt.\end{aligned} \tag{2.9}$$

From (2.9), we get

$$\begin{aligned}\frac{\eta_x(x,y)}{w(\psi^{-1}(\eta(x,y)))} &\le b(X,y)\sum_{i=1}^{n}\alpha'_i(x)\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)\varphi(\psi^{-1}(\eta(\alpha_i(x),t)))\\ &\quad + g_i(\alpha_i(x),t)]dt,\end{aligned} \tag{2.10}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$. Integrating (2.10) from x_0 to x, by the definition of W in (2.2), we get for all $(x,y) \in [x_0,X] \times [y_0,y_1)$,

$$\begin{aligned}W(\eta(x,y)) &\le W(\eta(x_0,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t)\varphi(\psi^{-1}(\eta(s,t))) + g_i(s,t)\right]dtds\\ &= W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t)\varphi(\psi^{-1}(\eta(s,t))) + g_i(s,t)\right]dtds\\ &\le W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds\\ &\quad + b(X,Y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\varphi(\psi^{-1}(\eta(s,t)))dtds\\ &= c(X,y) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)\varphi(\psi^{-1}(\eta(s,t)))dtds,\end{aligned} \tag{2.11}$$

where

$$c(X,y) = W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds. \tag{2.12}$$

Now, define a function $\Gamma(x,y)$ by the right-hand side of (2.11). Clearly, $\Gamma(x,y)$ is a positive and nondecreasing function in each variable, $\Gamma(x_0,y) = c(X, y) > 0$. then, (2.11) is equivalent to

$$\eta(x,y) \le W^{-1}(\Gamma(x,y)), \tag{2.13}$$

for all $(x,y) \in [x_0,X] \times [y_0,Y_1)$, where Y_1 is defined in (2.6). By the fact that $\alpha_i(x) \le x$ for $x \in [x_0,x_1)$, $\beta_i(y) \le y$ for $y \in [y_0,y_1)$, $i = 1, 2,...,n$, and the monotonicity of φ, ψ^{-1}, W^{-1}, Γ, we have for all $(x,y) \in [x_0,X] \times [y_0,Y_1)$,

$$\begin{aligned}\Gamma_x(x,y) &= b(X,y)\sum_{i=1}^{n}\alpha_i{}'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)\varphi(\psi^{-1}(\eta(\alpha_i(x),t)))dt \\ &\le b(X,y)\varphi(\psi^{-1}(W^{-1}(\Gamma(x,y))))\sum_{i=1}^{n}\alpha_i{}'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt.\end{aligned} \tag{2.14}$$

From (2.14), we have for all $(x,y) \in [x_0,X] \times [y_0,Y_1)$,

$$\frac{\Gamma_x(x,y)}{\varphi(\psi^{-1}(W^{-1}(\Gamma(x,y))))} \le b(X,y)\sum_{i=1}^{n}\alpha_i{}'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(\alpha_i(x),t)dt. \tag{2.15}$$

Integrating (2.15) from x_0 to x, by the definition of Φ in (2.3), we get

$$\begin{aligned}\Phi(\Gamma(x,y)) &\le \Phi(\Gamma(x_0 y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds \\ &= \Phi(c(X,Y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds,\end{aligned} \tag{2.16}$$

for all $(x,y) \in [x_0,X] \times [y_0,Y_1)$. From (2.12) and (2.16), we find

$$\begin{aligned}\Gamma(x,y) &\le \Phi^{-1}\left(\Phi(c(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds\right) \\ &= \Phi^{-1}\left(\Phi\left(W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds\right)\right. \\ &\quad \left. + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds\right),\end{aligned} \tag{2.17}$$

for all $(x, y) \in [x_0, X] \times [y_0, Y_1)$. From (2.8), (2.13), and (2.17), we get

$$\begin{aligned} u(x,y) &\le \psi^{-1}(\eta(x,y)) \le \psi^{-1}(W^{-1}(\Gamma(x,y))) \\ &\le \psi^{-1}\left(W^{-1}\left(\Phi^{-1}\left(\Phi\left(W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds\right)\right.\right.\right. \\ &\quad \left.\left.\left. + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds\right)\right)\right), \end{aligned} \tag{2.18}$$

for all $(x, y) \in [x_0,X] \times [y_0,Y_1)$. Let $x = X$, from (2.18), we observe that

$$\begin{aligned} u(X,y) &\le \psi^{-1}\left(W^{-1}\left(\Phi^{-1}\left(\Phi\left(W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds|\right)\right.\right.\right. \\ &\quad \left.\left.\left. + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds\right)\right)\right), \end{aligned} \tag{2.19}$$

for all $(X, y) \in [x_0, X_1) \times [y_0, Y_1)$, where X_1 is defined by (2.6). Since $X \in [x_0, X_1)$ is arbitrary, from (2.19), we get the required estimations

$$\begin{aligned} u(x,y) &\le \psi^{-1}\left(W^{-1}\left(\Phi^{-1}\left(\Phi\left(W(a(x,y)) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds\right)\right.\right.\right. \\ &\quad \left.\left.\left. + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)dtds\right)\right)\right), \end{aligned}$$

for all $(x,y) \in [x_0,X_1) \times [y_0,Y_1)$. Theorem 1 is proved.

Remark that Theorem 1 generalizes Theorem 2.1 in [3].

Theorem 2. *Suppose that* $(H_1\text{-}H_5)$ *hold and* $u(x,y)$ *is a nonnegative and continuous function on* Δ *satisfying* (1.2). *Then*

(*i*) *if* $\varphi_1(u) \ge \varphi_2(\log(u))$, *we have*

$$u(x,y) \le \psi^{-1}\left[W^{-1}\left(\psi_1^{-1}\left(D_1(x,y)\right)\right)\right], \tag{2.20}$$

for all $(x,y) \in [x_0,X_2) \times [y_0,Y_2)$,

(*ii*) *if* $\varphi_1(u) < \varphi_2(\log(u))$, *we have*

$$u(x,y) \le \psi^{-1}\left[W^{-1}\left(\Psi_2^{-1}(D_2(x,y))\right)\right], \tag{2.21}$$

for all $(x,y) \in [x_0,X_3) \times [y_0,Y_3)$, *where* W *is defined by* (2.2) *in Theorem* 1,

$$\begin{aligned} D_j(x,y) &:= \Psi_j(W(a(x,y))) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]dtds, \\ \Psi_j(r) &:= \int_1^r \frac{ds}{\varphi_j(\psi^{-1}(W^{-1}(s)))}, \qquad \Psi_j(0) := \lim_{r\to 0+}\Psi_j(r), \end{aligned} \tag{2.22}$$

$j = 1, 2$, ψ^{-1}, W^{-1}, Ψ_1^{-1} and Ψ_2^{-1} denote the inverse function of ψ, W, Ψ_1 and Ψ_2, respectively, (X_2,Y_2) is arbitrarily given on the boundary of the planar region

$$\mathcal{R}_1 := \left\{(x,y) \in \Delta : D_1(x,y) \in \mathrm{Dom}\left(\Psi_1^{-1}\right), \Psi_1^{-1}(D_1(x,y)) \in \mathrm{Dom}(W^{-1})\right\}, \quad (2.23)$$

and (X_3,Y_3) is arbitrarily given on the boundary of the planar region

$$\mathcal{R}_2 := \left\{(x,y) \in \Delta : D_2(x,y) \in \mathrm{Dom}\left(\Psi_2^{-1}\right), \Psi_2^{-1}(D_2(x,y)) \in \mathrm{Dom}(W^{-1})\right\}. \quad (2.24)$$

Proof. From the inequality (1.2), we have

$$\begin{aligned}\psi(u(x,y)) \le a(X,y) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))\left[f_i(s,t)\varphi_1(u(s,t))\right.\\ \left.+ g_i(s,t)\varphi_2(\log(u(s,t)))\right]dtds,\end{aligned} \quad (2.25)$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$, where $x_0 \le X \le X_2$ is chosen arbitrarily. Let $\Xi(x,y)$ denote the right-hand side of (2.25), which is a positive and nondecreasing function in each variable, $\Xi(x_0,y) = a(X,y)$. Then, (2.25) is equivalent to $u(x,y) \le \psi^{-1}(\Xi(x,y))$. By the fact that $\alpha_i(x) \le x$ for $x \in [x_0, x_1)$, $\beta_i(y) \le y$ for $y \in [y_0, y_1)$, $i = 1, 2,..., n$, and the monotonicity of w,ψ^{-1},Ξ, we have for all $(x,y) \in [x_0,X] \times [y_0,y_1)$,

$$\begin{aligned}\Xi_x(x,y) &= b(X,y)\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)} w(u(\alpha_i(s),t))\left[f_i(\alpha_i(x),t)\ \varphi_1(u(\alpha_i(x),t))\right.\\ &\quad \left.+g_i(\alpha_i(x),t)\varphi_2(\log(u(\alpha_i(x),t)))\right]dt\\ &\le b(X,y)w(\psi^{-1}(\Xi(x,y)))\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(\alpha_i(x),t)\varphi_1(\psi^{-1}(\Xi(\alpha_i(x),t)))\right.\\ &\quad \left.+g_i(\alpha_i(x),t)\varphi_2(\log(\psi^{-1}(\Xi(\alpha_i(x),t))))\right]dt,\end{aligned} \quad (2.26)$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$. From (2.26), we have

$$\begin{aligned}\frac{\Xi_x(x,y)}{w(\psi^{-1}(\Xi(x,y)))} \le b(X,y)\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(\alpha_i(x),t)\varphi_1(\psi^{-1}(\Xi(\alpha_i(x),t)))\right.\\ \left.+g_i(\alpha_i(x),t)\varphi_2(\log(\psi^{-1}(\Xi(\alpha_i(x),t))))\right]dt,\end{aligned} \quad (2.27)$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$. Integrating (2.27) from x_0 to x, by the definition of W in (2.2), we get

$$\begin{aligned}W(\Xi(x,y)) &\le W(\Xi(x_0,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t)\varphi_1(\psi^{-1}(\Xi(s,t)))\right.\\ &\quad \left.+g_i(s,t)\varphi_2(\log(\psi^{-1}(\Xi(s,t))))\right]dtds\\ &= W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t)\ \varphi_1(\psi^{-1}(\Xi(s,t)))\right.\\ &\quad \left.+g_i(s,t)\varphi_2(\log(\psi^{-1}(\Xi(s,t))))\right]dtds,\end{aligned} \quad (2.28)$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$.

When $\varphi_1(u) \geq \varphi_2(\log(u))$, from the inequality (2.28), we have

$$W(\Xi(x,y)) \leq W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]\varphi_1(\psi^{-1}(\Xi(s,t)))dtds, \tag{2.29}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$. Now, define a function $\Theta(x,y)$ by the right-hand side of (2.29). Clearly, $\Theta(x,y)$ is a positive and nondecreasing function in each variable, $\Theta(x_0,y) = W(a(X,y)) > 0$. Then, (2.29) is equivalent to

$$\Xi(x,y) \leq W^{-1}(\Theta(x,y)), \quad \forall (x,y) \in [x_0,X] \times \left[y_0, Y_2\right), \tag{2.30}$$

where Y_2 is defined by (2.23). Differentiating $\Theta(x,y)$ in x for any fixed $y \in [y_0,Y_2)$, we have

$$\begin{aligned}\Theta_x(x,y) &= b(X,y)\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(\alpha_i(x),t) + g_i(\alpha_i(x),t)\right]\varphi_1(\psi^{-1}(\Xi(\alpha_i(x),t)))dt \\ &\leq b(X,y)\varphi_1(\psi^{-1}(W^{-1}(\Theta(x,y))))\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(\alpha_i(x),t) + g_i(\alpha_i(x),t)\right]dt,\end{aligned} \tag{2.31}$$

for all $(x,y) \in [x_0,X] \times [y_0,Y_2)$. From (2.31), we have

$$\frac{\Theta_x(x,y)}{\varphi_1(\psi^{-1}(W^{-1}(\Theta(x,y))))} \leq b(X,y)\sum_{i=1}^{n}\alpha_i'(x)\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(\alpha_i(x),t) + g_i(\alpha_i(x),t)\right]dt, \tag{2.32}$$

for all $(x,y) \in [x_0,X] \times [y_0,Y_2)$. Integrating (2.32) from x_0 to x, by the definition of Ψ_1 in (2.22), we obtain

$$\begin{aligned}\Psi_1(\Theta(x,y)) &\leq \Psi_1(\Theta(x_0,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]dtds \\ &= \Psi_1\left(W(a(X,y))\right) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]dtds.\end{aligned} \tag{2.33}$$

From (2.30) and (2.33), we conclude

$$\begin{aligned}u(x,y) &\leq \psi^{-1}(\Xi(x,y)) \leq \psi^{-1}(W^{-1}(\Theta(x,y))) \leq \psi^{-1}\left[W^{-1}\left(\Psi_1^{-1}\left(\right.\right.\right. \\ &\left.\left.\left.\Psi_1(W(a(X,y))) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]dtds\right)\right)\right],\end{aligned} \tag{2.34}$$

for all $(x,y) \in [x_0,X] \times [y_0,Y_2)$. Let $x = X$, from (2.34), we get

$$\begin{aligned}u(X,y) \leq \psi^{-1}&\left[W^{-1}\left(\Psi_1^{-1}\left(\Psi_1(W(a(X,y)))\right.\right.\right. \\ &\left.\left.\left.+b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)}\left[f_i(s,t) + g_i(s,t)\right]dtds\right)\right)\right].\end{aligned} \tag{2.35}$$

Since $X \in [x_0,X_2)$ is arbitrary, from the inequality (2.35), we obtain the required inequality in (2.20).

When $\varphi_1(u) \leq \varphi_2(\log(u))$, from the inequality (2.28), we have

$$\begin{aligned} W(\Xi(x,y)) &\leq W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t) \\ &\quad + g_i(s,t)\big]\varphi_2(\log(\psi^{-1}(\Xi(s,t))))dtds, \\ &\leq W(a(X,y)) + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t) \\ &\quad + g_i(s,t)\big]\varphi_2(\psi^{-1}(\Xi(s,t)))dtds, \end{aligned} \tag{2.36}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$, where $x_0 \leq X \leq X_3$. Similarly to the above process from (2.29) to (2.35), from (2.36), we obtain

$$\begin{aligned} u(X,y) \leq \psi^{-1}\Big[W^{-1}\Big(\Psi_2^{-1}\Big(\Psi_2(W(a(X,y))) \\ + b(X,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(X)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t) + g_i(s,t)\big]dtds\Big)\Big)\Big]. \end{aligned} \tag{2.37}$$

Since $X \in [x_0,X_3)$ is arbitrary, where X_3 is defined by (2.24), from the inequality (2.37), we obtain the required inequality in (2.21).

Theorem 3. *Suppose that* $(H_1\text{-}H_5)$ *hold and that* $L,M \in C\left(\mathbb{R}_+^3, \mathbb{R}_+\right)$ *satisfy*

$$0 \leq L(s,t,u) - L(s,t,v) \leq M(s,t,v)(u-v), \tag{2.38}$$

for s, t, u, $v \in \mathbb{R}_+$ *with* $u > v \geq 0$. *If u(x,y) is a nonnegative and continuous function on* Δ *satisfying* (1.3), *then we have*

$$u(x,y) \leq \psi^{-1}\left[W^{-1}\left(\Psi_3^{-1}\left(E\left(x,y\right)\right)\right)\right], \tag{2.39}$$

for all $(x,y) \in [x_0,X_4) \times [y_0,Y_4)$, *where W is defined by* (2.2),

$$\Psi_3(r) := \int_1^r \frac{ds}{\psi^{-1}(W^{-1}(s))}, \quad r > 0, \quad \Psi_3(0) := \lim_{r\to 0+}\Psi_3(r), \tag{2.40}$$

$$E(x,y) := \Psi_3(F(x,y)) + b(x,y)\sum_{i=1}^{n}\int_{i=1}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\big[f_i(s,t)M(s,t,0) + g_i(s,t)\big]dtds,$$

$$F(x,y) := W(a(x,y)) + b(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)L(s,t,0)dtds,$$

ψ^{-1}, W^{-1} *and* Ψ_3^{-1} *denote the inverse function of* ψ, *W and* Ψ_3, *respectively, and* $(X_4,Y_4) \in \Delta$ *is arbitrarily given on the boundary of the planar region*

$$\mathcal{R} := \{(x,y) \in \Delta : E(x,y) \in \mathrm{Dom}(\Psi_3^{-1}), \Psi_3^{-1}(E(x,y)) \in \mathrm{Dom}(W^{-1})\}. \tag{2.41}$$

Proof. From the inequality (1.3), we have

$$\begin{aligned} \psi(u(x,y)) \le a(X,y) + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t)) \left[f_i(s,t) L(s,t,u(s,t)) \right. \\ \left. + g_i(s,t) u(s,t) \right] dtds, \end{aligned} \tag{2.42}$$

for all $(x,y) \in [x_0,X] \times [y_0,y_1)$, where $x_0 \le X \le X_4$ is chosen arbitrarily. Let $P(x,y)$ denote the right-hand side of (2.42), which is a positive and nondecreasing function in each variable, $P(x_0,y) = a(X,y)$. Similarly to the process in the proof of Theorem 2 from (2.25) to (2.28), we obtain

$$\begin{aligned} W(P(x,y)) \le W(a(X,y)) + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \left[f_i(s,t) L(s,t,\psi^{-1}(P(s,t))) \right. \\ \left. + g_i(s,t) \psi^{-1}(P(s,t)) \right] dtds, \quad \forall (x,y) \in [x_0 X] \times \left[y_0, y_1\right). \end{aligned} \tag{2.43}$$

From the inequality (2.38) and (2.43), we get

$$\begin{aligned} W(P(x,y)) \le W(a(X,y)) + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(X)} \int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t) L(s,t,0) dtds \\ + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \left[f_i(s,t) M(s,t,0) + g_i(s,t) \right] \psi^{-1}(P(s,t)) dtds, \end{aligned}$$

for all $(x, y) \in [x_0,X] \times [y_0,y_1)$. Similarly to the process in the proof of Theorem 2 from (2.29) to (2.35), we obtain

$$\begin{aligned} u(X,y) \le \psi^{-1} \left[W^{-1} \left(\Psi_3^{-1} \left(\Psi_3 \left(W(a(X,y)) + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(X)} \int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t) L(s,t,0) dtds \right) \right. \right. \right. \\ \left. \left. \left. + b(X,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(X)} \int_{\beta_i(y_0)}^{\beta_i(y)} \left[f_i(s,t) M(s,t,0) + g_i(s,t) \right] dtds \right) \right) \right], \end{aligned} \tag{2.44}$$

where Ψ_3 is defined by (2.40). Since $X \in [x_0,X_4)$ is arbitrary, where X_4 is defined by (2.41), from the inequality (2.44), we obtain the required inequality in (2.39).

3 Applications to BVP

In this section we use our result to study certain properties of solution of the following boundary value problem (simply called BVP):

$$\begin{cases} \dfrac{\partial^2 \psi(z(x,y))}{\partial x \partial y} = F(x,y,z(\alpha_1(x),\beta_1(y)), z(\alpha_2(x),\beta_2(y)), \ldots, z(\alpha_n(x),\beta_n(y))), \\ z(x,y_0) = a_1(x), \quad z(x_0,y) = a_2(y), a_1(x_0) = a_2(y_0) = 0, \end{cases} \tag{3.1}$$

for $x \in I, y \in J$, where $x_0,y_0,x_1,y_1 \in \mathbb{R}_+$ are constants, $I := [x_0,x_1)$, $J := [y_0,y_1)$, $F \in C(I \times J \times \mathbb{R}^n, \mathbb{R})$, $\psi: \mathbb{R} \to \mathbb{R}$ is strictly increasing on $\mathbb{R}_+$ with $\psi(0) = 0$, $|\psi(r)| = \psi(|r|) > 0$, for |

$r| > 0$ and $\psi(t) \to \infty$ as $t \to \infty$; functions $\alpha_i \in C^1(I,I);\beta_i \in C^1(J,J), i = 1,2,...,n$ are nondecreasing such that $\alpha_i(x) \le x$, $\beta_i(y) \le y, \alpha_i(x_0) = x_0$, $\beta_i(y_0) = y_0$; $|a_1| \in C^1(I,\mathbb{R}_+)$, $|a_2| \in C^1(J,\mathbb{R}_+)$ are both nondecreasing. Though this equation is similar to the equation discussed in Section 3 in [3], our results are more general than the results obtained in [3].

We first give an estimate for solutions of the BVP (3.1) so as to obtain a condition for boundedness.

Corollary 1. *Consider BVP* (3.1) *and suppose that* $F \in C(I \times J \times \mathbb{R}^n,\mathbb{R})$ *satisfies*

$$\left|F(x,y,u_1,u_2,...,u_n)\right| \le \sum_{i=1}^{n} w\left(|u_i|\right)\left[f_i(x,y)\varphi\left(|u_i|\right) + g_i(x,y)\right], \quad (x,y) \in I \times J, \tag{3.2}$$

where $f_i,g_i \in C(I \times J,\mathbb{R}_+)$ *and* $w,\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *are nondecreasing such that* $w(u) > 0,\varphi(u) > 0$ *for* $u > 0$. *Then all solutions* $z(x,y)$ *of BVP* (3.1) *have the estimate*

$$\left|z(x,y)\right| \le \psi^{-1}\left(W^{-1}\left(\Phi^{-1}\left(B(x,y)\right)\right)\right), \tag{3.3}$$

for all $(x,y) \in [x_0,X_1) \times [y_0,Y_1)$, *where*

$$B(x,y) := \Phi(A(x,y)) + \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} \frac{f_i\left(\alpha_i^{-1}(s),\beta_i^{-1}(t)\right)}{\alpha_i{}'\left(\alpha_i^{-1}(s)\right)\beta_i{}'\left(\beta_i^{-1}(t)\right)} dtds,$$

$$A(x,y) := W\left(\psi\left(\left|a_1(x)\right|\right) + \psi\left(\left|a_2(y)\right|\right)\right) + \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i^{-1}} \frac{g_i\left(\alpha_i^{-1}(s),\beta_i^{-1}(t)\right)}{\alpha_i{}'\left(\alpha_i^{-1}(s)\right)\beta_i{}'\left(\beta_i^{-1}(t)\right)} dtds,$$

for all $(x,y) \in [x_0,X_1) \times [y_0,Y_1)$, *where functions* W, W^{-1}, Φ, Φ^{-1} *and real numbers* X_1, Y_1 *are given as in Theorem* 1.

Proof. The equivalent integral equation of BVP (3.1) is

$$\begin{aligned}\psi(z(x,y)) = \psi(a_1(x)) + \psi(a_2(y)) + \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} F\big(s,t,z(\alpha_1(s),\beta_1(t)),z(\alpha_2(s),\beta_2(t))\,,...,\\ z(\alpha_n(s),\beta_n(t))\big)\,dtds.\end{aligned} \tag{3.4}$$

By (3.2) and (3.4), we get that

$$\begin{aligned}
&\psi\left(\left|z(x,y)\right|\right)\\
&\quad\le \psi\left(\left|a_1(x)\right|\right) + \psi\left(\left|a_2(y)\right|\right)\\
&\qquad+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} \left|F\left(s,t,z(\alpha_1(s),\beta_1(t)),z(\alpha_2(s),\beta_2(t)),...,z(\alpha_2(s),\beta_n(t))\right)\right| dtds\\
&\quad\le \psi\left(\left|a_1(x)\right|\right) + \psi\left(\left|a_2(y)\right|\right)\\
&\qquad+ \int\limits_{x_0}^{x}\int\limits_{y_0}^{y} \sum_{i=1}^{n} w\left(\left|z(\alpha_i(s),\beta_i(t))\right|\right)\left[f_i(s,t)\varphi\left(\left|z(\alpha_i(s),\beta_i(t))\right|\right) + g_i(s,t)\right] dtds\\
&\quad= \psi\left(\left|a_1(x)\right|\right) + \psi\left(\left|a_2(y)\right|\right) + \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)}\\
&\qquad\frac{w\left(\left|z(s_1,t_1)\right|\right)\left[f_i\left(\alpha_i^{-1}(s_1),\beta_i^{-1}(t_1)\right)\varphi\left(\left|z(s_1,t_1)\right|\right) + g_i\left(\alpha_i^{-1}(s_1),\beta_i^{-1}(t_1)\right)\right]}{\alpha_i{}'\left(\alpha_i^{-1}(s_1)\right)\beta_i{}'\left(\beta_i^{-1}(t_1)\right)} dt_1ds_1,
\end{aligned} \tag{3.5}$$

where a change of variables $s_1 = \alpha_i(s)$, $t_1 = \beta_i(t), i = 1,2,\ldots,n$ are made. Clearly, the inequality (3.5) is in the form of (1.1). Thus the estimate (3.3) of the solution $z(x,y)$ in this corollary is obtained immediately by our Theorem 1.

Our Corollary 1 actually gives a condition of boundedness for solutions. Concretely, if

$$\begin{aligned}
&\psi\left(|a_1(x)|\right) + \psi\left(|a_2(y)|\right) < \infty,\\
&\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} \frac{f_i\left(\alpha_i^{-1}(s), \beta_i^{-1}(t)\right)}{\alpha_i'\left(\alpha_i^{-1}(s)\right)\beta_i'\left(\beta_i^{-1}(t)\right)}dtds < \infty,\\
&\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} \frac{g_i\left(\alpha_i^{-1}(s), \beta_i^{-1}(t)\right)}{\alpha_i'\left(\alpha_i^{-1}(s)\right)\beta_i'\left(\beta_i^{-1}(t)\right)}dtds < \infty,
\end{aligned}$$

on $[x_0,X_1) \times [y_0,Y_1)$, then every solution $z(x,y)$ of BVP (3.1) is bounded on $[x_0,X_1) \times [y_0,Y_1)$.

Next, we discuss the uniqueness of solutions for BVP (3.1).

Corollary 2. *Consider BVP* (3.1) *and suppose that* $F \in C(I \times J \times \mathbb{R}^n, \mathbb{R})$ *satisfies*

$$\left|F(x,y,u_1,u_2,\ldots,u_n) - F(x,y,v_1,v_2,\ldots,v_n)\right| \le \sum_{i=1}^{n} f_i(x,y)\left|\psi(u_i) - \psi(v_i)\right|, \quad (3.6)$$

for all $(x,y) \in I \times J$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1, 2,\ldots, n$, *where* $I = [x_0, x_1]$, $J = [y_0, y_1]$ *are two finite intervals, and* $f_i \in C(I \times J, \mathbb{R}_+), i = 1,2,\ldots,n$. *Then BVP* (3.1) *has at most one solution on* $I \times J$.

Proof. Assume that both $z(x,y)$ and $\tilde{z}(x,y)$ are solutions of BVP (3.1). From the equivalent integral Equations (3.4) and (3.6), we have

$$\begin{aligned}
&\left|\psi(z(x,y)) - \psi(\tilde{z}(x,y))\right|\\
&\le \int_{x_0}^{x}\int_{y_0}^{y} \big|F(s,t,z(\alpha_1(s),\beta_1(t)), z(\alpha_2(s),\beta_2(t)),\ldots,z(\alpha_n(s),\beta_n(t)))\\
&\quad -F(s,t,\tilde{z}(\alpha_1(s),\beta_1(t)), \tilde{z}(\alpha_2(s),\beta_2(t)),\ldots,\tilde{z}(\alpha_n(s),\beta_n(t)))\big|\,dtds\\
&\le \int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n} f_i(s,t)\left|\psi(z(\alpha_i(s),\beta_i(t))) - \psi(\tilde{z}(\alpha_i(s),\beta_i(t)))\right|dtds\\
&\le \varepsilon + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\sum_{i=1}^{n}\frac{f_i(\alpha_i^{-1}(s_1),\beta_i^{-1}(t_1))\left|\psi(z(s_1,t_1)) - \psi(\tilde{z}(s_1,t_1))\right|}{\alpha_i'(\alpha_i^{-1}(s_1))\beta_i'(\beta_i^{-1}(t_1))}dt_1ds_1,
\end{aligned} \quad (3.7)$$

for all $(x,y) \in I \times J$, where changes of variables $s_1 = \alpha_i(s)$, $t_1 = \beta_i(t)$ are made, $\varepsilon > 0$ is an arbitrary small number. Clearly, the inequality (3.7) is in the form of (1.1). Suitably applying our Theorem 1 to (3.7), we get an estimate of the form (2.1) for all $(x,y) \in I \times J$,

$$\left|\psi(z(x,y)) - \psi(\tilde{z}(x,y))\right| \le \varepsilon \exp\left(\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}\sum_{i=1}^{n}\frac{f_i\left(\alpha_i^{-1}(s),\beta_i^{-1}(t)\right)}{\alpha_i'\left(\alpha_i^{-1}(s)\right)\beta_i'\left(\beta_i^{-1}(t)\right)}dtds\right). \quad (3.8)$$

Letting $\varepsilon \to 0_+$, since $\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} \sum_{i=1}^{n} \frac{f_i\left(\alpha_i^{-1}(s), \beta_i^{-1}(t)\right)}{\alpha_i{}'\left(\alpha_i^{-1}(s)\right)\beta_i{}'\left(\beta_i^{-1}(t)\right)} dtds$ is finite on finite intervals I and J, ψ is a strictly increasing function, from (3.8), we conclude that $\left|\psi(z(x,y)) - \psi(\tilde{z}(x,y))\right| \leq 0$, implying that $z(x,y) = \tilde{z}(x,y)$ for all $(x,y) \in I \times J$. The uniqueness is proved.

Remark Suppose that $F \in C(I \times J \times \mathbb{R}^n, \mathbb{R})$ in BVP (3.1) satisfies

$$\left|F(x,y,u_1,u_2,\ldots,u_n)\right| \leq \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))\left[f_i(s,t)\varphi_1(u(s,t)) + g_i(s,t)\varphi_2(\log(u(s,t)))\right] dtds.$$

By using Theorem 2, we can give an estimate for solutions of the BVP (3.1).

Suppose that $F \in C(I \times J \times \mathbb{R}^n, \mathbb{R})$ in BVP (3.1) satisfies

$$\left|F(x,y,u_1,u_2,\ldots,u_n)\right| \leq \sum_{i=1}^{n} \int\limits_{\alpha_i(x_0)}^{\alpha_i(x)} \int\limits_{\beta_i(y_0)}^{\beta_i(y)} w(u(s,t))\left[f_i(s,t)L(s,t,u(s,t)) + g_i(s,t)u(s,t)\right] dtds.$$

By using Theorem 3, we can give an estimate for solutions of the BVP (3.1) too.

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Competing interests

The author declares that they have no competing interests.

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