Yu et al. *Journal of Inequalities and Applications* 2012, **2012**:308
http://www.journalofinequalitiesandapplications.com/content/2012/1/308

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Boundedness for multilinear commutator of Marcinkiewicz operator with variable kernels on Hardy and Herz-Hardy spaces

Wenxin Yu[1*], Yigang He[1,2], Qiwu Luo[1], Jian Zheng[1] and Xianming Wu[1]

*Correspondence: slowbird@sohu.com
[1]College of Electrical and Information Engineering, Hunan University, Changsha, Hunan 410082, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, the $(H^p_{\vec{b}}, L^p)$- and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$-type boundedness for the multilinear commutator related to the Marcinkiewicz operator with variable kernels is obtained.
MSC: 42B20; 42B25

Keywords: Marcinkiewicz operator; multilinear commutator; BMO; Hardy space; Herz-Hardy space

1 Introduction and definitions

Let T be the Calderón-Zygmund operator and $b \in BMO(R^n)$. The commutator $[b, T]$ generated by T and b is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [1, 2]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^p_b(R^n)$, then $[b, T]$ maps continuously $H^p_b(R^n)$ into $L^p(R^n)$ (see [3]). In addition, we easily know that $H^p_b(R^n) \subset H^p(R^n)$. In recent years, the theory of Herz-type Hardy spaces have been developed (see [4–7]). The main purpose of this paper is to consider the continuity of multilinear commutators related to Marcinkiewicz operators with variable kernels and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces. Let us first introduce some definitions (see [3–14]).

Given a positive integer m and $1 \le j \le m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C^m_j$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C^m_j$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Definition 1 Let b_i $(i = 1, \dots, m)$ be a locally integrable function and $0 < p \le 1$. A bounded measurable function a on R^n is said to be a $(p, \vec{b})$ atom if

(1) $\operatorname{supp} a \subset B = B(x_0, r)$,
(2) $\|a\|_{L^\infty} \le |B|^{-1/p}$,
(3) $\int_B a(y)\,dy = \int_B a(y) \prod_{l\in\sigma} b_l(y)\,dy = 0$ for any $\sigma \in C^m_j$, $1 \le j \le m$.

A temperate distribution f is said to belong to $H^p_{\vec{b}}(R^n)$ if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where every a_j is $(p,\vec{b})$ atom, $\lambda \in C$ and $\sum_{j=1}^{\infty} |\lambda|^p < \infty$. Moreover, $\|f\|_{H^p_{\vec{b}}(R^n)} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Given a set $E \subset R^n$, the characteristic function of E is defined by χ_E. Let $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$and $\chi_k = \chi_{B_k}$, $k \in Z$.

Definition 2 Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and by χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q(R^n) = \{f \in L^q_{\mathrm{loc}}(R^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q}\right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{\mathrm{loc}}(R^n) : \|f\|_{K^{\alpha,p}_q} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q} + \|f\chi_0\|^p_{L^q}\right]^{1/p}.$$

Definition 3 Let $\alpha \in R$, $1 < q < \infty$, $0 < \alpha < n(1-1/q)$, $b_i \in BMO(R^n)$, $1 \le i \le m$. A function a on R^n is called a central $(\alpha, q, \vec{b})$-atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type) if

(1) $\operatorname{supp} a \subset B = B(0,r)$ (or for some $r \ge 1$),

(2) $\|a\|_{L^q} \le |B|^{-\alpha/n}$,

(3) $\int_B a(x)\,dx = \int_B a(x) \prod_{l\in\sigma} b_l(x)\,dx = 0$ for any $\sigma \in C^m_j$, $1 \le j \le m$.

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ (or $HK^{\alpha,p}_{q,\vec{b}}(R^n)$) if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the Schwartz distribution sense, where a_j is a central $(\alpha, q, \vec{b})$-atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover, $\|f\|_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}$(or $\|f\|_{HK^{\alpha,p}_{q,\vec{b}}}$) $\approx (\sum_j |\lambda_j|^p)^{1/p}$.

Definition 4 Let Ω be homogeneous of degree zero on R^n such that $\omega_r(\delta)$ is defined

$$\omega_r(\delta) = \sup_{|\rho|<\delta} \left(\int_{S^{n-1}} \left|\Omega(\rho\acute{x}) - \Omega(\acute{x})\right|^r d\sigma(\acute{x})\right)^{1/r},$$

where $|\rho| = \sup_{\acute{x}\in S^{n-1}} |\rho\acute{x} - \acute{x}|$.

If $\int_0^1 \frac{\omega_r(\delta)}{\delta}\,d\delta < \infty$, we say $\Omega(x)$ satisfied L^r-*Dini* condition.

Definition 5 Let $0 < \epsilon < n$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\epsilon^{\vec{b}}(f)(x) = \left(\int_0^\infty \left|F_t^{\vec{b}}(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\epsilon}} \left[\prod_{j=1}^m \bigl(b_j(x) - b_j(y)\bigr)\right] f(y)\,dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\epsilon}} f(y)\,dy,$$

we also define that

$$\mu_\epsilon(f)(x) = \left(\int_0^\infty \left|F_t(f)(x)\right|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [9, 13, 14]).

2 Theorems and proofs

We begin with three preliminary lemmas.

Lemma 1 (see [12]) *Let* $1 < r < \infty$, $b_j \in BMO(R^n)$ *for* $j = 1, \dots, k$ *and* $k \in N$. *Then we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k \left|b_j(y) - (b_j)_Q\right| dy \le C \prod_{j=1}^k \|b_j\|_{BMO(R^n)}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k \left|b_j(y) - (b_j)_Q\right|^r dy\right)^{1/r} \le C \prod_{j=1}^k \|b_j\|_{BMO(R^n)}.$$

Lemma 2 (see [14]) *Let* $0 < \epsilon < n$, $1 < s < n/\epsilon$ *and* $1/r = 1/s - \epsilon/n$. *Then* $\mu_\epsilon^{\vec{b}}$ *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$.

Lemma 3 (see [15]) *Let* $0 < \mu < n$, $\Omega(x,z) \in L^\infty(R^n)$ *satisfy* $L^r(S^{n-1})$ $(r \ge 1)$ *conditions, that is, there exists a constant* $0 < a_0 < 1/2$ *such that for* $|y_0| < a_0 R$,

$$\left(\int_{R<|x|<2R} \left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x,x)}{|x-y|^{n-\mu}}\right|^r dx\right)^{1/r} \le CR^{n/r-(n-\mu)}\left(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta}\,d\delta\right).$$

Theorem 1 *Let* $0 < \epsilon < n$, $n/(n+1/2-\epsilon) < q \le 1$, $1/q = 1/p - \epsilon/n$, $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $1 \le i \le m$. *Then* $\mu_\epsilon^{\vec{b}}$ *is bounded from* $H_{\vec{b}}^p(R^n)$ *to* $L^q(R^n)$.

Proof It suffices to show that there exists a constant $C > 0$ such that for every $(p, \vec{b})$ atom a,

$$\|\mu_\epsilon^{\vec{b}}(a)\|_{L^q} \le C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, 2d)$. We write

$$\int_{R^n} |\mu_\epsilon^{\vec{b}}(a)(x)|^q\,dx = \int_{|x-x_0|\le 2d} |\mu_\epsilon^{\vec{b}}(a)(x)|^q\,dx + \int_{|x-x_0|>2d} |\mu_\epsilon^{\vec{b}}(a)(x)|^q\,dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\epsilon$ and $1/r = 1/s - \epsilon/n$, by Hölder's inequality and the (L^s, L^r)-boundedness of $\mu_\epsilon^{\vec{b}}$, we get

$$I \le C\|\mu_\epsilon^{\vec{b}}(a)\|_{L^r}^q |B(x_0, 2d)|^{1-q/r} \le C\|a\|_{L^s}^q |B|^{1-q/r} \le C|B|^{-q/p+q/s+1-q/r} \le C.$$

For II, denoting $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_i = (b_i)_B$, $1 \le i \le m$, where $(b_i)_B = |B(x_0, 2d)|^{-1} \times \int_{B(x_0,2d)} b_i(x)\,dx$, by Hölder's inequality and the vanishing moment of a, we get

$$\begin{aligned} II &\le \left[\int_0^{|x-x_0|+2d} \left|\int_{|x-y|<t} \prod_{j=1}^m \big(b_j(x) - b_j(y)\big) a(y) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}}\,dy\right|^2 \frac{dt}{t^3}\right]^{1/2} \\ &\quad + \left[\int_{|x-x_0|+2d}^\infty \left|\int_{|x-y|<t} \prod_{j=1}^m \big(b_j(x) - b_j(y)\big) a(y) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}}\,dy\right|^2 \frac{dt}{t^3}\right]^{1/2} \\ &= II_1 + II_2. \end{aligned}$$

Note that $|x-y| \sim |x-x_0| \sim |x-x_0| + 2d$ for $|x-x_0| > 2d$, $y \in B$. For $1/t + 1/r = 1$, we have

$$\begin{aligned} II_1 &\le C\int_{R^n} \left(\int_{|x-y|}^{|x-x_0|+2d} \frac{dt}{t^3}\right)^{1/2} \prod_{j=1}^m |b_j(x) - b_j(y)||a(y)| \frac{|\Omega(x, x-y)|}{|x-y|^{n-1-\epsilon}}\,dy \\ &\le C\int_{R^n} \left|\frac{1}{|x-y|^2} - \frac{1}{(|x-x_0| + 2d)^2}\right|^{1/2} \prod_{j=1}^m |b_j(x) - b_j(y)||a(y)| \frac{|\Omega(x, x-y)|}{|x-y|^{n-1-\epsilon}}\,dy \\ &\le C\int_{R^n} \prod_{j=1}^m |b_j(x) - b_j(y)||a(y)| \frac{|\Omega(x, x-y)|}{|x-y|^{n-1-\varepsilon}} \frac{|y-x_0|^{1/2}}{|x-x_0|^{3/2}}\,dy \\ &\le C\int_{R^n} \prod_{j=1}^m |b_j(x) - b_j(y)||a(y)| \frac{|\Omega(x, x-y)|}{|x-x_0|^{n+1/2-\epsilon}} |y-x_0|^{1/2}\,dy \\ &\le C\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|x-x_0|^{n+1/2-\epsilon}} \left(\int_B |(\vec{b}(y) - \lambda)_{\sigma^c}||a(y)||\Omega(x, x-y)||y-x_0|^{1/2}\,dy\right) \\ &\quad \times |(\vec{b}(x) - \lambda)_\sigma| \\ &\le C\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|x-x_0|^{n+1/2-\epsilon}} \left(\int_B \big(|(\vec{b}(y) - \lambda)_{\sigma^c}||a(y)||y-x_0|^{1/2}\big)^t\,dy\right)^{1/t} \end{aligned}$$

$$\begin{aligned}
&\times \left(\int_B \left(\left| \Omega(x, x-y) \right| \right)^r dy \right)^{1/r} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right| \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{d^{n(1-1/p+1/t+1/r+1/2n)}}{|x-x_0|^{n+1/2-\epsilon}} \|\vec{b}_{\sigma^c}\|_{BMO} \cdot \|\Omega\|_{L^\infty \times L^r} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right| \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{d^{n(-1/p+1/t+1/r+1/2n)}}{|x-x_0|^{n+1/2-\epsilon}} \|\vec{b}_{\sigma^c}\|_{BMO} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right|,
\end{aligned}$$

so we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{2^{k+1} d \geq |x-x_0| > 2^k d} \left| \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{d^{n(-1/p+1/t+1/r+1/2n)}}{|x-x_0|^{n+1/2-\epsilon}} \|\vec{b}_{\sigma^c}\|_{BMO} \left(\vec{b}(x) - \lambda \right)_\sigma \right|^q dx \\
&\quad \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_{\sigma^c}\|_{BMO}^q \sum_{k=1}^{\infty} \int_{C_k} \left(\frac{d^{n(1-1/p+1/2n)}}{|x-y|^{n+1/2-\epsilon}} \right)^q \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right|^q dx \\
&\quad \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_{\sigma^c}\|_{BMO}^q \sum_{k=1}^{\infty} \left(\frac{d^{qn(1+1/2n-1/q-\delta/n)}}{|2^k d|^{q(n+1/2-\epsilon)}} \right) \left(2^k d \right)^n \\
&\qquad \times \left(\frac{1}{|2^k B|} \int_{2^k B} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right| dx \right)^q \\
&\quad \leq C \sum_{k=1}^{\infty} k^q 2^{-k(q(n+1/2-\epsilon)-n)} \|\vec{b}\|_{BMO}^q \\
&\quad \leq C \|\vec{b}\|_{BMO}^q.
\end{aligned}$$

For II_2, we can obtain

$$\begin{aligned}
II_2 &= \int_{R^n} \left(\int_{|x-x_0|+2d}^{\infty} \frac{dt}{t^3} \right)^{1/2} \left| \prod_{j=1}^{m} \left(b_j(x) - b_j(y) \right) a(y) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}} dy \right| \\
&\leq C \left| \int_{R^n} \prod_{j=1}^{m} \left(b_j(x) - b_j(y) \right) a(y) \left| \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}} - \frac{\Omega(x, x)}{|x|^{n-1-\epsilon}} \right| dy \right| \frac{1}{|x-x_0| + 2d} \\
&\leq C \int_{R^n} \prod_{j=1}^{m} \left| b_j(x) - b_j(y) \right| \left| a(y) \right| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\epsilon}} - \frac{\Omega(x, x)}{|x|^{n-\epsilon}} \right| dy \\
&\leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right| \int_B \left| \left(\vec{b}(y) - \lambda \right)_{\sigma^c} \right| \left| a(y) \right| \left| \frac{\Omega(x, x-y)}{|x-y|^{n-\epsilon}} - \frac{\Omega(x, x)}{|x|^{n-\epsilon}} \right| dy.
\end{aligned}$$

Thus, by Lemma 3, we have

$$\begin{aligned}
&\left\{ \sum_{k=1}^{\infty} \int_{2^{k+1} d \geq |x-x_0| > 2^k d} |II_2|^q dx \right\}^{1/q} \\
&\quad \leq C \left\{ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \sum_{k=1}^{\infty} \int_{C_k} \left| \left(\vec{b}(x) - \lambda \right)_\sigma \right|^q \right.
\end{aligned}$$

$$\times \left|\int_B \left|(\vec{b}(y)-\lambda)_{\sigma^c}\right|\left|a(y)\right|\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\epsilon}}-\frac{\Omega(x,x)}{|x|^{n-\epsilon}}\right|dy\right|^q dx\Bigg\}^{1/q}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}\int_B \left|(\vec{b}(y)-\lambda)_{\sigma^c}\right|\left|a(y)\right|$$

$$\times\left(\int_{C_k}\left|(\vec{b}(x)-\lambda)_{\sigma}\right|^q\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\epsilon}}-\frac{\Omega(x,x)}{|x|^{n-\epsilon}}\right|^q dx\right)^{1/q}dy$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}\left|2^{k+1}B\right|^{1/q-1/r}\|\vec{b}_\sigma\|_{BMO}$$

$$\times\int_B \left|(\vec{b}(y)-\lambda)_{\sigma^c}\right|\left|a(y)\right|\left(\int_{C_k}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\epsilon}}-\frac{\Omega(x,x)}{|x|^{n-\epsilon}}\right|^r dx\right)^{1/r}dy$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}\left|2^{k+1}d\right|^{n(1/q-1/r)}\left|2^k d\right|^{n/r-(n-\epsilon)}\|\vec{b}_\sigma\|_{BMO}\int_B\left|(\vec{b}(y)-\lambda)_{\sigma^c}\right|\left|a(y)\right|$$

$$\times\left\{\frac{|y|}{|2^k d|}+\int_{|y|/|2^{k+1}d|<\delta<|y|/|2^k d|}\frac{\omega_r(\delta)}{\delta}\,d\delta\right\}dy$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}\left|2^k d\right|^{n/q-n/r}\left|2^k d\right|^{n/r-(n-\epsilon+1/2)}\|\vec{b}_\sigma\|_{BMO}$$

$$\times\int_B\left|(\vec{b}(y)-\lambda)_{\sigma^c}\right|\left|a(y)\right|dy$$

$$\le C\sum_{k=1}^{\infty}|B|\left|2^k d\right|^{n/q-(n-\epsilon+1/2)}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}$$

$$\le C\|\vec{b}\|_{BMO}.$$

This finishes the proof of Theorem 1. □

Theorem 2 *Let* $0<\epsilon<n$, $0<p<\infty$, $1<q_1$, $q_2<\infty$, $1/q_1-1/q_2=\epsilon/n$, $n(1-1/q_1)\le\alpha<n(1-1/q_1)+1/2+\epsilon$ *and* $b_i\in BMO(R^n)$, $1\le i\le m$, $\vec{b}=(b_1,\dots,b_m)$. *Then* $\mu_\epsilon^{\vec{b}}$ *is bounded from* $H\dot{K}^{\alpha,p}_{q_1,\vec{b}}(R^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Proof Let $f\in H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$\begin{aligned}
&\left\|\mu_\epsilon^{\vec{b}}(f)(x)\right\|^p_{\dot{K}^{\alpha,p}_{q_2}}\\
&\le\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\left\|\mu_\epsilon^{\vec{b}}(a_j)(x)\chi_k\right\|_{L^{q_2}}\right)^p\\
&\quad+\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\left\|\mu_\epsilon^{\vec{b}}(a_j)(x)\chi_k\right\|_{L^{q_2}}\right)^p\\
&=J+JJ.
\end{aligned}$$

For JJ, by the (L^{q_1}, L^{q_2})-boundedness of $\mu_\epsilon^{\vec{b}}$, we get

$$\begin{aligned}
JJ &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}} \right)^p \\
&\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha} \right)^p \\
&\le \begin{cases} C\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p}), & 0 < p \le 1, \\ C\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p/2})(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2})^{p/p'}, & p > 1, \end{cases} \\
&\le \begin{cases} C\sum_{j=-\infty}^{\infty} |\lambda_j|^p (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}), & 0 < p \le 1, \\ C\sum_{j=-\infty}^{\infty} |\lambda_j|^p (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2})(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2})^{p/p'}, & p > 1, \end{cases} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C\|f\|^p_{H\dot{K}^{\alpha,p}_{q_1,\vec{b}}}.
\end{aligned}$$

For J, let $x \in B_k \setminus B_{k-1}$, $b_j^i = |B_j|^{-1} \int_{B_j} b_i(x)\,dx$, $1 \le i \le m$, $\vec{b}' = (b_j^1, \dots, b_j^m)$, we have

$$\begin{aligned}
\mu_\epsilon^{\vec{b}}(a_j)(x) &= \left(\int_0^\infty \left| \int_{|x-y|<t} \prod_{i=1}^m (b_i(x) - b_i(y)) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}} a_j(y)\,dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&= \left(\int_0^{|x|+2^j} \left| \int_{|x-y|<t} \prod_{i=1}^m (b_i(x) - b_i(y)) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}} a_j(y)\,dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_{|x|+2^j}^\infty \left| \int_{|x-y|<t} \prod_{i=1}^m (b_i(x) - b_i(y)) \frac{\Omega(x, x-y)}{|x-y|^{n-1-\epsilon}} a_j(y)\,dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&= G + H.
\end{aligned}$$

For G, noting that $y \in B_j$, $x \in B(0, 2^k) \setminus B(0, 2^{k-1})$, $j \le k-3$, we know $|x-y| \sim |x| \sim |x| + 2^j$. Then, similar to the proof of Theorem 1, we obtain

$$\begin{aligned}
G &\le C \int_{B_j} \left| \int_{|x-y|}^{|x|+2^j} \frac{dt}{t^3} \right|^{1/2} \prod_{i=1}^m |b_i(x) - b_i(y)| \frac{|\Omega(x, x-y)||a_j(y)|}{|x-y|^{n-1-\epsilon}}\,dy \\
&\le \int_{B_j} \left| \frac{1}{|x-y|^2} - \frac{1}{(|x| + 2^j)^2} \right|^{1/2} \prod_{i=1}^m |b_i(x) - b_i(y)| \frac{|\Omega(x, x-y)||a_j(y)|}{|x-y|^{n-1-\epsilon}}\,dy \\
&\le C2^{j(1/2+\epsilon)} \int_{B_j} \frac{1}{|x|^{n+1/2}} \prod_{i=1}^m |b_i(x) - b_i(y)| |\Omega(x, x-y)| |a_j(y)|\,dy \\
&\le C \frac{2^{j(1/2+\epsilon)}}{|x|^{n+1/2}} \sum_{i=0}^m \sum_{\sigma \in C_i^m} |(\vec{b}(x) - \vec{b}')_\sigma| \int_{B_j} |\Omega(x, x-y)| |a_j(y)| |(\vec{b}(y) - \vec{b})_{\sigma^c}|\,dy \\
&\le C \frac{2^{j(1/2+\epsilon)}}{|x|^{n+1/2}} \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\int_{B_j} (|(\vec{b}(y) - \vec{b})_{\sigma^c}| |a_j(y)|)^t\,dy \right)^{1/t}
\end{aligned}$$

$$\begin{aligned}
&\times \left(\int_B \left(\left| \Omega(x, x-y) \right| \right)^r dy \right)^{1/r} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \\
&\leq C \frac{2^{j(1/2+\epsilon+n(1/t+1/r-1/q_1-\alpha))}}{|x|^{n+1/2}} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma^c}\|_{BMO} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right|,
\end{aligned}$$

$$\begin{aligned}
H &\leq \left(\int_{|x|+2^j}^{\infty} \left| \int_{|x-y|<t} \prod_{i=1}^{m} \left(b_i(x) - b_i(y) \right) \left| \frac{\Omega(x-y,x)}{|x-y|^{n-1-\epsilon}} - \frac{\Omega(x,x)}{|x|^{n-1-\epsilon}} \right| a_j(y)\, dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq \int_{B_j} \frac{1}{|x|+2^j} \left| \frac{\Omega(x-y,x)}{|x-y|^{n-1-\epsilon}} - \frac{\Omega(x,x)}{|x|^{n-1-\epsilon}} \right| \prod_{i=1}^{m} \left| b_i(x) - b_i(y) \right| \left| a_j(y) \right| dy \\
&\leq C \frac{1}{|x|+2^j} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \int_{B_j} \left| \frac{\Omega(x-y,x)}{|x-y|^{n-1-\epsilon}} - \frac{\Omega(x,x)}{|x|^{n-1-\epsilon}} \right| \left| a_j(y) \right| \left| \left(\vec{b}(y) - \vec{b} \right)_{\sigma^c} \right| dy \\
&\leq C \frac{1}{|x|+2^j} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \left(\int_{B_j} \left| \frac{\Omega(x-y,x)}{|x-y|^{n-1-\epsilon}} - \frac{\Omega(x,x)}{|x|^{n-1-\epsilon}} \right|^r dy \right)^{1/r} \\
&\quad \times \left(\int_{B_j} \left(\left| a_j(y) \right| \left| \left(\vec{b}(y) - \vec{b} \right)_{\sigma^c} \right)^t \right| dy \right)^{1/t} \\
&\leq C \frac{1}{|x|+2^j} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|a\|_{L^{q_1}} |B_j|^{1/t-1/q_1} \|\vec{b}_{\sigma^c}\|_{BMO} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \left(\frac{1}{2^j} + \int_{B_j} \frac{\omega_r(\delta)}{\delta}\, d\delta \right) \\
&\leq C \frac{2^{jn(-1/q_1+1/t-\alpha)}}{|x|+2^j} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma^c}\|_{BMO} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right|,
\end{aligned}$$

thus

$$\begin{aligned}
&\left\| \mu_\epsilon^{\vec{b}}(a_j) \chi_k \right\|_{L^{q_2}} \\
&\quad \leq C 2^{j(1/2+\epsilon+n(1-1/q_1-\alpha))} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma^c}\|_{BMO} \left[\int_{C_k} \left(|x|^{-(n+1/2)} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \right)^{q_2} \right]^{1/q_2} \\
&\qquad + C 2^{jn(1-1/q_1-\alpha)} \sum_{i=0}^{m} \sum_{\sigma \in C_i^m} \|\vec{b}_{\sigma^c}\|_{BMO} \left[\int_{C_k} \left(|x|^{-1} \left| \left(\vec{b}(x) - \vec{b}' \right)_\sigma \right| \right)^{q_2} \right]^{1/q_2} \\
&\quad \leq C 2^{j(1/2+\epsilon+n(1-1/q_1)-\alpha)+kn(1/q_1-\epsilon/n-1)} \|\vec{b}\|_{BMO} \\
&\qquad + C 2^{jn(1-1/q_1-\alpha)+kn(1/q_1-\epsilon/n-1/n)} \|\vec{b}\|_{BMO}.
\end{aligned}$$

For the sake of simplicity, we denote

$$\begin{aligned}
&W(j,k) \\
&\quad = 2^{j(1/2+\epsilon+n(1-1/q_1)-\alpha)+kn(1/q_1-\epsilon/n-1)} \\
&\qquad + 2^{jn(1-1/q_1-\alpha)+kn(1/q_1-\epsilon/n-1/n)},
\end{aligned}$$

then

$$\left\| \mu_\epsilon^{\vec{b}}(a_j) \chi_k \right\|_{L^{q_2}} \leq C \|\vec{b}\|_{BMO} W(j,k),$$

we obtain

$$J \le C\|\vec{b}\|_{BMO}^{p} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| W(j,k) \right)^{p}$$

$$\le \begin{cases} C\|\vec{b}\|_{BMO}^{p} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} W(j,k)^p, & 0 < p \le 1, \\ C\|\vec{b}\|_{BMO}^{p} \sum_{j=-\infty}^{\infty} |\lambda_j|^p [\sum_{k=j+3}^{\infty} W(j,k)^{p/2}][\sum_{k=j+3}^{\infty} W(j,k)^{p'/2}]^{p/p'}, & p > 1 \end{cases}$$

$$\le C\|\vec{b}\|_{BMO}^{p} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \le C\|\vec{b}\|_{BMO}^{p} \|f\|_{H\dot{K}_{q_1,\vec{b}}^{\alpha,p}}^{p}.$$

This completes the proof of Theorem 2. □

Remark Theorem 2 also holds for nonhomogeneous Herz-type spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
In this paper, WY carried out the $(H_{\vec{b}}^{p}, L^p)$ and $(H\dot{K}_{q,\vec{b}}^{\alpha,p}, \dot{K}_q^{\alpha,p})$-type boundedness for the multilinear commutator related to the Marcinkiewicz operator with variable kernels. YH, QL, JZ, XW participated in the analysis. All authors read and revised the final manuscript.

Author details
[1]College of Electrical and Information Engineering, Hunan University, Changsha, Hunan 410082, P.R. China. [2]School of Electrical and Automation Engineering, Hefei University of Technology, Hefei, Anhui 230009, P.R. China.

Acknowledgements
This work was supported by the National Natural Science Funds of China for Distinguished Young Scholar under Grant No. 50925727, National Natural Science Foundation of China under Grant No. 60876022, The National Defense Advanced Research Project Grant No. C1120110004, Hunan Provincial Science and Technology Foundation of China under Grant No. 2010J4 and 2011JK2023, the Key Grant Project of Chinese Ministry of Education under Grant No. 313018.

Received: 30 March 2012 Accepted: 3 December 2012 Published: 27 December 2012

References
1. Chanillo, S: A note on commutators. Indiana Univ. Math. J. **31**, 7-16 (1982)
2. Coifman, R, Rocherg, R, Weiss, G: Factorization theorem for Hardy space in several variables. Ann. Math. **103**, 611-635 (1976)
3. Alvarez, J: Continuity properties for linear commutators of Calderón-Zygmund operators. Collect. Math. **49**, 17-31 (1998)
4. Garcia-Cuerva, J, Herrero, ML: A theory of Hardy spaces associated to Herz spaces. Proc. Lond. Math. Soc. **69**, 605-628 (1994)
5. Lu, S, Yang, D: The local versions of $H^p(R^n)$ spaces at the origin. Stud. Math. **116**, 147-158 (1995)
6. Lu, SZ, Yang, DC: The decomposition of the weighted Herz spaces and its applications. Sci. China Ser. A **38**, 147-158 (1995)
7. Lu, SZ, Yang, DC: The weighted Herz type Hardy spaces and its applications. Sci. China Ser. A **38**, 662-673 (1995)
8. Liu, LZ: The continuity of commutators on Triebel-Lizorkin spaces. Integral Equ. Oper. Theory **49**(1), 65-76 (2004)
9. Liu, LZ, Wu, BS: Weighted boundedness for commutator of Marcinkiewicz integral on some Hardy spaces. Southeast Asian Bull. Math. **28**, 643-650 (2004)
10. Pérez, C: Endpoint estimate for commutators of singular integral operators. J. Funct. Anal. **128**, 163-185 (1995)
11. Pérez, C, Trujillo-Gonzalez, R: Sharp weighted estimates for multilinear commutators. J. Lond. Math. Soc. **65**, 672-692 (2002)
12. Stein, EM: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)
13. Torchinsky, A, Wang, S: A note on the Marcinkiewicz integral. Colloq. Math. **60/61**, 235-243 (1990)
14. Yu, WX, Liu, LZ: Sharp weighted inequalities for multilinear commutator of Marcinkiewicz operator (to appear)
15. Yu, X, Tao, XX: The boundedness of Marcinkiewicz integration with variable kernels on Herz-type Hardy space. J. Ningbo Univ. (NSEE) **19**, 200-203 (2006)

doi:10.1186/1029-242X-2012-308
Cite this article as: Yu et al.: **Boundedness for multilinear commutator of Marcinkiewicz operator with variable kernels on Hardy and Herz-Hardy spaces.** *Journal of Inequalities and Applications* 2012 **2012**:308.