Ostrovska *Journal of Inequalities and Applications* 2012, **2012**:297
http://www.journalofinequalitiesandapplications.com/content/2012/1/297

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Functions whose smoothness is not improved under the limit q-Bernstein operator

Sofiya Ostrovska*

*Correspondence: ostrovsk@atilim.edu.tr; ostrovskasofiya@yahoo.com
Department of Mathematics, Atilim University, Ankara, Turkey

Abstract

The limit q-Bernstein operator B_q emerges naturally as a modification of the Szász-Mirakyan operator related to the Euler probability distribution. At the same time, this operator serves as the limit for a sequence of the q-Bernstein polynomials with $0 < q < 1$. Over the past years, the limit q-Bernstein operator has been studied widely from different perspectives. Its approximation, spectral, and functional-analytic properties, probabilistic interpretation, the behavior of iterates, and the impact on the analytic characteristics of functions have been examined. It has been proved that under a certain regularity condition, B_q improves the smoothness of a function which does not satisfy the Hölder condition. The purpose of this paper is to exhibit 'exceptional' functions whose smoothness is not improved under the limit q-Bernstein operator.

MSC: 26A15; 26A16; 41A36

Keywords: limit q-Bernstein operator; Hölder condition; modulus of continuity

1 Introduction

The *limit q-Bernstein operator B_q* comes out as an analogue of the Szász-Mirakyan operator related to the Euler probability distribution, also called the q-deformed Poisson distribution (see [1]). On the other hand, this operator is important for approximation theory as it provides the limit for a sequence of the q-Bernstein polynomials in the case $0 < q < 1$ (*cf.* [2]). A comprehensive review of the results on the q-Bernstein polynomials can be found in [3]. It should be pointed out that operators whose nature is similar to that of B_q typically emerge as the limit for a sequence of positive operators based on the q-integers; see, for example, [4–8] making B_q a principal exemplary model. A general approach to this problem based on the Korovkin-type theorem has been developed by Wang in [9].

In the sequel, we employ the notation (see, *e.g.*, [10], Ch. 10):

$$(a;q)_0 := 1, \qquad (a;q)_k := \prod_{s=0}^{k-1}(1 - aq^s), \qquad (a;q)_\infty := \prod_{s=0}^{\infty}(1 - aq^s).$$

In addition, we adopt the writing

$$f(\delta) \asymp g(\delta), \quad \delta \in I$$

meaning

$$C_1 f(\delta) \le g(\delta) \le C_2 f(\delta), \quad \delta \in I$$

for some positive constants C_1 and C_2 independent of δ.

Definition 1.1 Given $q \in (0,1)$, $f \in C[0,1]$, the *limit q-Bernstein operator* is defined by $f \mapsto B_q f$, where

$$(B_q f)(x) = B_q(f;x) := \begin{cases} (x;q)_\infty \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)x^k}{(q;q)_k} & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1. \end{cases}$$

The limit q-Bernstein operator has been undergoing an intensive scrutiny lately. Among other facts, it has been established that B_q is a shape-preserving positive linear operator on $C[0,1]$ with $\|B_q\| = 1$, which possesses the end-point interpolation property, leaves the linear functions invariant, and maps a polynomial of degree m to a polynomial of degree m. Moreover, it takes a binomial $(1-x)^m$ to the corresponding q-binomial, that is,

$$B_q\big((1-x)^m\big) = (x;q)_m, \quad m = 0,1,2,\ldots.$$

The object of study of this paper is the impact of B_q on the analytic properties of functions. This problem has been considered in [11–13], where some direct and inverse theorems have been proved. In general, it can be stated that functions become 'better' after applying B_q. It is not difficult to see that for any $f \in C[0,1]$, the function $B_q f$ being continuous on $[0,1]$ admits an analytic continuation into the open unit disc centered at 0. As a result, possible 'bad' smoothness of f will be removed for its image. What can be concluded about the smoothness at 1? To describe the behavior of $f(x)$ as x approaches 1^-, the local modulus of continuity at 1 is applied:

$$\Omega(f;\delta) := \max_{1-\delta \le x \le 1} \big|f(x) - f(1)\big|.$$

It has been demonstrated that if f satisfies the Hölder condition at 1, then $B_q f$ is a 'better' function than f in terms of its analytic properties unless f is a polynomial. A detailed analysis of this situation is provided in [12]. How about functions without the Hölder condition at 1? Theorem 5.1 of [12] states that for any $f \in C[0,1]$, there is $C > 0$ such that

$$\frac{\Omega(B_q f;\delta)}{\delta} \le C \int_{q^{1/\delta}}^{1} \frac{\Omega(f;t)}{t}\,dt, \quad \delta \in (0,1]. \tag{1.1}$$

A detailed analysis of formula (1.1) implies that under the following regularity condition:

$$\exists b \in (0,1), \quad \lim_{\delta \to 0^+} \frac{\delta \int_{b^{1/\delta}}^{1} \frac{\Omega(f;t)}{t}\,dt}{\Omega(f;\delta)} = 0 \tag{1.2}$$

one has $\Omega(B_q f;\delta) = o(\Omega(f;\delta))$ as $\delta \to 0^+$; see [12], Theorem 5.7. That is, if condition (1.2) is satisfied, $B_q f$ approaches its value at 1 faster than the function f. It is natural to ask whether the same is true for functions which do not satisfy (1.2). The aim of the present paper is to prove the existence of continuous functions different from polynomials whose analytic properties are not improved under the limit q-Bernstein operator. To be specific, as a main result, it is proved that there are continuous functions on $[0,1]$ which do not satisfy the Hölder condition at 1 such that the following holds:

$$\Omega(B_q f;\delta) \asymp \Omega(f;\delta), \quad \delta \in [0,1].$$

It should be mentioned here that the last equality is satisfied whenever f is an eigenfunction of B_q corresponding to a non-zero eigenvalue. Consequently, the findings of this article encourage the search for non-polynomial eigenfunctions of the limit q-Bernstein operator whose existence as yet remains an open problem (*cf.* [3]).

2 Some auxiliary results

In the sequel, we denote by the letter C - with or without indices - a positive constant whose value does not need to be addressed. The same letter does not necessary mean the equal values of the constant. This section presents some technical lemmas needed for the proof of main Theorem 3.1.

Lemma 2.1 *Let* $p \in (0,1)$. *Consider a sequence* $\{a_n\}_{n=0}^{\infty}$ *defined by*

$$a_0 = 1, \qquad a_{n+1} = p^{1/a_n} \quad \text{for } n = 1,2,\ldots \quad \text{and} \quad a_1 = \begin{cases} p & \text{if } p < e^{-1/e}, \\ h & \text{if } p \geq e^{-1/e}, \end{cases} \tag{2.1}$$

where $h \in (0,L)$ *and* L *is the least positive root of the equation* $x^x = p$, *and an associated sequence*

$$m_n = \int_{a_{n+1}}^{a_n} \frac{dt}{t}, \quad n = 0,1,\ldots. \tag{2.2}$$

Then (i) *the sequence* $\{a_n\}$ *is decreasing with* $\lim_{n\to\infty} a_n = 0$; (ii) $\lim_{n\to\infty} \frac{m_{n+1}}{m_n} = \infty$.

Proof (i) It can be readily seen by the induction on n that $\{a_n\}$ is a strictly decreasing sequence of positive real numbers, hence $\{a_n\} \to A \in [0,h]$. If $A > 0$, then A satisfies $A = p^{1/A}$. For $p < e^{-1/e}$, this is impossible since $p < x^x$ for all $x \in [0,1]$. For $p \geq e^{-1/e}$, this contradicts the choice of a_1. Thus, $A = 0$.

(ii) Obviously, $m_n = \ln(a_n) + (1/a_n)\ln(1/p)$, whence

$$\frac{m_{n+1}}{m_n} = \frac{(1/a_{n+1})}{(1/a_n)} \cdot \frac{\ln(1/p) + a_{n+1}\ln(a_{n+1})}{\ln(1/p) + a_n \ln(a_n)}.$$

As $\{a_n \ln(a_n)\} \to 0$, one has

$$\lim_{n\to\infty} \frac{m_{n+1}}{m_n} = \lim_{n\to\infty} \frac{a_n}{p^{1/a_n}} = \infty,$$

because $\{a_n\} \to 0$. □

Lemma 2.2 *For every* $p \in (0,1)$, *there exists a function* $\omega : [0,1] \to \mathbb{R}$ *satisfying the following conditions*:

(a) $\omega \not\equiv 0$ *is a continuous non-decreasing function on* $[0,1]$ *with* $\omega(0) = 0$;
(b) *there exists* $C > 0$ *such that*

$$\omega(p^{1/\delta}) \geq C\omega(\delta), \quad \text{for all } \delta \in (0,1]. \tag{2.3}$$

Proof Consider the sequence of intervals $(a_{n+1}, a_n]$, $n = 0,1,2,\ldots$, where a_n are given by (2.1). Since $\{a_n\} \to 0$, it follows that $\bigcup_{n=1}^{\infty}(a_{n+1}, a_n] = (0,1]$. As $\omega(0) = 0$, it is left to construct $\omega(\delta)$ on the intervals $(a_{n+1}, a_n]$, $n = 0,1,2,\ldots$. Let $\{\nu_n\}_{n=0}^{\infty}$ be a sequence of positive numbers satisfying the conditions

$$\lim_{n\to\infty} \nu_n = 0 \quad \text{and} \quad 1 \geq \frac{\nu_{n+1}}{\nu_n} \geq \rho > 0, \quad n = 0,1,2,\ldots. \tag{2.4}$$

Define a continuous piecewise linear function ω on each $(a_{n+1}, a_n]$, $n = 0,1,2,\ldots$ as follows:

$$\omega(\delta) = \nu_{n+1} + \frac{\nu_n - \nu_{n+1}}{a_n - a_{n+1}}(\delta - a_{n+1}) \quad \text{if } \delta \in (a_{n+1}, a_n] \tag{2.5}$$

and put $\omega(0) = 0$. It follows directly from (2.4) and (2.5) that $\omega(\delta)$ is both continuous and non-decreasing on $[0,1]$. Further, since for $n \geq 1$,

$$\delta \in (a_{n+1}, a_n] \quad \Leftrightarrow \quad p^{1/\delta} \in (a_{n+2}, a_{n+1}],$$

while

$$\delta \in (a_1, a_0] \quad \Rightarrow \quad p^{1/\delta} \in (a_2, a_0],$$

it can be derived that for $\delta \in (a_{n+1}, a_n]$ and all $n \geq 0$,

$$\omega(p^{1/\delta}) \geq \nu_{n+2} \geq \rho^2 \nu_n \geq \rho^2 \omega(\delta).$$

Thus, ω satisfies the conditions (a) and (b) of Lemma 2.2 with $C = \rho^2$. □

For the sequel, we need the next statement.

Lemma 2.3 *Given* $p \in (0,1)$, *let* ω *be a function on* $[0,1]$ *obeying the conditions* (a) *and* (b) *of Lemma* 2.2. *Then* ω *does not satisfy the Hölder condition of any order* α *at* 0.

Proof In order to establish the absence of the Hölder condition, it suffices to show that $\frac{\omega(a_n)}{(a_n)^\alpha} \to \infty$. For the given value of p, we construct the sequence $\{a_n\}$ as defined by (2.1) and denote $\omega(a_n) = u_n$. By the condition (b), $\frac{u_{n+1}}{u_n} \geq C > 0$ for all $n \in \mathbb{N}$. Now, consider

$$\frac{\omega(a_n)}{(a_n)^\alpha} = \frac{u_n}{(a_n)^\alpha} =: r_n.$$

Since

$$\frac{r_{n+1}}{r_n} = \frac{u_{n+1}}{u_n} \cdot \left(\frac{a_n}{p^{1/a_n}}\right)^\alpha \geq C\left(\frac{a_n}{p^{1/a_n}}\right)^\alpha \to +\infty \quad \text{as } n \to \infty,$$

it follows that

$$\frac{\omega(a_n)}{(a_n)^\alpha} \to +\infty \quad \text{as } n \to \infty. \qquad \square$$

Analyzing the constructions of the preceding lemmas, the following observations can be derived. The notation is the same as adopted in Lemmas 2.1 and 2.2, and ω is a function satisfying the conditions (a), (b) of Lemma 2.2.

Lemma 2.4 *For all $n = 1, 2, \ldots$, the following estimate holds*:

$$\sum_{i=0}^{n} \omega(a_i) m_i \le C\omega(a_n) m_n, \tag{2.6}$$

where $C > 0$ is independent of n.

Proof Indeed, one concludes by virtue of (2.3) that $\frac{\omega(a_{n-i})}{\omega(a_n)} \le C^{-i}$, while (ii) of Lemma 2.1 implies $\frac{m_{n-i}}{m_n} \le C_1(\frac{C}{2})^i$, $i = 0, \ldots, n$, where C_1 does not depend on n. As a result,

$$\sum_{i=0}^{n} \omega(a_i) m_i \le C_1 \omega(a_n) m_n \cdot \sum_{i=0}^{n} 2^{-i} \le 2C_1 \omega(a_n) m_n. \qquad \square$$

Lemma 2.5 *Let ω be a function satisfying the conditions of Lemma* 2.2. *Then*

$$\omega(\delta) \asymp \delta \int_{p^{1/\delta}}^{1} \frac{\omega(t)}{t}\, dt, \quad \delta \in [0,1]. \tag{2.7}$$

Proof Clearly,

$$\delta \int_{p^{1/\delta}}^{1} \frac{\omega(t)}{t}\, dt \ge \delta\omega\bigl(p^{1/\delta}\bigr) \int_{p^{1/\delta}}^{1} \frac{dt}{t} = \omega\bigl(p^{1/\delta}\bigr) \ln(1/p) \ge C\omega(\delta),$$

where the last inequality is a consequence of (2.3).

To prove the converse inequality, we use the sequence (2.1) and select $\delta_0 = \delta_0(p) > 0$ in such a way that $\delta_0 \in (a_{m+1}, a_m]$, where $a_m \le \ln(1/p)$ and, in addition, $p^{1/\delta} < \delta$ for $\delta \in (0, \delta_0]$. Then, for $\delta \in (0, \delta_0]$, one has

$$\int_{p^{1/\delta}}^{1} \frac{\omega(t)}{t}\, dt = \int_{p^{1/\delta}}^{\delta} \frac{\omega(t)}{t}\, dt + \int_{\delta}^{1} \frac{\omega(t)}{t}\, dt =: I_1(\delta) + I_2(\delta).$$

Now, let $\delta < \delta_0$ and $\delta \in (a_{n+1}, a_n]$, $n \ge m$. Since $a_m \le \ln(1/p)$, the function $\int_{p^{1/\delta}}^{\delta} \frac{dt}{t}$ is decreasing in δ for $\delta < \delta_0$ and hence

$$I_1(\delta) \ge \omega\bigl(p^{1/\delta}\bigr) \int_{p^{1/\delta}}^{\delta} \frac{dt}{t} \ge \omega(a_{n+2}) \int_{a_{n+1}}^{a_n} \frac{dt}{t} = \omega(a_{n+2}) m_n \ge C^2 \omega(a_n) m_n. \tag{2.8}$$

To estimate $I_2(\delta)$, we write for $\delta \in (a_{n+1}, a_n]$

$$I_2(\delta) \le \int_{a_{n+1}}^{1} \frac{\omega(t)}{t}\, dt \le \sum_{i=0}^{n} \omega(a_i) m_i \le C_1 \omega(a_n) m_n \tag{2.9}$$

by virtue of Lemma 2.4. Combining (2.8) and (2.9), we derive

$$\begin{aligned}\int_{p^{1/\delta}}^{1} \frac{\omega(t)}{t}\,dt &\le C_2 \int_{p^{1/\delta}}^{\delta} \frac{\omega(t)}{t}\,dt \le C_2\omega(\delta)\int_{p^{1/\delta}}^{\delta} \frac{dt}{t} \\ &= C_2\frac{\omega(\delta)}{\delta}\bigl[\delta\ln\delta + \ln(1/p)\bigr] \le C_3\frac{\omega(\delta)}{\delta}, \quad \delta\in(0,\delta_0].\end{aligned}$$

The statement now follows. □

3 Main theorem

The aim of this section is to study the comparative behavior of $\Omega(f;\delta)$ and $\Omega(B_qf;\delta)$ in the situation when the local modulus of continuity of f at 1 is a function satisfying the conditions of Lemma 2.2. It is not difficult to see that such a function f does not satisfy the regularity condition (1.2). The examination carried out here implies that f is a continuous function on $[0,1]$ without the Hölder condition at 1 for which the order of $\Omega(f;\delta)$ as $\delta\to 0^+$ is not changed under B_q. Such examples have not been known previously. The next theorem constitutes the main result of this work.

Theorem 3.1 *For every* $q\in(0,1)$, *there exists a function* $f\in C[0,1]$ *such that*

(i) f *does not satisfy the Hölder condition at* 1;

(ii) *the following relation is true*:

$$\Omega(B_qf;\delta)\asymp\Omega(f;\delta), \quad \delta\in[0,1].$$

Proof Given $q\in(0,1)$, let $\omega(t)$ be a function satisfying the conditions of Lemma 2.2 with $p=\sqrt{q}$. Set $f(x):=\omega(1-x)$ for $x\in[0,1]$. By virtue of Lemma 2.3, f does not satisfy the Hölder condition at $x=1$. Besides, $\Omega(f;\delta)=\omega(\delta)$ due to the monotonicity of f.

Step 1. First, we prove that for some $C_1>0$, the following holds:

$$\Omega(B_qf;\delta)\ge C_1\Omega(f;\delta). \tag{3.1}$$

Applying Definition 1.1 for $x\in[0,1)$, one has

$$(B_qf)(x)=(x;q)_\infty\sum_{k=0}^{\infty}\frac{\omega(q^k)}{(q;q)_k}x^k \ge (1-x)(q;q)_\infty\sum_{k=0}^{\infty}\omega\bigl(q^k\bigr)x^k.$$

Hence,

$$\begin{aligned}\Omega(B_qf;\delta) &\ge \delta(q;q)_\infty\sum_{k=0}^{\infty}\omega\bigl(q^k\bigr)(1-\delta)^k \\ &\ge \delta(q;q)_\infty\int_0^{\infty}\omega\bigl(q^x\bigr)(1-\delta)^x\,dx \ge \delta(q;q)_\infty(1-\delta)^A\int_0^{A}\omega\bigl(q^x\bigr)\,dx\end{aligned}$$

for any $A>0$. With $A=\frac{1}{\ln(1/(1-\delta))}$, one obtains

$$\Omega(B_qf;\delta)\ge\frac{\delta(q;q)_\infty}{e}\int_0^{1/\ln(1/(1-\delta))}\omega\bigl(q^x\bigr)\,dx=\frac{\delta(q;q)_\infty}{e}\int_{q^{1/\ln(1/(1-\delta))}}^{1}\frac{\omega(t)}{\ln(1/q)t}\,dt.$$

Now, we restrict our attention to the case $\delta \in (0,1/2)$. In this case, $\ln(1/(1-\delta)) \leq 2\delta$ and as a result,

$$\Omega(B_q f;\delta) \geq \frac{\delta(q;q)_\infty}{e\ln(1/q)} \int_{q^{1/(2\delta)}}^1 \frac{\omega(t)}{t}\,dt = \frac{\delta(q;q)_\infty}{e\ln(1/q)} \int_{p^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt. \tag{3.2}$$

By Lemma 2.5,

$$\int_{p^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt \geq C_2 \frac{\omega(\delta)}{\delta} = C_2 \frac{\Omega(f;\delta)}{\delta}.$$

Substituting the last inequality into (3.2), one obtains (3.1) on $[0,1/2]$ and (with a different constant) on $[0,1]$.

Step 2. Now, we are going to prove that for the function f and some $C_3 > 0$, the following inequality is true:

$$\Omega(B_q f;\delta) \leq C_3 \Omega(f;\delta), \quad \delta \in [0,1].$$

Theorem 5.1 of [12] - we refer to formula (5.2) therein - claims that for any $f \in C[0,1]$, one has

$$\Omega(B_q f;\delta) \leq C_4 \delta \int_{q^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt, \quad \delta \in [0,1]. \tag{3.3}$$

Since $q = p^2 < p$, in order to apply (3.3), we need to estimate $\int_{q^{1/\delta}}^{p^{1/\delta}} \frac{\omega(t)}{t}\,dt$ as follows:

$$\begin{aligned}\int_{q^{1/\delta}}^{p^{1/\delta}} \frac{\omega(t)}{t}\,dt &\leq \omega\big(p^{1/\delta}\big) \int_{q^{1/\delta}}^{p^{1/\delta}} \frac{dt}{t} = \frac{\ln(1/q)}{2} \cdot \frac{\omega(p^{1/\delta})}{\delta} \\ &\leq \frac{\ln(1/q)}{2} \cdot \frac{1}{\ln(1/p)} \int_{p^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt = \int_{p^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt.\end{aligned}$$

Combining the last inequality with (3.3), we obtain

$$\Omega(B_q f;\delta) \leq 2C_4 \delta \int_{p^{1/\delta}}^1 \frac{\omega(t)}{t}\,dt.$$

Finally, the application of Lemma 2.5 yields

$$\Omega(B_q f;\delta) \leq C_5 \omega(\delta) = C_5 \Omega(f;\delta),$$

as claimed. □

Competing interests
The author declares that she has no competing interests.

Received: 17 September 2012 Accepted: 26 November 2012 Published: 14 December 2012

References

1. Charalambides, CA: The q-Bernstein basis as a q-binomial distribution. J. Stat. Plan. Inference **140**(8), 2184-2190 (2010)
2. Il'inskii, A, Ostrovska, S: Convergence of generalized Bernstein polynomials. J. Approx. Theory **116**, 100-112 (2002)
3. Ostrovska, S: The first decade of the q-Bernstein polynomials: results and perspectives. J. Math. Anal. Approx. Theory **2**(1), 35-51 (2007)
4. Gupta, V, Wang, H: The rate of convergence of q-Durrmeyer operators for $0 < q < 1$. Math. Methods Appl. Sci. **31**(16), 1946-1955 (2008)
5. Mahmudov, NI, Sabancigil, P: Voronovskaja type theorem for the Lupas q-analogue of the Bernstein operators. Math. Commun. **17**(1), 83-91 (2012)
6. Videnskii, VS: On some classes of q-parametric positive operators. In: Operator Theory, Advances and Applications, vol. 158, pp. 213-222 (2005)
7. Wang, H: Properties of convergence for the q-Meyer-Konig and Zeller operators. J. Math. Anal. Appl. **335**(2), 1360-1373 (2007)
8. Wang, H: Properties of convergence for ω, q-Bernstein polynomials. J. Math. Anal. Appl. **340**(2), 1096-1108 (2008)
9. Wang, H: Korovkin-type theorem and application. J. Approx. Theory **132**(2), 258-264 (2005)
10. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
11. Ostrovska, S: On the limit q-Bernstein operator. In: Proceedings of the 1st Congress of MASSEE, Mathematica Balkanica, N.S., vol. 18, pp. 165-172 (2004)
12. Ostrovska, S: On the improvement of analytic properties under the limit q-Bernstein operator. J. Approx. Theory **138**(1), 37-53 (2006)
13. Ostrovska, S: On the properties of the limit q-Bernstein operator. Studia Sci. Math. Hung. **48**(2), 160-179 (2011)

doi:10.1186/1029-242X-2012-297
Cite this article as: Ostrovska: Functions whose smoothness is not improved under the limit q-Bernstein operator. *Journal of Inequalities and Applications* 2012 **2012**:297.