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On a half-discrete inequality with a generalized homogeneous kernel

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Abstract

By introducing a real number homogeneous kernel and estimating the weight function through the real function techniques, a half-discrete inequality with a best constant factor is established. In addition, the operator expressions, equivalent forms, reverse inequalities and some particular cases are given.

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1. Introduction

One hundred years ago, Hilbert proved the following classic inequality [1]

$$\sum_n \sum_m \frac{a_m b_n}{m+n} \le \pi \left(\sum_n a_n^2 \right)^{1/2} \left(\sum_n b_n^2 \right)^{1/2}. \tag{1.1}$$

During the past century, ever since the advent the inequality (1.1), numerous related results have been obtained. The inequality (1.1) may be classified into several types (discrete and integral etc.), being the following integral form:

If f, g are real functions such that $0 < \int_0^\infty f^2(x)\,\mathrm{d}x < \infty, 0 < \int_0^\infty g^2(x)\,\mathrm{d}x < \infty$, then we have [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi \left\{ \int_0^\infty f^2(x)\mathrm{d}x \int_0^\infty g^2(x)\mathrm{d}x \right\}^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor π is the best possible. Inequality (1.2) had been generalized by Hardy-Riesz in 1925 as [1]:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \ge 0$ such that $0 < \int_0^\infty f^p(x)\mathrm{d}x, \int_0^\infty g^q(x)\mathrm{d}x < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{ \int_0^\infty f^p(x)\mathrm{d}x \right\}^{1/p} \left\{ \int_0^\infty g^q(x)\mathrm{d}x \right\}^{1/q}, \tag{1.3}$$

where the constant factor $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible. Inequality (1.3) is named as Hardy-Hilbert's integral inequality, which is of great importance in analysis and its applications [2-4]. Its generalization can be seen in [5-11].

Until now, we only studied the related inequalities with pure discrete or integral inequalities, but half-discrete inequality is very rare in the literature [12-14]. Now we attempt investigation for it, lots of related results will appear in the coming future.

The main purpose of this article is to establish a half-discrete inequality with the mixed homogeneous kernel of real number degree. For example: If $0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x < \infty, 0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$, then

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{f(x)}{(\max\{x,n\})^{\alpha}} \mathrm{d}x < \frac{\alpha}{\lambda_1\lambda_2} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}, \tag{1.4}$$

where $\alpha = \lambda_1 + \lambda_2$, $0 <\lambda_1 <\alpha$ and the constant factor $\frac{\alpha}{\lambda_1\lambda_2}$ is the best possible. Meanwhile, the extended inequality, operator expressions, reverse inequality, and equivalent forms are given. We hope this work will expand our understanding of inequality and the scope of the study.

2. Lemmas

LEMMA 2.1. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $\lambda_1 + \lambda_2 = \alpha - \beta$, $-\beta <\lambda_1 <\alpha$, $\lambda_2 \le 1 - \beta$, $k_{\lambda_1} := \frac{1}{\alpha - \lambda_1} + \frac{1}{\beta + \lambda_1}$ *define the weight function and the weight coefficient as follows*

$$\omega(n) := n^{\lambda_2} \int_0^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \cdot x^{\lambda_1 - 1} dx, n \in \mathbb{N}_+, \tag{2.1}$$

$$\varpi(x) := x^{\lambda_1} \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \cdot n^{\lambda_2 - 1}, x \in (0,\infty), \tag{2.2}$$

then

$$0 < k_{\lambda_1}(1 - \theta_\lambda(x)) < \varpi(x) < \omega(n) = k_{\lambda_1}, \tag{2.3}$$

where

$$\theta_\lambda(x) := \frac{1}{k_{\lambda_1}} \int_0^{1/x} \frac{(\min\{1,t\})^{\beta}}{(\max\{1,t\})^{\alpha}} \cdot t^{\lambda_2 - 1}\mathrm{d}t = \begin{cases} \frac{1}{k_{\lambda_1}}\left(\frac{1}{\beta+\lambda_2} + \frac{1 - x^{\alpha-\lambda_2}}{\alpha - \lambda_2}\right), x \in (0,1), \\ \frac{x^{-(\beta+\lambda_2)}}{(\beta+\lambda_2)k_{\lambda_1}} = O\left(\frac{1}{x^{\beta+\lambda_2}}\right), \quad x \in [0,\infty). \end{cases}$$

Proof. For fixed n, let $t = \frac{x}{n}$, substituting into $\omega(n)$ gives

$$\begin{aligned}\omega(n) &= \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha}\cdot t^{\lambda_1-1}\mathrm{d}t \\ &= \int_0^1 t^{\beta+\lambda_1-1}\mathrm{d}t + \int_1^\infty t^{-\alpha+\lambda_1-1}\mathrm{d}t \\ &= \frac{1}{\alpha-\lambda_1}+\frac{1}{\beta+\lambda_1} = k_{\lambda_1}.\end{aligned}$$

In view of $\lambda_2 \le 1 - \beta$, $\alpha - \lambda_2 = \beta + \lambda_1 > 0$, for fixed $x > 0$, the function

$$\frac{(\min\{x,y\})^\beta}{(\max\{x,y\})^\alpha}\cdot y^{\lambda_2-1} = \begin{cases} x^{-\alpha}y^{\beta+\lambda_2-1}, & 0<y<x, \\ x^\beta y^{-\alpha+\lambda_2-1}, & y \ge x\end{cases}$$

is monotonically decreasing with respect to y, then

$$\begin{aligned}\varpi(x) &< x^{\lambda_1}\int_0^\infty \frac{(\min\{x,y\})^\beta}{(\max\{x,y\})^\alpha}\cdot y^{\lambda_2-1}\mathrm{d}y \quad (t = y/x) \\ &= \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha}\cdot t^{\lambda_2-1}\mathrm{d}t = \int_0^1 t^{\beta+\lambda_2-1}\mathrm{d}t + \int_1^\infty t^{-\alpha+\lambda_2-1}\mathrm{d}t \\ &= \frac{1}{\beta+\lambda_2}+\frac{1}{\alpha-\lambda_2} = \frac{1}{\alpha-\lambda_1}+\frac{1}{\beta+\lambda_1} = k_{\lambda_1}.\end{aligned}$$

$$\begin{aligned}\varpi(x) &> x^{\lambda_1}\int_0^\infty \frac{(\min\{x,y\})^\beta}{(\max\{x,y\})^\alpha}\cdot y^{\lambda_2-1}\mathrm{d}y \quad (t = y/x) \\ &= \int_{1/x}^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha}\cdot t^{\lambda_2-1}\mathrm{d}t \\ &= k_{\lambda_1} - \int_0^{1/x} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha}\cdot t^{\lambda_2-1}\mathrm{d}t = k_{\lambda_1}(1-\theta_\lambda(x)) > 0.\end{aligned}$$

where $\theta_\lambda(x) = \frac{1}{k_{\lambda_1}}\int_0^{1/x} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha}\cdot t^{\lambda_2-1}\mathrm{d}t$. If $x \in (0,1)$, then

$$\theta_\lambda(x) = \frac{1}{k_{\lambda_1}}\left(\int_0^1 t^{\beta+\lambda_2-1}\mathrm{d}t + \int_1^{1/x} t^{-\alpha+\lambda_2-1}\mathrm{d}t\right) = \frac{1}{k_{\lambda_1}}\left[\frac{1}{\beta+\lambda_2}+\frac{1-x^{\alpha-\lambda_2}}{\alpha-\lambda_2}\right];$$

if $x \in [1,\infty)$, then

$$\theta_\lambda(x) = \frac{1}{k_{\lambda_1}}\int_0^{1/x} t^{\beta+\lambda_2-1}\mathrm{d}t = \frac{x^{-(\beta+\lambda_2)}}{(\beta+\lambda_2)k_{\lambda_1}} = O\left(\frac{1}{x^{\beta+\lambda_2}}\right).$$

Thus (2.3) is valid.

In what follows, α, β will be real numbers such that $\lambda_1 + \lambda_2 = \alpha - \beta$, $-\beta < \lambda_1 < \alpha$, $\lambda_2 \leq 1 - \beta$.

LEMMA 2.2. *Suppose that* $p > 0 (p \neq 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $f(x)$ *is a non-negative measurable function in* $(0, \infty)$, *then*

(a) if $p > 1$, *then the following two inequalities hold:*

$$J := \sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left[\int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} f(x) \mathrm{d}x \right]^p \leq k_{\lambda_1}^p \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x, \tag{2.4}$$

$$L := \int_0^{\infty} x^{q\lambda_1 - 1} \left[\sum_{n=1}^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} a_n \right]^q \mathrm{d}x \leq k_{\lambda_1}^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q; \tag{2.5}$$

(b) if $0 < p < 1$, *then we have*

$$J \geq k_{\lambda_1}^p \int_0^{\infty} (1 - \theta_{\lambda}(x)) x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x, \tag{2.6}$$

$$\widetilde{L} := \int_0^{\infty} \frac{x^{q\lambda_1 - 1}}{[1 - \theta_{\lambda}(x)]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} a_n \right]^q \mathrm{d}x \leq k_{\lambda_1}^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \tag{2.7}$$

Proof. (a) Using Hölder's inequality with weight [15] and (2.3) gives

$$\begin{aligned}
&\left[\int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} f(x) \mathrm{d}x \right]^p \\
&= \left\{ \int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \left[\frac{x^{(1-\lambda_1)/q}}{n^{(1-\lambda_2)/p}} f(x) \right] \left[\frac{n^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}} \right] \mathrm{d}x \right\}^p \\
&\leq \int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x) \mathrm{d}x \left[\int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} \mathrm{d}x \right]^{p-1} \\
&= \int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x) \mathrm{d}x \left[n^{q(1-\lambda_2)-1} \omega(n) \right]^{p-1} \\
&= n^{-p\lambda_2 + 1} k_{\lambda_1}^{p-1} \int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x) \mathrm{d}x.
\end{aligned}$$

$$\begin{aligned} J &\le k_{\lambda_1}^{p-1} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x)\mathrm{d}x \\ &= k_{\lambda_1}^{p-1} \int_0^{\infty} \left[x^{\lambda_1} \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} n^{\lambda_2 - 1} \right] x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x \\ &= k_{\lambda_1}^{p-1} \int_0^{\infty} \varpi(x) x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x \le k_{\lambda_1}^{p} \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x. \end{aligned} \tag{2.8}$$

Hence (2.4) is valid. By similar reasoning to the above it may be shown that

$$\begin{aligned} &\left[\sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} a_n \right]^q \\ &= \left\{ \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \left[\frac{x^{(1-\lambda_1)/q}}{n^{(1-\lambda_2)/p}} \right] \left[\frac{n^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}} a_n \right] \right\}^q \\ &\le [\varpi(x) x^{p(1-\lambda_1)-1}]^{q-1} \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q \\ &= k_{\lambda_1}^{q-1} x^{-q\lambda_1+1} \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q. \end{aligned} \tag{2.9}$$

$$\begin{aligned} L &\le k_{\lambda_1}^{q-1} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q \mathrm{d}x \\ &= k_{\lambda_1}^{q-1} \sum_{n=1}^{\infty} \left[n^{\lambda_2} \int_0^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} x^{\lambda_1 - 1} \mathrm{d}x \right] n^{q(1-\lambda_2)-1} a_n^q \\ &= k_{\lambda_1}^{q-1} \sum_{n=1}^{\infty} \omega(n) n^{q(1-\lambda_2)-1} a_n^q = k_{\lambda_1}^{q} \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \end{aligned}$$

Thus (2.5) is valid.

(b) Similarly, using the reverse Hölder's inequality with weight [15] and (2.3) gives (2.6) and (2.7).

LEMMA 2.3. *Suppose that* $0 < q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \ge 0$, $f(x)$ *is a non-negative measurable function in* $(0, \infty)$, *then (Let J, L be as in Lemma 2.2)*

$$J \le k_{\lambda_1}^{p} \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x, \tag{2.10}$$

$$L \ge k_{\lambda_1}^{q} \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \tag{2.11}$$

Proof. Applying Hölder's inequality [15] and (2.3), where $p < 0$ gives

$$\begin{aligned}
&\left[\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} f(x)\mathrm{d}x\right]^p \\
&= \left\{\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\left[\frac{x^{(1-\lambda_1)/q}}{n^{(1-\lambda_2)/p}} f(x)\right]\left[\frac{n^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}}\right]\mathrm{d}x\right\}^p \\
&\le \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x)\mathrm{d}x\left[\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}}\mathrm{d}x\right]^{p-1} \\
&= \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\min\{x,n\})^\alpha}\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x)\mathrm{d}x\left[n^{q(1-\lambda_2)-1}\omega(n)\right]^{p-1} \\
&= n^{-p\lambda_2+1} k_{\lambda_1}^{p-1}\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x)\mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
J &\le k_{\lambda_1}^{p-1}\sum_{n=1}^\infty \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{x^{(1-\lambda_1)(p-1)}}{n^{1-\lambda_2}} f^p(x)\mathrm{d}x \\
&= k_{\lambda_1}^{p-1}\int_0^\infty \left[x^{\lambda_1}\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} n^{\lambda_2-1}\right] x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x \\
&= k_{\lambda_1}^{p-1}\int_0^\infty \varpi(x) x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x \le k_{\lambda_1}^{p}\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x
\end{aligned} \tag{2.12}$$

Hence (2.10) is valid. By similar reasoning to the above, in view of $0 < q < 1$, it may be shown that

$$\begin{aligned}
&\left[\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} a_n\right]^q \\
&= \left\{\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\left[\frac{x^{(1-\lambda_1)/q}}{n^{(1-\lambda_2)/p}}\right]\left[\frac{n^{(1-\lambda_2)/p}}{x^{(1-\lambda_1)/q}} a_n\right]\right\}^q \\
&\ge [\varpi(x) x^{p(1-\lambda_1)-1}]^{q-1}\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q \\
&\ge k_{\lambda_1}^{q-1} x^{-q\lambda_1+1}\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q.
\end{aligned} \tag{2.13}$$

$$L \geq k_{\lambda_1}^{q-1} \int_0^\infty \sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} \frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}} a_n^q \mathrm{d}x$$
$$= k_{\lambda_1}^{q-1} \sum_{n=1}^\infty \left[n^{\lambda_2} \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} x^{\lambda_1-1} \mathrm{d}x \right] n^{q(1-\lambda_2)-1} a_n^q$$
$$= k_{\lambda_1}^{q-1} \sum_{n=1}^\infty \omega(n) n^{q(1-\lambda_2)-1} a_n^q = k_{\lambda_1}^q \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q.$$

Thus (2.11) is valid.

3. Main results

THEOREM 3.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $f(x) \geq 0$ *such that* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$*and* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$, *then we have the following equivalent inequalities*

$$I := \sum_{n=1}^\infty a_n \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} f(x) \mathrm{d}x = \int_0^\infty f(x) \sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} a_n \mathrm{d}x$$
$$< k_{\lambda_1} \left\{ \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^p n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}, \tag{3.1}$$

$$J = \sum_{n=1}^\infty n^{p\lambda_2-1} \left[\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} f(x) \mathrm{d}x \right]^p < k_{\lambda_1}^p \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x, \tag{3.2}$$

$$L = \int_0^\infty x^{q\lambda_1-1} \left[\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} a_n \right]^q \mathrm{d}x < k_{\lambda_1}^q \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q, \tag{3.3}$$

where the constant factor $k_{\lambda_1} = \frac{1}{\alpha - \lambda_1} + \frac{1}{\beta + \lambda_1}, k_{\lambda_1}^p, k_{\lambda_1}^q$, *are the best possible.*

Proof. Using Lebesgue term-by-term integration theorem, there are two forms of I of (3.1). In view of $0 < \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x < \infty$, (2.8) takes the strict inequality, thus (3.1) is valid. On one hand, using Hölder's inequality [15] gives

$$I = \sum_{n=1}^\infty \left[n^{\lambda_2 - \frac{1}{p}} \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} f(x) \mathrm{d}x \right] \left[n^{\frac{1}{p} - \lambda_2} a_n \right] \leq J^{1/p} = \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}. \tag{3.4}$$

By (3.2), (3.1) is valid. On the other hand, suppose that (3.1) is valid. Let

$$a_n := n^{p\lambda_2-1} \left[\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha} f(x) \mathrm{d}x \right]^{p-1}, n \in \mathbb{N}_+,$$

then from (3.1), it follows

$$\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q = J = I \le k_{\lambda_1} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q} . \quad (3.5)$$

By (2.8) and the conditions, it follows that $J < \infty$. If $J = 0$, then (3.2) is naturally valid. If $J > 0$, in view of the conditions of (3.1), then (3.5) takes the strict inequality, and

$$J^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/p} < k_{\lambda_1} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} .$$

Hence (3.2) is valid, which is equivalent to (3.1).

On one hand, in view of the conditions, (2.9) takes the strict inequality, thus (3.3) is valid. Using Hölder's inequality [15] gives

$$\begin{aligned} I &= \int_0^{\infty} \sum_{n=1}^{\infty} a_n \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} f(x) \mathrm{d}x \\ &= \int_0^{\infty} \left[x^{\frac{1}{q}-\lambda_1} f(x) \right] \left[x^{\lambda_1 - \frac{1}{q}} a_n \sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} \right] \mathrm{d}x \\ &\ge \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \right\}^{1/p} L^{1/q}. \end{aligned} \quad (3.6)$$

By (3.3), (3.1) is valid. On the other hand, suppose that (3.1) is valid. Let

$$f(x) := x^{q\lambda_1 - 1} \left[\sum_{n=1}^{\infty} \frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}} a_n \right]^{q-1}, x \in (0, \infty).$$

Applying (3.1) gives

$$\begin{aligned} \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) &= L = I \\ \le k_{\lambda_1} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} & \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q} . \end{aligned} \quad (3.7)$$

By (2.9) and the conditions, it follows that $L < \infty$. If $L = 0$, then (3.3) is naturally valid. If $L > 0$, in view of the conditions of (3.1), then (3.7) takes the strict inequality, and

$$L^{1/q} = \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \right\}^{1/q} < k_{\lambda_1} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q} .$$

Hence (3.3) is valid, which is equivalent to (3.1). Thus (3.1), (3.2), and (3.3) are equivalent to each other.

For any $0 < \varepsilon < q\lambda_2$, suppose that $\widetilde{f}(x) = 0, x \in (0, 1); \widetilde{f}(x) = x^{\lambda_1 - \frac{\varepsilon}{p} - 1}, x \in [1, \infty)$ and $\widetilde{a}_n = n^{\lambda_2 - \frac{\varepsilon}{q} - 1}, n \in \mathbb{N}_+$. Assuming there exists a positive number k with $k \leq k_{\lambda_1}$, such that (3.1) is still valid by changing k_{λ_1} to k. In particular, on one hand,

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty} \widetilde{a}_n \int_0^{\infty} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \widetilde{f}(x) \mathrm{d}x \\
&< k \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} \widetilde{f}^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} \widetilde{a}_n^q \right\}^{1/q} \\
&= k \left(\int_1^{\infty} x^{-1-\varepsilon} \mathrm{d}x \right)^{1/p} \left(1 + \sum_{n=2}^{\infty} n^{-1-\varepsilon} \right)^{1/q} \\
&< k \left(\frac{1}{\varepsilon} \right)^{1/p} \left(1 + \int_1^{\infty} x^{-1-\varepsilon} \mathrm{d}x \right)^{1/q} = \frac{k}{\varepsilon} (\varepsilon + 1)^{1/q}.
\end{aligned} \tag{3.8}$$

On the other hand, by monotonicity and Fubini theorem, it follows that

$$\begin{aligned}
\widetilde{I} &= \int_1^{\infty} x^{\lambda_1 - \frac{\varepsilon}{p} - 1} \left[\sum_{n=1}^{\infty} n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{x, n\})^{\beta}}{(\max\{x, n\})^{\alpha}} \right] \mathrm{d}x \\
&\geq \int_1^{\infty} x^{\lambda_1 - \frac{\varepsilon}{p} - 1} \left[\int_1^{\infty} y^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{x, y\})^{\beta}}{(\max\{x, y\})^{\alpha}} \mathrm{d}y \right] \mathrm{d}x, \ (t = y/x) \\
&= \int_1^{\infty} x^{-\varepsilon - 1} \left[\int_{1/x}^{\infty} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{x, t\})^{\beta}}{(\max\{x, t\})^{\alpha}} \mathrm{d}t \right] \mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
&= \int_1^{\infty} x^{-\varepsilon - 1} \left[\int_{1/x}^{1} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \mathrm{d}t \right] \mathrm{d}x + \frac{1}{\varepsilon} \int_1^{\infty} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \mathrm{d}t \\
&= \int_0^1 \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \left(\int_{1/t}^{\infty} x^{-\varepsilon - 1} \mathrm{d}x \right) t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \mathrm{d}t + \frac{1}{\varepsilon} \int_1^{\infty} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \mathrm{d}t \\
&= \frac{1}{\varepsilon} \int_0^1 \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} t^{\lambda_2 + \frac{\varepsilon}{p} - 1} \mathrm{d}t + \frac{1}{\varepsilon} \int_1^{\infty} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \mathrm{d}t.
\end{aligned} \tag{3.9}$$

Applying (3.8) and (3.9) gives

$$\int_0^1 \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} t^{\lambda_2 + \frac{\varepsilon}{p} - 1} \mathrm{d}t + \int_1^{\infty} t^{\lambda_2 - \frac{\varepsilon}{q} - 1} \frac{(\min\{1, t\})^{\beta}}{(\max\{1, t\})^{\alpha}} \mathrm{d}t < k(\varepsilon + 1)^{1/q}.$$

Using Fatou theorem gives

$$\begin{aligned} k_{\lambda_1} &= \int_0^1 \lim_{\varepsilon\to 0^+} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\lambda_2+\frac{\varepsilon}{p}-1} \mathrm{d}t + \int_1^\infty \lim_{\varepsilon\to 0^+} t^{\lambda_2-\frac{\varepsilon}{q}-1} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} \mathrm{d}t \\ &\le \frac{\lim}{\varepsilon \to 0^+} \left[\int_0^1 \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\lambda_2+\frac{\varepsilon}{p}-1} \mathrm{d}t + \int_1^\infty t^{\lambda_2-\frac{\varepsilon}{q}-1} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} \mathrm{d}t \right] \\ &\le \frac{\lim}{\varepsilon \to 0^+} k(\varepsilon+1)^{1/q} = k. \end{aligned}$$

Hence $k = k_{\lambda_1}$ is the best constant factor of (3.1). It is obvious that the constant factor in (3.2) (or (3.3)) is the best possible. Otherwise, by (3.4) (or (3.6)), we may get a contradiction that the constant factor in (3.1) is not the best possible. This completes the proof.

Remark 1. Let $\Phi(x) = x^{p(1-\lambda_1)-1}$, $x \in (0,\infty)$ and $\Psi(n) = n^{q(1-\lambda_2)-1}$, $n \in \mathbb{N}_+$, then $[\Phi(x)]^{1-q} = x^{q\lambda_1-1}$, $[\Psi(n)]^{1-p} = n^{p\lambda_2-1}$.

(i) A half-discrete Hilbert's operator $T : L_{p,\Phi}(0,\infty) \to l_{p,\Psi^{1-p}}$ is defined by:

$$Tf(n) = \int_0^\infty h_\lambda(x,n) f(x)\mathrm{d}x, n \ge 1.$$

where $f \in L_{p,\Phi}(0,\infty), Tf \in l_{p,\Psi^{1-p}}, h_\lambda(x,n) = \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}, \lambda =: \alpha - \beta$. Then by (3.2), it follows that: $\|Tf\|_{p,\Psi^{1-p}} \le k_{\lambda_1} \|f\|_{p,\Phi}$, i.e., T is a bounded operator with $\|T\| = k_{\lambda_1}$. Since the constant factor in (3.2) is the best possible, we have $\|T\| = k_{\lambda_1}$.

(ii) Similarly, another half-discrete Hilbert's operator $\widetilde{T} : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(0,\infty)$ is defined by:

$$\widetilde{T}a(x) = \sum_1^\infty h_\lambda(x,n)a_n, x \in (0,\infty).$$

where $a \in l_{q,\Psi}, \widetilde{T}a \in L_{q,\Phi^{1-q}}(0,\infty)$. Then by (3.3), it follows that: $\|\widetilde{T}a\|_{q,\Phi^{1-q}} \le k_{\lambda_1} \|a\|_{q,\Psi}$. In another word, T is a bounded operator with $\|\widetilde{T}\| \le k_{\lambda_1}$. Since the constant factor in (3.3) is the best possible, we obtain $\|\widetilde{T}\| \le k_{\lambda_1}$.

THEOREM 3.2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1, \theta_\lambda(x) := \frac{1}{k_{\lambda_1}} \int_0^{1/x} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} \cdot t^{\lambda_2-1} (x \in (0,\infty))$, $a_n \ge 0$, $f(x) \ge 0$ *such that* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$*and* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q < \infty$, *then we have the following equivalent inequalities (Let I, J be as in Theorem 3.1)*

$$I > k_{\lambda_1} \left\{ \int_0^\infty (1-\theta_\lambda(x)) x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}, \quad (3.10)$$

$$J > k_{\lambda_1}^p \int_0^\infty (1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x, \tag{3.11}$$

$$\widetilde{L} = \int_0^\infty \frac{x^{q\lambda_1-1}}{[1-\theta_\lambda(x)]^{q-1}}\left[\sum_{n=1}^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}a_n\right]^q \mathrm{d}x < k_{\lambda_1}^q \sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q, \tag{3.12}$$

where the constant factor $k_{\lambda_1} = \frac{1}{\alpha-\lambda_1} + \frac{1}{\beta+\lambda_1}$, $k_{\lambda_1}^p$, $k_{\lambda_1}^q$ *are the best possible.*

Proof. Similar to (2.8), by the reverse Hölder's inequality [15], (2.3) and the conditions, we have

$$J \geq k_{\lambda_1}^{p-1} \int_0^\infty \varpi(x)x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x > k_{\lambda_1}^p \int_0^\infty (1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x, \tag{3.13}$$

thus (2.11) is valid. On one hand, by the reverse Hölder's inequality [15], we obtain the reverse form of (3.4) as follows

$$I = \sum_{n=1}^\infty \left[n^{\lambda_2-\frac{1}{p}} \int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}f(x)\mathrm{d}x\right]\left[n^{\frac{1}{p}-\lambda_2}a_n\right] \geq J^{1/p}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q}. \tag{3.14}$$

by (3.11), (3.10) is valid. On the other hand, suppose that (3.10) is valid. Let

$$a_n = n^{p\lambda_2-1}\left[\int_0^\infty \frac{(\min\{x,n\})^\beta}{(\max\{x,n\})^\alpha}f(x)\mathrm{d}x\right]^{p-1}, n \in \mathbb{N}_+.$$

Applying (3.10) gives

$$\begin{aligned}\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q &= J = I \\ &\geq k_{\lambda_1}\left\{\int_0^\infty (1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/p}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q},\end{aligned} \tag{3.15}$$

By (3.13) and the conditions, it follows that $J > 0$. If $J = \infty$, then (3.11) is naturally valid. If $J < \infty$, in view of the conditions and (3.10), then (3.15) takes the strict inequality, and

$$J^{1/p} = \left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q} > k_{\lambda_1}\left\{\int_0^\infty (1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/p}.$$

Hence (3.11) is valid, which is equivalent to (3.10).

On one hand, similar to (2.9), by the reverse Hölder's inequality [15], (2.3) and the conditions, in view of $q < 0$, we have

$$\begin{aligned}&\left[\sum_{n=1}^{\infty}\frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}}a_n\right]^q\\&\le[\omega(x)x^{p(1-\lambda_1)-1}]^{q-1}\sum_{n=1}^{\infty}\frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}}\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}}a_n^q\\&<k_{\lambda_1}^{q-1}(1-\theta_\lambda(x))x^{-q\lambda_1+1}\sum_{n=1}^{\infty}\frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}}\frac{n^{(1-\lambda_2)(q-1)}}{x^{1-\lambda_1}}a_n^q.\end{aligned}\tag{3.16}$$

Similarly, we get (3.12). Applying the reverse Hölder's inequality [15] gives

$$\begin{aligned}I&=\int_0^{\infty}\left[(1-\theta_\lambda(x))^{1/p}x^{\frac{1}{q}-\lambda_1}f(x)\right]\left[\frac{x^{\lambda_1-\frac{1}{q}}}{(1-\theta_\lambda(x))^{1/p}}\sum_{n=1}^{\infty}\frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}}\cdot a_n\right]\mathrm{d}x\\&\ge\left\{\int_0^{\infty}(1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\right\}^{1/p}\widetilde{L}^{1/q}.\end{aligned}\tag{3.17}$$

By (3.12), (3.10) is valid. On the other hand, suppose that (3.10) is valid. Let

$$f(x)=\frac{x^{q\lambda_1-1}}{(1-\theta_\lambda(x))^{q-1}}\left[\sum_{n=1}^{\infty}\frac{(\min\{x,n\})^{\beta}}{(\max\{x,n\})^{\alpha}}a_n\right]^{q-1},x\in(0,\infty),$$

applying (3.10) gives

$$\begin{aligned}&\int_0^{\infty}(1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)=\widetilde{L}=I\\&\le k_{\lambda_1}\left\{\int_0^{\infty}(1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q}.\end{aligned}\tag{3.18}$$

By (3.16) and the conditions, it follows that $\widetilde{L}<\infty$. If $\widetilde{L}=0$, then (3.12) is naturally valid. If $\widetilde{L}>0$, in view of the conditions of (3.10), then (3.18) takes the strict inequality, and

$$\widetilde{L}^{1/q}=\left\{\int_0^{\infty}(1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/q}>k_{\lambda_1}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q}.$$

In view of $q < 0$, hence (3.12) is valid, which is equivalent to (3.10). Thus (3.10), (3.11) and (3.12) are equivalent to each other.

For any $0 <\varepsilon <p(\lambda_1 + \beta)$, suppose that $\widetilde{f}(x)=0,x\in(0,1);\widetilde{f}(x)=x^{\lambda_1-\frac{\varepsilon}{p}-1},x\in[1,\infty)$ and $\widetilde{a}_n=n^{\lambda_2-\frac{\varepsilon}{q}-1},n\in\mathbb{N}_+$. Assuming there exists a positive number K with $K\ge k_{\lambda_1}$, such that (3.10) is still valid by changing k_{λ_1} to K. In particular, on one hand,

$$\begin{aligned}\widetilde{I} &> K\left\{\int_0^\infty (1-\theta_\lambda(x))x^{p(1-\lambda_1)-1}\widetilde{f}^p(x)\mathrm{d}x\right\}^{1/p}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}\widetilde{a}_n^q\right\}^{1/q}\\ &= K\left(\int_1^\infty (1-\theta_\lambda(x))x^{-1-\varepsilon}\mathrm{d}x\right)^{1/p}\left(1+\sum_{n=1}^\infty n^{-1-\varepsilon}\right)^{1/q}\\ &= K\left(\int_1^\infty \left(x^{-1-\varepsilon}-O\left(\frac{1}{x^{\beta+\lambda_2+\varepsilon+1}}\right)\right)\mathrm{d}x\right)^{1/p}\left(1+\sum_{n=2}^\infty n^{-1-\varepsilon}\right)^{1/q}\\ &> K\left(\frac{1}{\varepsilon}-O(1)\right)^{1/p}\left(1+\int_1^\infty x^{-1-\varepsilon}\mathrm{d}x\right)^{1/q} = \frac{K}{\varepsilon}(1-\varepsilon O(1))^{1/p}(\varepsilon+1)^{1/q}.\end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned}\tilde{I} &\le \sum_{n=1}^\infty n^{\lambda_2-\frac{\varepsilon}{q}-1}\int_0^\infty x^{\lambda_1-\frac{\varepsilon}{p}-1}\frac{(\min\{x,n\})^\beta}{(\min\{x,n\})^\alpha}\mathrm{d}x\\ &= \sum_{n=1}^\infty n^{\lambda_2-\frac{\varepsilon}{q}-1}\left(n^{-\alpha}\int_0^n x^{\lambda_1+\beta-\frac{\varepsilon}{p}-1}\mathrm{d}x + n^\beta\int_n^\infty x^{\lambda_1-\alpha-\frac{\varepsilon}{p}-1}\mathrm{d}x\right)\\ &= \left(\frac{1}{\lambda_1+\beta-\frac{\varepsilon}{p}}+\frac{1}{\alpha-\lambda_1+\frac{\varepsilon}{p}}\right)\sum_{n=1}^\infty n^{-1-\varepsilon} = (k_{\lambda_1}+o(1))\sum_{n=1}^\infty n^{-1-\varepsilon}\\ &\le (k_{\lambda_1}+o(1))\left(1+\int_1^\infty x^{-1-\varepsilon}\mathrm{d}x\right) = \frac{1+\varepsilon}{\varepsilon}(k_{\lambda_1}+o(1)).\end{aligned} \tag{3.20}$$

By (3.19) and (3.20), it may be shown that

$$(1+\varepsilon)(k_{\lambda_1}+o(1)) > K(1+\varepsilon O(1))^{1/p}(\varepsilon+1)^{1/q}. \tag{3.21}$$

Let $\varepsilon \to 0^+$, then $k_{\lambda_1} \ge K$. Hence $K = k_{\lambda_1}$ is the best constant factor of (3.10). It is obviously that the constant factor in (3.11) (or (3.12)) is the best possible. Otherwise, by (3.14) (or (3.17)), we may get a contradiction that the constant factor in (3.10) is not the best possible. This completes the proof.

THEOREM 3.3. *If* $0 < q < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n \ge 0$, $f(x) \ge 0$ *such that* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q < \infty$*and* $0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q < \infty$, *then the following inequalities hold and are equivalent (Let I, J, L be as in Theorem 3.1):*

$$I > k_{\lambda_1}\left\{\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x\right\}^{1/p}\left\{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right\}^{1/q}, \tag{3.22}$$

$$J < k_{\lambda_1}^q\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)\mathrm{d}x, \tag{3.23}$$

$$L > k_{\lambda_1}^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q, \tag{3.24}$$

where the constant factor $k_{\lambda_1} = \frac{1}{\alpha - \lambda_1} + \frac{1}{\beta + \lambda_1}$, $k_{\lambda_1}^p$, $k_{\lambda_1}^q$ *are the best possible.*

Proof In view of $p < 0$, the proof can be completed by following the same steps as in the proof of Theorem 3.2, thus we omit the details.

Remark 2. (1) In particular, if $\beta = 0$, $0 < \lambda_1 < \alpha$, $\lambda_2 \le 1$, $\lambda_1 + \lambda_2 = \alpha$, then (3.1) reduces to (1.4), (3.2), and (3.3) reduce to the following inequalities respectively, which are equivalent to (1.4):

$$\sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left[\int_0^{\infty} \frac{f(x)}{(\max\{x,n\})^{\alpha}} \mathrm{d}x \right]^p < \left(\frac{\alpha}{\lambda_1 \lambda_2} \right)^p \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x, \tag{3.25}$$

$$\int_0^{\infty} x^{q\lambda_1 - 1} \left[\sum_{n=1}^{\infty} \frac{a_n}{(\max\{x,n\})^{\alpha}} \right]^q \mathrm{d}x < \left(\frac{\alpha}{\lambda_1 \lambda_2} \right)^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \tag{3.26}$$

(2) If $\alpha = 0$, $-\beta < \lambda_1 < 0$, $\lambda_2 \le 1 - \beta$, $\lambda_1 + \lambda_2 = -\beta$, then (3.1)-(3.3), respectively, reduce to the following equivalent inequalities:

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \int_0^{\infty} (\min\{x,n\})^{\beta} f(x) \mathrm{d}x \\ &< \frac{\beta}{\lambda_1 \lambda_2} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}, \end{aligned} \tag{3.27}$$

$$\sum_{n=1}^{\infty} n^{p\lambda_2 - 1} \left[\int_0^{\infty} (\min\{x,n\})^{\beta} f(x) \mathrm{d}x \right]^p < \left(\frac{\beta}{\lambda_1 \lambda_2} \right)^p \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x, \tag{3.28}$$

$$\int_0^{\infty} x^{q\lambda_1 - 1} \left[\sum_{n=1}^{\infty} (\min\{x,n\})^{\beta} a_n \right]^q \mathrm{d}x < \left(\frac{\beta}{\lambda_1 \lambda_2} \right)^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \tag{3.29}$$

(3) If $\alpha = \beta$, $-\alpha < \lambda_1 < \alpha$, $\lambda_2 \le 1 - \alpha$, $\lambda_1 + \lambda_2 = 0$, then (3.1)-(3.3), respectively, reduce to the following equivalent inequalities:

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \int_0^{\infty} \left(\frac{\min\{x,n\}}{\max\{x,n\}} \right)^{\alpha} f(x) \mathrm{d}x \\ &< \frac{2\alpha}{\alpha^2 - \lambda_1^2} \left\{ \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x) \mathrm{d}x \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q \right\}^{1/q}, \end{aligned} \tag{3.30}$$

$$\sum_{n=1}^{\infty} n^{p\lambda_2-1}\left[\int_0^{\infty}\left(\frac{\min\{x,n\}}{\max\{x,n\}}\right)^{\alpha} f(x)\mathrm{d}x\right]^p < \left(\frac{2\alpha}{\alpha^2-\lambda_1^2}\right)^p \int_0^{\infty} x^{p(1-\lambda_1)-1} f^p(x)\mathrm{d}x, \quad (3.31)$$

$$\int_0^{\infty} x^{q\lambda_1-1}\left[\sum_{n=1}^{\infty}\left(\frac{\min\{x,n\}}{\max\{x,n\}}\right)^{\alpha} a_n\right]^q \mathrm{d}x < \left(\frac{2\alpha}{\alpha^2-\lambda_1^2}\right)^q \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} a_n^q. \quad (3.32)$$

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Authors' contributions
BH drafted the manuscript. BY conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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