Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On strengthened form of Copson's inequality

Qian Xu*

*Correspondence:
xq32153215@yahoo.com.cn
Jiaxing Radio & TV University,
Jiaxing, Zhejiang 314000, P.R. China

Abstract

In this paper, the famous Copson inequality has been improved. We obtain some new results by a different method.

MSC: 26D15; 25D05

Keywords: inequality; Copson's inequality; analytic inequalities

1 Introduction

Suppose that $a_k > 0$, $p > 1$, then we obtain the following Hardy inequality:

$$\left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p > \sum_{n=1}^{\infty} \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p. \tag{1.1}$$

Hardy's inequality plays an important role in the field of analysis; see [1–4]. In recent decades, some generalizations and strengthening of Hardy's inequality have been obtained in [1–5]. We list some previous results as follows.

Copson's inequality [4, 5] Suppose that $a_n > 0$ $(n = 1, 2, 3 \ldots)$. If $p > 1$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k}\right)^p < p^p \sum_{n=1}^{\infty} {a_n}^p. \tag{1.2}$$

If $0 < p < 1$, then

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k}\right)^p > p^p \sum_{n=1}^{\infty} a_n^p. \tag{1.3}$$

And in [4] and [6], the authors pay much attention to the generalization of Copson's inequality.

In this paper, inequalities (1.2) and (1.3) were strengthened by using a new method.

2 Relational lemmas and definitions

In this section, some relational lemmas and definitions will be introduced.

Theorem A [7, Th. 1.1] *Suppose that* $a, b \in \mathbb{R}$, $a < b$, $c \in [a, b]$, *and* $f : [a, b]^n \to \mathbb{R}$ *has continuous partial derivatives, and*

$$D_i = \left\{(x_1, x_2, \ldots, x_{n-1}, c) \Big| \min_{1 \le k \le n-1} \{x_k\} \ge c, x_i = \min_{1 \le k \le n-1} \{x_k\} \ne c\right\}, \quad i = 1, 2, \ldots, n-1.$$

If $\frac{\partial f(x)}{\partial x_i} > 0$ *holds for all* $x \in D_i$ $(i = 1, 2, \ldots, n-1)$, *then*

$$f(y_1, y_2, \ldots, y_{n-1}, c) \geq f(c, c, \ldots, c, c),$$

where $y_i \in [c, b]$ $(i = 1, 2, \ldots, n-1)$.

Theorem B [7, Cor. 1.3] *Suppose that* $a, b \in \mathbb{R}$, $a < b$, *and* $f : [a, b]^n \to \mathbb{R}$ *has continuous partial derivatives and*

$$D_i = \Big\{(x_1, x_2, \ldots, x_n) \Big| a \leq \min_{1\leq k\leq n}\{x_k\} < x_i = \max_{1\leq k\leq n}\{x_k\} \leq b\Big\}, \quad i = 1, 2, \ldots, n.$$

If $\frac{\partial f(x)}{\partial x_i} > 0$ *holds for all* $x \in D_i$, *where* $i = 1, 2, \ldots, n$, *then*

$$f(x_1, x_2, \ldots, x_n) \geq f(x_{\min}, x_{\min}, \ldots, x_{\min}),$$

where $x_i \in [a, b]$ $(i = 1, 2, \ldots, n)$, $x_{\min} = \min_{1\leq k\leq n}\{x_k\}$.

Definition 1 [1] Let $G \subseteq \mathbb{R}^n$ be a convex set, $\phi : H \to \mathbb{R}$ be a continuous function. If

$$\phi\big(\alpha x + (1-\alpha)y\big) \leq (\geq)\alpha\phi(x) + (1-\alpha)\phi(y)$$

holds for all $x, y \in G$, $\alpha \in [0, 1]$, then the function φ is convex (concave).

Lemma 1 (Hermite-Hadamard's inequality) *Let* $\phi : [a, b] \to \mathbb{R}$ *be a convex* (*concave*) *function. Then*

$$\phi\left(\frac{a+b}{2}\right) \leq (\geq)\frac{1}{b-a}\int_a^b \phi(x)\,dx \leq (\geq)\frac{\phi(a)+\phi(b)}{2}, \tag{2.1}$$

and the equality holds if and only if ϕ *is linear.*

Lemma 2 *Suppose that* $p > 0$.

(1) *If* $p \geq 1$, *or* $0 < p \leq \frac{1}{2}$, *then*

$$2p^p \geq 2^p; \tag{2.2}$$

(2) *If* $\frac{1}{2} < p < 1$, *then*

$$2^p > 2p^p; \tag{2.3}$$

(3) *If* $\frac{1}{2} < p < 1$, *then*

$$p^p + (p-1)2^p > 0. \tag{2.4}$$

Proof Set $f_1 : p \in (0, +\infty) \to p\ln p + \ln 2 - p\ln 2$. Then we have

$$f_1{}'(p) = \ln p + 1 - \ln 2 = \ln\left(\frac{ep}{2}\right).$$

Obviously, f_1 is monotone decreasing for $p \in (0, \frac{2}{e})$, f_1 is monotone increasing for $p \in (\frac{2}{e}, +\infty)$, and $f_1(1) = 0$, $f_1(\frac{1}{2}) = -\frac{1}{2}\ln 2 + \ln 2 - \frac{1}{2}\ln 2 = 0$, then (2.2) and (2.3) hold. Let

$$f_2 : p \in (0,1) \to p\ln p - p\ln 2 - \ln(1-p).$$

We get

$$f_2'(p) = \frac{1}{1-p}\bigl[(1-p)\ln p + (1-p)(1-\ln 2) + 1\bigr] \stackrel{\text{Def.}}{=} \frac{1}{1-p}h(p),$$
$$h'(p) = -\ln p + \frac{1}{p} - 2 + \ln 2$$

and

$$h''(p) = -\frac{1}{p} - \frac{1}{p^2} < 0.$$

Then h is concave for $p \in (0,1)$. Because $h(\frac{1}{2}) > 0$ and $\lim_{p\to 1^-} h(p) > 0$, then $h(p) > 0$ and $f_2'(p) > 0$ hold for $p \in [\frac{1}{2}, 1)$. From $f_2(\frac{1}{2}) = 0$, we have $f_2(p) > 0$ for $p \in (\frac{1}{2}, 1)$. Inequality (2.4) is proved. □

Lemma 3

(1) *If $p > 1$, then the equation*

$$p^p(1-x)\left(\frac{1}{2} - x\right)^{p-1} = 1 \tag{2.5}$$

has only a positive root for $x \in (0, \frac{1}{2})$.

(2) *If $\frac{1}{2} < p < 1$, then the equation*

$$p^p(1+x) = \left(\frac{1}{2} + x\right)^{1-p} \tag{2.6}$$

has only a positive root for $x \in (0, \frac{1}{2})$.

Proof

(1) Let $g_1 : x \in [0, \frac{1}{2}] \to p^p(1-x)(\frac{1}{2} - x)^{p-1} - 1$. Then g_1 is monotone decreasing. According to inequality (2.2), we have

$$g_1(0) = p^p\left(\frac{1}{2}\right)^{p-1} - 1 = \left(\frac{1}{2}\right)^{p-1}\bigl[p^p - 2^{p-1}\bigr] > 0$$

and $g_1(\frac{1}{2}) = -1$. So, equation (2.5) has only a positive root for $x \in (0, \frac{1}{2})$.

(2) Let $g_2 : x \in [0, \frac{1}{2}] \to p^p(1+x) - (\frac{1}{2} + x)^{1-p}$. Thus,

$$g_2'(x) = p^p - (1-p)\left(\frac{1}{2} + x\right)^{-p} > \left(\frac{1}{2} + x\right)^{-p}\left[\frac{1}{2^p} \cdot p^p - (1-p)\right].$$

By inequality (2.4), g_2 is monotone increasing. According to inequality (2.3), we get

$$\lim_{x\to 0^+} g_2(x) = p^p - \frac{1}{2^{1-p}} < 0$$

and

$$\lim_{x\to (1/2)^+} g_2(x) = \frac{3}{2}p^p - 1 \geq \frac{3}{2}\cdot\left(\frac{1}{2}\right)^{\frac{1}{2}} - 1 > 0.$$

Therefore, equation (2.6) has only a positive root for $x \in (0, \frac{1}{2})$. □

Lemma 4 *If $p > 1$, $m > 0$, $m \in \mathrm{N}$ and $c \in (0, \frac{1}{2})$ is the only one positive root of equation* (2.5), *then*

$$\sum_{n=1}^{m}\left[\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+\frac{1}{p}}}\right]^{p-1} < p^p(m-c)^{\frac{1}{p}} \tag{2.7}$$

and

$$\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right]^{p} < p^p\sum_{n=1}^{m}\frac{n^p}{(n-c)^{p+1}}. \tag{2.8}$$

Proof (1) If $m = 1$, by Lemma 1, we get

$$\begin{aligned}\sum_{n=1}^{m}\left[\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right]^{p-1} &= \left[\sum_{k=1}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right]^{p-1} \\ &< \left[\int_{\frac{1}{2}}^{\infty}\frac{1}{(x-c)^{1+1/p}}\,dx\right]^{p-1} = p^{p-1}\left(\frac{1}{2}-c\right)^{-(p-1)/p}.\end{aligned}$$

If $m \geq 2$, by Lemma 1, we get

$$\begin{aligned}\sum_{n=1}^{m}\left[\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right]^{p-1} &< \sum_{n=1}^{m}\left[\int_{n-\frac{1}{2}}^{\infty}\frac{1}{(x-c)^{1+1/p}}\right]^{p-1} \\ &= p^{p-1}\sum_{n=1}^{m}\left(n-\frac{1}{2}-c\right)^{-(p-1)/p} \\ &= p^{p-1}\left[\left(\frac{1}{2}-c\right)^{-(p-1)/p} + \sum_{n=2}^{m}\left(n-\frac{1}{2}-c\right)^{-(p-1)/p}\right] \\ &< p^{p-1}\left[\left(\frac{1}{2}-c\right)^{-(p-1)/p} + \int_{\frac{3}{2}}^{m+\frac{1}{2}}\left(x-\frac{1}{2}-c\right)^{-(p-1)/p}dx\right] \\ &= p^{p-1}\left[\left(\frac{1}{2}-c\right)^{-(p-1)/p} + p(m-c)^{1/p} - p(1-c)^{1/p}\right].\end{aligned}$$

So,

$$\sum_{n=1}^{m}\left[\sum_{k=n}^{\infty}\frac{1}{(k-c)^{1+1/p}}\right]^{p-1} < p^{p-1}\left[\left(\frac{1}{2}-c\right)^{-(p-1)/p} + p(m-c)^{1/p} - p(1-c)^{1/p}\right] \tag{2.9}$$

holds for every $m > 0$ and $m \in \mathrm{N}$. Since inequalities (2.9), (2.10) and

$$\left(\frac{1}{2}-c\right)^{-(p-1)/p} = p(1-c)^{1/p}, \tag{2.10}$$

inequality (2.7) holds.

(2)

$$\begin{aligned}
&\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right]^{p} \\
&\quad = p\sum_{n=1}^{m}\int_0^{\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}} x^{p-1}\,dx \\
&\quad = p\sum_{n=1}^{m}\left[\int_0^{\frac{1}{(m-c)^{1+1/p}}} x^{p-1}\,dx + \int_{\frac{1}{(m-c)^{1+1/p}}}^{\frac{1}{(m-c)^{1+1/p}}+\frac{1}{(m-c-1)^{1+1/p}}} x^{p-1}\,dx + \cdots \right.\\
&\qquad \left. + \int_{\sum_{k=n+1}^{m}\frac{1}{(k-c)^{1+1/p}}}^{\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}} x^{p-1}\,dx\right] \\
&\quad < p\sum_{n=1}^{m}\left[\frac{1}{(m-c)^{1+1/p}}\cdot\left(\frac{1}{(m-c)^{1+1/p}}\right)^{p-1} + \frac{1}{(m-c-1)^{1+1/p}}\right.\\
&\qquad \left.\cdot\left(\frac{1}{(m-c)^{1+1/p}}+\frac{1}{(m-c-1)^{1+1/p}}\right)^{p-1} + \cdots + \frac{1}{(n-c)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}\right] \\
&\quad = p\left[\frac{m}{(m-c)^{1+1/p}}\cdot\left(\frac{1}{(m-c)^{1+1/p}}\right)^{p-1} + \frac{m-1}{(m-c-1)^{1+1/p}}\left(\frac{1}{(m-c)^{1+1/p}}\right.\right.\\
&\qquad \left.\left. + \frac{1}{(m-c-1)^{1+1/p}}\right)^{p-1} + \cdots + \frac{1}{(1-c)^{1+1/p}}\left(\sum_{k=1}^{m}\frac{1}{k^{1+1/p}}\right)^{p-1}\right] \\
&\quad = p\sum_{n=1}^{m}\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\left(\sum_{i=k}^{m}\frac{1}{(i-c)^{1+1/p}}\right)^{p-1} = p\sum_{n=1}^{m}\frac{n}{(n-c)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p-1}.
\end{aligned}$$

Let $q > 1$ and $1/p + 1/q = 1$. Using Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right]^{p} \\
&\quad < p\left[\sum_{n=1}^{m}\left(\frac{n}{(n-c)^{1+1/p}}\right)^{p}\right]^{1/p}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{(p-1)q}\right]^{1/q} \\
&\quad = p\left[\sum_{n=1}^{m}\frac{n^p}{(n-c)^{p+1}}\right]^{1/p}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right]^{1/q}.
\end{aligned}$$

Since

$$\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right]^{1/p} < p\left[\sum_{n=1}^{m}\frac{n^p}{(n-c)^{p+1}}\right]^{1/p},$$

inequality (2.8) holds. □

Lemma 5 *If* $\frac{1}{2} < p < 1$, $m > 0$, $m \in \mathrm{N}$ *and* $d \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.6), *then*

$$\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right]^{p} > p^p\sum_{n=1}^{m}\frac{n^p}{(n+d)^{p+1}}. \tag{2.11}$$

Proof

$$\begin{aligned}
&\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right]^{p} \\
&\quad = p\sum_{n=1}^{m}\int_0^{\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}} x^{p-1}\,dx \\
&\quad = p\sum_{n=1}^{m}\left[\int_0^{\frac{1}{(m+d)^{1+1/p}}} x^{p-1}\,dx + \int_{\frac{1}{(m+d)^{1+1/p}}}^{\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}} x^{p-1}\,dx + \int_{\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}}^{\sum_{k=m-2}^{m}\frac{1}{(k+d)^{1+1/p}}} x^{p-1}\,dx + \cdots\right. \\
&\qquad \left. + \int_{\sum_{k=n+1}^{m}\frac{1}{(k+d)^{1+1/p}}}^{\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}} x^{p-1}\,dx\right] \\
&\quad > p\sum_{n=1}^{m}\left[\frac{1}{(m+d)^{1+1/p}}\left(\frac{1}{(m+d)^{1+1/p}}\right)^{-(1-p)}\right. \\
&\qquad + \frac{1}{(m-1+d)^{1+1/p}}\left(\sum_{k=m-1}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)} + \cdots \\
&\qquad \left. + \frac{1}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right] \\
&\quad = p\sum_{n=1}^{m}\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}.
\end{aligned} \tag{2.12}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{m}\left[\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)^{p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right] \\
&\quad < \left[\sum_{n=1}^{m}\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)\right]^{p}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right]^{1-p}.
\end{aligned}$$

And by using inequality (2.12), we obtain

$$\begin{aligned}
&\sum_{n=1}^{m}\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\cdot\left[\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}\right]^{\frac{1-p}{p}}\\
&\quad>\left\{\sum_{n=1}^{m}\left[\left(\frac{n}{(n+d)^{1+1/p}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)}\right)^{p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right]\right\}^{1/p}\\
&\quad=\left\{\sum_{n=1}^{m}\left[\frac{n^{p}}{(n+d)^{p+1}}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{-(1-p)p}\cdot\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p(1-p)}\right]\right\}^{1/p}\\
&\quad=\left(\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}\right)^{1/p}.
\end{aligned}\tag{2.13}$$

From inequality (2.12) and inequality (2.13), we get

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right)^{p}>p\frac{[\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}]^{1/p}}{\{\sum_{n=1}^{m}[\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}]^{p}\}^{\frac{1-p}{p}}}$$

and

$$\left\{\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{1}{(k+d)^{1+1/p}}\right]^{p}\right\}^{1/p}>p\left[\sum_{n=1}^{m}\frac{n^{p}}{(n+d)^{p+1}}\right]^{1/p}.$$

Then inequality (2.11) holds. □

3 Strengthened Copson's inequality ($p > 1$)

Theorem 1 *Assume that* $p > 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0$ $(n = 1,2,\ldots,m)$, $c \in (0,\frac{1}{2})$ *is the only one positive root of equation* (2.5) *and* $B_m = \min_{1\le n\le m}\{(n-c)^{1/p}a_n\}$. *Then*

$$\begin{aligned}
&p^{p}\sum_{n=1}^{m}a_n^{p}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k-c}\right)^{p}\\
&\quad\ge B_m^{p}\left[p^{p}\sum_{n=1}^{m}\frac{1}{n-c}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right].
\end{aligned}\tag{3.1}$$

Proof Set $b_n = (n-c)^{1/p}a_n$ $(n = 1,2,\ldots,m)$. Then inequality (3.1) is equivalent to

$$\begin{aligned}
&p^{p}\sum_{n=1}^{m}\frac{b_n^{p}}{n-c}-\sum_{n=1}^{m}\left[\sum_{k=n}^{m}\frac{b_k}{(k-c)^{1+1/p}}\right]^{p}\\
&\quad\ge B_m^{p}\left[p^{p}\sum_{n=1}^{m}\frac{1}{n-c}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{1}{(k-c)^{1+1/p}}\right)^{p}\right],
\end{aligned}\tag{3.2}$$

where $B_m = \min_{1\le n\le N}\{b_n\}$. Let

$$f:b=(b_1,b_2,\ldots,b_m)\in[0,+\infty)^{m}\to p^{p}\sum_{n=1}^{m}\frac{b_n^{p}}{n-c}-\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{b_k}{(k-c)^{1+1/p}}\right)^{p}$$

and

$$D_i = \left\{(b_1, b_2, \ldots, b_n) | 0 \le \min_{1\le n\le m}\{b_n\} < b_i = \max_{1\le n\le m}\{b_n\}\right\}.$$

If $(b_1, b_2, \ldots, b_n) \in D_i$, then

$$\begin{aligned}\frac{\partial f}{\partial b_i} &= p^p \frac{pb_i^{p-1}}{i-c} - \frac{p}{(i-c)^{1+1/p}} \sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{b_k}{(k-c)^{1+1/p}}\right)^{p-1} \\ &> \frac{pb_i^{p-1}}{(i-c)^{1+\frac{1}{p}}} \left[p^p (i-c)^{1/p} - \sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{1}{(k-c)^{1+1/p}}\right)^{p-1}\right] \\ &> \frac{pb_i^{p-1}}{(i-c)^{1+\frac{1}{p}}} \left[p^p (i-c)^{1/p} - \sum_{n=1}^{i} \left(\sum_{k=n}^{\infty} \frac{1}{(k-c)^{1+1/p}}\right)^{p-1}\right].\end{aligned}$$

By inequality (2.7), we know $\frac{\partial f}{\partial b_i} > 0$. By Theorem B, inequality (3.2) holds, the proof is completed. □

Corollary 1 *If* $p > 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0$ $(n = 1, 2, \ldots, m)$, $c \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.5), *and* $B_m = \min_{1\le n\le m}\{(n-c)^{1/p} a_n\}$, *then*

$$p^p \sum_{n=1}^{m} a_n^p - \sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{a_k}{k-c}\right)^p > -p^p B_m^p \sum_{n=1}^{m} \frac{n^p - (n-c)^p}{(n-c)^{p+1}}. \tag{3.3}$$

Proof By (3.1) and (2.8), we can obtain

$$\begin{aligned}p^p \sum_{n=1}^{m} a_n^p - \sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{a_k}{k-c}\right)^p &> p^p B_m^p \left[\sum_{n=1}^{m} \frac{1}{n-c} - \sum_{n=1}^{m} \frac{n^p}{(n-c)^{p+1}}\right] \\ &= -p^p B_m^p \sum_{n=1}^{m} \frac{n^p - (n-c)^p}{(n-c)^{p+1}}.\end{aligned}$$

□

Corollary 2 *If* $p > 1$, $a_n > 0$ $(n = 1, 2, \ldots)$, $\sum_{n=1}^{\infty} a_n^p < +\infty$ *and* $c \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.5), *then*

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k-c}\right)^p \le p^p \sum_{n=1}^{\infty} a_n^p. \tag{3.4}$$

Proof Because of $\sum_{n=1}^{\infty} a_n^p < +\infty$, the infimum of $\{(n-c)^{1/p} a_n\}_{n=1}^{\infty}$ is zero. Then there exists a sequence $\{m_i | m_i \in \mathrm{N}\}$ such that $\{(m_i - c)^{1/p} a_{m_i}\}_{i=1}^{\infty}$ decrease to zero. Since (3.3), we have

$$p^p \sum_{n=1}^{m_i} a_n^p - \sum_{n=1}^{m_i} \left(\sum_{k=n}^{m_i} \frac{a_k}{k-c}\right)^p > -p^p \left[(m_i - c)^{1/p} a_{m_i}\right]^p \sum_{n=1}^{m_i} \frac{n^p - (n-c)^{p+1}}{(n-c)^{p+1}}. \tag{3.5}$$

Let $i \to +\infty$ in inequality (3.5), we have $m_i \to +\infty$ and

$$\lim_{i\to+\infty} \left[(m_i - c)^{1/p} a_{m_i}\right]^p \sum_{n=1}^{m_i} \frac{n^p - (n-c)^{p+1}}{(n-c)^{p+1}} = 0.$$

Then by (3.5), we can obtain

$$p^p \sum_{n=1}^{m_i} a_n^p - \sum_{n=1}^{m_i} \left(\sum_{k=n}^{m_i} \frac{a_k}{k-c} \right)^p \geq 0.$$

Therefore, inequality (3.4) holds. □

Remark Obviously, inequality (3.4) strengthens inequality (1.2).

4 Strengthened Copson's inequality (1/2 < p < 1)

Theorem 2 *If* $\frac{1}{2} < p < 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0$ $(n = 1, 2, \ldots, m)$, $d \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.6) *and* $B_m = \min_{1 \leq n \leq m}\{(n+d)^{1/p} a_n\}$. *Then*

$$\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{a_k}{k+d} \right)^p - p^p \sum_{n=1}^{m} a_n^p \geq B_m^p \left[\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{1}{(k+d)^{1+1/p}} \right)^p - p^p \sum_{n=1}^{m} \frac{1}{n+d} \right]. \tag{4.1}$$

Proof Let $b_n = (n+d)^{1/p} a_n$ $(n = 1, 2, \ldots, m)$. Then inequality (4.1) is equivalent to

$$\sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p}} \right)^p - p^p \sum_{n=1}^{m} \frac{b_n^p}{n+d} \geq B_m^p \left[p^p \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{1}{k^{1+1/p}} \right)^p \right], \tag{4.2}$$

where $B_m = \min_{1 \leq n \leq m}\{b_n\}$. Set

$$f : b \in (0, +\infty)^m \to \sum_{n=1}^{m} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p}} \right)^p - p^p \sum_{n=1}^{m} \frac{b_n^p}{n+d}$$

and $D_i = \{(b_1, b_2, \ldots, b_n) | 0 \leq \min_{1 \leq n \leq m}\{b_n\} < b_i = \max_{1 \leq n \leq m}\{b_n\}\}$. If $(b_1, b_2, \ldots, b_n) \in D_i$, then

$$\begin{aligned}
\frac{\partial f}{\partial b_i} &= \frac{p}{(i+d)^{1+1/p}} \sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p}} \right)^{p-1} - p^{p+1} \frac{b_i^{p-1}}{i+d} \\
&= \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{b_k}{(k+d)^{1+1/p} b_i} \right)^{-(1-p)} - p^p (i+d)^{1/p} \right] \\
&> \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{m} \frac{1}{(k+d)^{1+1/p}} \right)^{-(1-p)} - p^p (i+d)^{1/p} \right] \\
&> \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\sum_{k=n}^{\infty} \frac{1}{(k+d)^{1+1/p}} \right)^{-(1-p)} - p^p (i+d)^{1/p} \right].
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
\frac{\partial f}{\partial b_i} &> \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[\sum_{n=1}^{i} \left(\int_{n-\frac{1}{2}}^{\infty} \frac{1}{(x+d)^{1+1/p}} \, dx \right)^{-(1-p)} - p^p (i+d)^{1/p} \right] \\
&= \frac{p b_i^{p-1}}{(i+d)^{1+1/p}} \left[p^{-(1-p)} \sum_{n=1}^{i} \left(n - \frac{1}{2} + d \right)^{(1-p)/p} - p^p (i+d)^{1/p} \right].
\end{aligned}$$

As $i = 1$, by the definition of d, we have

$$\frac{\partial f}{\partial b_1} > \frac{pb_1^{p-1}}{(1+d)^{1+1/p}}\left[p^{-(1-p)}\left(\frac{1}{2}+d\right)^{(1-p)/p} - p^p(1+d)^{1/p}\right] = 0.$$

As $2 \le i \le m$, because $\frac{1}{2} < p < 1$, $0 < \frac{p-1}{p} \le 1$ and $g : x \in (0, +\infty) \to x^{(1-p)/p}$ is concave, we have

$$\begin{aligned}
\frac{\partial f}{\partial b_i} &> \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p} + \sum_{n=2}^{i}\left(n-\frac{1}{2}+d\right)^{(1-p)/p}\right) - p^p(i+d)^{1/p}\right] \\
&> \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p}\right.\right. \\
&\quad \left.\left. + \int_{\frac{3}{2}}^{i+\frac{1}{2}}\left(x-\frac{1}{2}+d\right)^{(1-p)/p}dx\right) - p^p(i+d)^{1/p}\right] \\
&= \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\left(\left(\frac{1}{2}+d\right)^{(1-p)/p} + p(i+d)^{1/p} - p(1+d)^{1/p}\right) - p^p(i+d)^{1/p}\right] \\
&= \frac{pb_i^{p-1}}{(i+d)^{1+1/p}}\left[p^{-(1-p)}\cdot p(i+d)^{1/p} - p^p(i+d)^{1/p}\right] = 0.
\end{aligned}$$

Thus, for every D_i, $\frac{\partial f}{\partial b_i} > 0$. By Theorem B, inequality (4.2) holds. □

Corollary 3 *If* $\frac{1}{2} < p < 1$, $m > 0$, $m \in \mathrm{N}$, $a_n > 0$ $(n = 1, 2, \ldots, m)$, $d \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.6) *and* $B_m = \min_{1\le n\le m}\{(n+d)^{1/p}a_n\}$. *Then*

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p - p^p\sum_{n=1}^{m}a_n^p \ge p^pB_m^p\sum_{n=1}^{m}\frac{n^p-(n+d)^p}{(n+d)^{p+1}}. \tag{4.3}$$

Proof From Theorem 2 and Lemma 5, we have

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p - p^p\sum_{n=1}^{m}a_n^p \ge B_m^p\left[p^p\sum_{n=1}^{m}\frac{n^p}{(n+d)^{p+1}} - p^p\sum_{n=1}^{m}\frac{1}{n+d}\right].$$

Then inequality (4.3) holds. □

Corollary 4 *If* $\frac{1}{2} < p < 1$, $a_n > 0$ $(n = 1, 2, \ldots)$, $d \in (0, \frac{1}{2})$ *is the only one positive root of equation* (2.6) *and series* $\sum_{n=1}^{\infty}(\sum_{k=n}^{\infty}\frac{a_k}{k+d})^p < +\infty$. *Then*

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k+d}\right)^p \ge p^p\sum_{n=1}^{\infty}a_n^p. \tag{4.4}$$

Proof According to inequality (4.3), we obtain

$$\sum_{n=1}^{m}\left(\sum_{k=n}^{m}\frac{a_k}{k+d}\right)^p + p^pB_m^p\sum_{n=1}^{m}\frac{(n+d)^p-n^p}{(n+d)^{p+1}} \ge p^p\sum_{n=1}^{m}a_n^p.$$

The following proof is the same as the relevant proof for Corollary 2, omitted here. □

Remark For $\frac{1}{2} < p < 1$, there is no doubt that inequality (4.4) strengthens inequality (1.3).

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The research is supported by the Nature Science Foundation of China (No. 110771069) and the NS Foundation of the Educational Committee of Zhejiang Province under Grant Y201223283.

Received: 11 July 2012 Accepted: 28 November 2012 Published: 19 December 2012

References
1. Hardy, GH, Littlewood, JE, Poly, AG: Inequalities. Cambridge University Press, Cambridge (1952)
2. Yang, B-C: Arithmetic Operators and Hilbert Type Inequality. Science Press, Beijing (2009) (in Chinese)
3. Kuang, J-C: Inequality of Regular. Shandong Science and Technology Press, Jinan (2004) (in Chinese)
4. Pachpatte, BG: Mathematical Inequalities, pp. 118-127. Elsevier, Amsterdam (2005)
5. Bullen, PS: A Dictionary of Inequalities, p. 65. Chapman & Hall, London (1998)
6. Gao, P: On weighted remainder form of Hadry-type inequalities. RGMIA **12**(3), 17-32 (2009) (http://www.staff.vu.edu.au/RGMIA/v12n3.asp)
7. Zhang, Xiao-Ming, Chu, Yu-Ming: A new method to study analytic inequalities. J. Inequal. Appl. **2010**, Article ID 698012 (2010)

doi:10.1186/1029-242X-2012-305
Cite this article as: Xu: On strengthened form of Copson's inequality. *Journal of Inequalities and Applications* 2012 **2012**:305.