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Multiple positive periodic solutions for a food-limited two-species Gilpin-Ayala competition patch system with periodic harvesting terms

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Abstract
By using Mawhin's coincidence degree theory and some inequality techniques, this paper establishes a new sufficient condition on the existence of at least eight positive periodic solutions for a food-limited two-species Gilpin-Ayala competition patch system with periodic harvesting terms. An example is given to illustrate the effectiveness of the result.

1 Introduction

In the past years, the study of population dynamics with harvesting in mathematical bioeconomics, due to its theoretical and practical significance in the optimal management of renewable resources, has attracted much attention [1–8]. Huusko and Hyvarinen in [9] pointed out that 'the dynamics of exploited populations are clearly affected by recruitment and harvesting, and the changes in harvesting induced a tendency to generation cycling in the dynamics of a freshwater fish population.' Recently, some researchers have paid much attention to the investigation of harvesting-induced multiple positive periodic solutions for some population systems under the assumption of periodicity of the parameters by using Mawhin's coincidence degree theory [5–8]. In 1973, Gilpin and Ayala in [10] firstly proposed and studied a few Gilpin-Ayala type competition models. Since then, many papers have been published on the dynamics of Gilpin-Ayala type competition models (for example, see [11–15]).

In this paper, we consider a food-limited two-species Gilpin-Ayala competition patch system with harvesting terms:

$$\begin{cases} x_1'(t) = \frac{x_1(t)}{k_1(t)+c_1(t)x_1(t)}[a_1(t) - a_{11}(t)x_1^{\theta_1}(t) - a_{13}(t)y^{\theta_3}(t)] \\ \qquad\quad + D_1(t)[x_2(t) - x_1(t)] - H_1(t), \\ x_2'(t) = \frac{x_2(t)}{k_2(t)+c_2(t)x_2(t)}[a_2(t) - a_{22}(t)x_2^{\theta_2}(t)] + D_2(t)[x_1(t) - x_2(t)] - H_2(t), \\ y'(t) = \frac{y(t)}{k_3(t)+c_3(t)y(t)}[a_3(t) - a_{33}(t)y^{\theta_3}(t) - a_{31}(t)x_1^{\theta_1}(t)] - H_3(t), \end{cases} \tag{1.1}$$

where x_1 and y are the population densities of species x and y in patch 1, and x_2 is the density of species x in patch 2. Species y is confined to patch 1, while species x can diffuse between two patches due to the spatial heterogeneity and unbalanced food resources.

$D_i(t)$ ($i = 1,2$) are diffusion coefficients of species x. $a_1(t)$ ($a_2(t)$) is the natural growth rate of species x in patch 1 (patch 2), $a_3(t)$ is the natural growth rate of species y, $a_{13}(t)$, $a_{31}(t)$ are the inter-species competition coefficients. $a_{ii}(t)$ ($i = 1,2,3$) are the density-dependent coefficients. $k_i(t)$ ($i = 1,2$) are the population numbers of species x at saturation in patch 1 (patch 2), and $k_3(t)$ is the population number of species y at saturation in patch 1, respectively. $H_i(t)$ ($i = 1,2,3$) denote the harvesting rates. θ_i ($i = 1,2,3$) represent a nonlinear measure of interspecific interference. When $c_i(t) \neq 0$ ($i = 1,2,3$), $\frac{a_i(t)}{k_i(t)c_i(t)}$ ($i = 1,2,3$) are the rate of replacement of mass in the population at saturation (including the replacement of metabolic loss and of dead organisms). In this case, system (1.1) is a food-limited population model. For other food-limited population models, we refer to [16–19].

To our knowledge, few papers have been published on the existence of multiple positive periodic solutions for Gilpin-Ayala type competition patch models. Motivated by the work of Chen [20], we study the existence of multiple positive periodic solutions of (1.1) by using Mawhin's coincidence degree theory. Since system (1.1) involves the diffusion terms, the rates of replacement and the interspecific interference, the methods used in [5–8] are not available to system (1.1).

2 Existence of multiple positive periodic solutions

For the sake of convenience and simplicity, we denote

$$\bar{g} = \frac{1}{T}\int_0^T g(t)\,dt, \qquad g^l = \min_{t\in[0,T]} g(t), \qquad g^u = \max_{t\in[0,T]} g(t),$$

where g is a nonnegative continuous T-periodic function.

Set

$$N_1 = \max\left\{\left[\left(\frac{a_1}{a_{11}}\right)^u\right]^{1/\theta_1}, \left[\left(\frac{a_2}{a_{22}}\right)^u\right]^{1/\theta_2}\right\}, \qquad N_2 = \left[\left(\frac{a_3}{a_{33}}\right)^u\right]^{1/\theta_3}.$$

From now on, we always assume that

(H$_1$) $k_i(t)$, $a_i(t)$, $a_{ii}(t)$, $H_i(t)$, $c_i(t)$ ($i = 1,2,3$), $a_{13}(t)$, $a_{31}(t)$, $D_i(t)$ ($i = 1,2$) are positive continuous T-periodic functions. θ_i ($i = 1,2,3$) are positive constants.

(H$_2$) $\frac{k_1^l}{k_1^l + c_1^u N_1}(\frac{a_1}{k_1})^l > (\frac{a_{13}}{k_1})^u(\frac{a_3}{a_{33}})^u + D_1^u + (1+\theta_1)[(\frac{a_{11}}{k_1})^u]^{\frac{1}{1+\theta_1}}[\frac{H_1^u}{\theta_1}]^{\frac{\theta_1}{1+\theta_1}}$.

(H$_3$) $\frac{k_2^l}{k_2^l + c_2^u N_1}(\frac{a_2}{k_2})^l > D_2^u + (1+\theta_2)[(\frac{a_{22}}{k_2})^u]^{\frac{1}{1+\theta_2}}[\frac{H_2^u}{\theta_2}]^{\frac{\theta_2}{1+\theta_2}}$.

(H$_4$) $\frac{k_3^l}{k_3^l + c_3^u N_2}(\frac{a_3}{k_3})^l > (\frac{a_{31}}{k_3})^u N_1^{\theta_1} + (1+\theta_3)[(\frac{a_{33}}{k_3})^u]^{\frac{1}{1+\theta_3}}[\frac{H_3^u}{\theta_3}]^{\frac{\theta_3}{1+\theta_3}}$.

(H$_5$) $H_i^l > D_i^u N_1$ ($i = 1,2$).

We first make the following preparations [21].

Let X, Z be normed vector spaces, $L : \operatorname{dom} L \subset X \to Z$ be a linear mapping, $N : X \times [0,1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. If we define $L_P : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ as the restriction $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$ of L to $\operatorname{dom} L \cap \operatorname{Ker} P$, then L_P is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega} \times [0,1]$ if $QN(\bar{\Omega} \times [0,1])$ is bounded and $K_P(I - Q)N : \bar{\Omega} \times [0,1] \to X$

is compact, *i.e.*, continuous and such that $K_P(I-Q)N(\bar{\Omega}\times[0,1])$ is relatively compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J:\operatorname{Im} Q\to\operatorname{Ker} L$.

For convenience, we introduce Mawhin's continuation theorem [21, p.29] as follows.

Lemma 2.1 *Let L be a Fredholm mapping of index zero and let $N:\bar{\Omega}\times[0,1]\to Z$ be L-compact on $\bar{\Omega}\times[0,1]$. Suppose*

(a) *$Lu\neq\lambda N(u,\lambda)$ for every $u\in\operatorname{dom} L\cap\partial\Omega$ and every $\lambda\in(0,1)$;*
(b) *$QN(u,0)\neq 0$ for every $u\in\partial\Omega\cap\operatorname{Ker} L$;*
(c) *Brouwer degree $\deg_B(JQN(\cdot,0)|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L,0)\neq 0$.*

Then $Lu=N(u,1)$ has at least one solution in $\operatorname{dom} L\cap\bar{\Omega}$.

Set

$$h(x)=b-ax^{\alpha}-\frac{c}{x},\quad x\in(0,+\infty).$$

Lemma 2.2 *Assume that a, b, c, α are positive constants and*

$$b>(1+\alpha)a^{\frac{1}{1+\alpha}}\left(\frac{c}{\alpha}\right)^{\frac{\alpha}{1+\alpha}}.$$

Then there exist $0<x^-<x^+$ such that

$$\begin{aligned}
&h(x^-)=h(x^+)=0,\\
&h(x)>0\quad\text{for } x\in(x^-,x^+),\qquad h(x)<0\quad\text{for } x\in(0,x^-)\cup(x^+,+\infty),\\
&h'(x^-)>0,\qquad h'(x^+)<0.
\end{aligned}$$

Proof Since

$$h'(x)=-a\alpha x^{\alpha-1}+\frac{c}{x^2}=0,\quad x\in(0,+\infty)$$

implies that

$$x=\left(\frac{c}{a\alpha}\right)^{\frac{1}{1+\alpha}},$$

we have

$$\sup_{x\in(0,+\infty)}h(x)=b-a\left(\frac{c}{a\alpha}\right)^{\frac{\alpha}{1+\alpha}}-\frac{c}{(\frac{c}{a\alpha})^{\frac{1}{1+\alpha}}}=b-(1+\alpha)a^{\frac{1}{1+\alpha}}\left(\frac{c}{\alpha}\right)^{\frac{\alpha}{1+\alpha}}>0.$$

From this, it is easy to see that the assertion holds.

Set

$$\begin{aligned}
M_1(x)={}&\left[\frac{k_1^l}{k_1^l+c_1^uN_1}\left(\frac{a_1}{k_1}\right)^l-\left(\frac{a_{13}}{k_1}\right)^u\left(\frac{a_3}{a_{33}}\right)^u-D_1^u\right]\\
&-\left(\frac{a_{11}}{k_1}\right)^u x^{\theta_1}-\frac{H_1^u}{x},\quad x\in(0,+\infty),
\end{aligned}$$

$$M_2(x) = \left[\frac{k_2^l}{k_2^l + c_2^u N_1}\left(\frac{a_2}{k_2}\right)^l - D_2^u\right] - \left(\frac{a_{22}}{k_2}\right)^u x^{\theta_2} - \frac{H_2^u}{x}, \quad x \in (0, +\infty),$$

$$M_3(x) = \left[\frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_3}{k_3}\right)^l - \left(\frac{a_{31}}{k_3}\right)^u N_1^{\theta_1}\right] - \left(\frac{a_{33}}{k_3}\right)^u x^{\theta_3} - \frac{H_3^u}{x}, \quad x \in (0, +\infty),$$

$$p_i(x) = \overline{\left(\frac{a_i}{k_i}\right)} - \overline{\left(\frac{a_{ii}}{k_i}\right)} x^{\theta_i} - \frac{\bar{H}_i}{x} \quad (i = 1, 2, 3), x \in (0, +\infty),$$

$$m_i(x) = \left(\frac{a_i}{k_i}\right)^u - \frac{k_i^l}{k_i^l + c_i^u N_1}\left(\frac{a_{ii}}{k_i}\right)^l x^{\theta_i} - \frac{H_i^l - D_i^u N_1}{x} \quad (i = 1, 2), x \in (0, +\infty),$$

$$m_3(x) = \left(\frac{a_3}{k_3}\right)^u - \frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_{33}}{k_3}\right)^l x^{\theta_3} - \frac{H_3^l}{x}, \quad x \in (0, +\infty).$$

□

Lemma 2.3 *Assume that* (H_1)-(H_5) *hold. Then the following assertions hold*:

(1) *There exist* $0 < u_i^- < u_i^+$ *such that*

$$M_i(u_i^-) = M_i(u_i^+) = 0,$$

and

$$M_i(x) > 0 \quad \textit{for } x \in (u_i^-, u_i^+), \qquad M_i(x) < 0 \quad \textit{for } x \in (0, u_i^-) \cup (u_i^+, +\infty), i = 1, 2, 3.$$

(2) *There exist* $0 < x_i^- < x_i^+$ *such that*

$$p_i(x_i^-) = p_i(x_i^+) = 0,$$

and

$$p_i(x) > 0 \quad \textit{for } x \in (x_i^-, x_i^+), \qquad p_i(x) < 0 \quad \textit{for } x \in (0, x_i^-) \cup (x_i^+, +\infty), i = 1, 2, 3.$$

(3) *There exist* $0 < l_i^- < l_i^+$ *such that*

$$m_i(l_i^-) = m_i(l_i^+) = 0,$$

and

$$m_i(x) > 0 \quad \textit{for } x \in (l_i^-, l_i^+), \qquad m_i(x) < 0 \quad \textit{for } x \in (0, l_i^-) \cup (l_i^+, +\infty), i = 1, 2, 3.$$

(4)

$$l_i^- < x_i^- < u_i^- < u_i^+ < x_i^+ < l_i^+, \quad i = 1, 2, 3. \tag{2.1}$$

Proof It follows from (H_1)-(H_5) and Lemma 2.2 that the assertions (1)-(3) hold. Noticing that

$$\frac{k_i^l}{k_i^l + c_i^u N_1}\left(\frac{a_{ii}}{k_i}\right)^l < \overline{\left(\frac{a_{ii}}{k_i}\right)} \le \left(\frac{a_{ii}}{k_i}\right)^u \quad (i = 1, 2),$$

$$\frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_{33}}{k_3}\right)^l < \overline{\left(\frac{a_{33}}{k_3}\right)} \le \left(\frac{a_{33}}{k_3}\right)^u,$$
$$\frac{k_1^l}{k_1^l + c_1^u N_1}\left(\frac{a_1}{k_1}\right)^l - \left(\frac{a_{13}}{k_1}\right)^u \left(\frac{a_3}{a_{33}}\right)^u - D_1^u < \overline{\left(\frac{a_1}{k_1}\right)} \le \left(\frac{a_1}{k_1}\right)^u,$$
$$\frac{k_2^l}{k_2^l + c_2^u N_1}\left(\frac{a_2}{k_2}\right)^l - D_2^u < \overline{\left(\frac{a_2}{k_2}\right)} \le \left(\frac{a_2}{k_2}\right)^u,$$
$$\frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_3}{k_3}\right)^l - \left(\frac{a_{31}}{k_3}\right)^u N_1^{\theta_1} < \overline{\left(\frac{a_3}{k_3}\right)} \le \left(\frac{a_3}{k_3}\right)^u,$$
$$H_i^l - D_i^u N_1 < \bar{H}_i \le H_i^u \quad (i = 1, 2),$$
$$H_3^l \le \bar{H}_3 \le H_3^u,$$

we have

$$m_i(x) < p_i(x) < M_i(x), \quad i = 1, 2, 3.$$

It follows from this and the assertions (1)-(3) that the assertion (4) also holds. □

Lemma 2.4 [22] *Assume that* $x \ge 0$, $y \ge 0$, $p > 1$, $q > 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequality holds*:

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \le \frac{x}{p} + \frac{y}{q}.$$

Now, we are ready to state the following main result of this paper.

Theorem 2.1 *Assume that* (H_1)-(H_5) *hold. Then system* (1.1) *has at least eight positive* T*-periodic solutions.*

Proof Since we are concerned with positive solutions of (1.1), we make the change of variables

$$x_j(t) = e^{u_j(t)} \quad (j = 1, 2), \qquad y(t) = e^{u_3(t)}.$$

Then (1.1) is rewritten as

$$\begin{cases} u_1'(t) = \frac{1}{k_1(t)+c_1(t)e^{u_1(t)}}[a_1(t) - a_{11}(t)e^{\theta_1 u_1(t)} - a_{13}(t)e^{\theta_3 u_3(t)}] \\ \qquad + D_1(t)[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1] - \frac{H_1(t)}{e^{u_1(t)}}, \\ u_2'(t) = \frac{1}{k_2(t)+c_2(t)e^{u_2(t)}}[a_2(t) - a_{22}(t)e^{\theta_2 u_2(t)}] + D_2(t)[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1] - \frac{H_2(t)}{e^{u_2(t)}}, \\ u_3'(t) = \frac{1}{k_3(t)+c_3(t)e^{u_3(t)}}[a_3(t) - a_{33}(t)e^{\theta_3 u_3(t)} - a_{31}(t)e^{\theta_1 u_1(t)}] - \frac{H_3(t)}{e^{u_3(t)}}. \end{cases} \tag{2.2}$$

Take

$$X = Z = \{u = (u_1, u_2, u_3)^T \in C(R, R^3) : u_i(t+T) = u_i(t), i = 1, 2, 3\}$$

and define

$$\|u\| = \max_{t\in[0,T]}\bigl|u_1(t)\bigr| + \max_{t\in[0,T]}\bigl|u_2(t)\bigr| + \max_{t\in[0,T]}\bigl|u_3(t)\bigr|, \quad u = (u_1, u_2, u_3)^T \in X \text{ or } Z.$$

Equipped with the above norm $\|\cdot\|$, it is easy to verify that X and Z are Banach spaces.

Set

$$\begin{aligned}
\Delta_1(u,t,\lambda) &= \left[\frac{k_1(t) + (1-\lambda)c_1(t)e^{u_1(t)}}{k_1(t) + c_1(t)e^{u_1(t)}}\right]\left[\frac{a_1(t)}{k_1(t)} - \frac{a_{11}(t)e^{\theta_1 u_1(t)}}{k_1(t)} - \frac{\lambda a_{13}(t)e^{\theta_3 u_3(t)}}{k_1(t)}\right] \\
&\quad + \lambda D_1(t)\left[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1\right] - \frac{H_1(t)}{e^{u_1(t)}}, \\
\Delta_2(u,t,\lambda) &= \left[\frac{k_2(t) + (1-\lambda)c_2(t)e^{u_2(t)}}{k_2(t) + c_2(t)e^{u_2(t)}}\right]\left[\frac{a_2(t)}{k_2(t)} - \frac{a_{22}(t)e^{\theta_2 u_2(t)}}{k_2(t)}\right] \\
&\quad + \lambda D_2(t)\left[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1\right] - \frac{H_2(t)}{e^{u_2(t)}}, \\
\Delta_3(u,t,\lambda) &= \left[\frac{k_3(t) + (1-\lambda)c_3(t)e^{u_3(t)}}{k_3(t) + c_3(t)e^{u_3(t)}}\right]\left[\frac{a_3(t)}{k_3(t)} - \frac{a_{33}(t)e^{\theta_3 u_3(t)}}{k_3(t)} - \frac{\lambda a_{31}(t)e^{\theta_1 u_1(t)}}{k_3(t)}\right] \\
&\quad - \frac{H_3(t)}{e^{u_3(t)}}.
\end{aligned}$$

For any $u \in X$, because of the periodicity, we can easily check that $\Delta_i(u,t,\lambda) \in C(R^2, R)$ $(i = 1,2,3)$ are T-periodic in t.

Let

$$\begin{aligned}
&L : \operatorname{dom} L = \bigl\{u \in X : u \in C\bigl(R, R^3\bigr)\bigr\} \ni u \longmapsto u' \in Z, \\
&P : X \ni u \longmapsto \frac{1}{T}\int_0^T u(t)\,dt \in X, \\
&Q : Z \ni u \longmapsto \frac{1}{T}\int_0^T u(t)\,dt \in Z, \\
&N : X \times [0,1] \ni (u,\lambda) \longmapsto \bigl(\Delta_1(u,t,\lambda), \Delta_2(u,t,\lambda), \Delta_3(u,t,\lambda)\bigr)^T \in Z.
\end{aligned}$$

Here, for any $k \in R^3$, we also identify it as the constant function in X or Z with the constant value k. It is easy to see that

$$\operatorname{Ker} L = R^3, \qquad \operatorname{Im} L = \left\{u \in X : \int_0^T u_i(t)\,dt = 0, i = 1,2,3\right\}$$

is closed in Z, $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 3$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q).$$

Therefore, L is a Fredholm mapping of index zero. On the other hand, $K_p : \operatorname{Im} L \longmapsto \operatorname{dom} L \cap \operatorname{Ker} P$ has the form

$$K_p(u) = \int_0^t u(s)\,ds - \frac{1}{T}\int_0^T \int_0^t u(s)\,ds\,dt.$$

Thus,

$$QN(u,\lambda) = \left(\frac{1}{T}\int_0^T \Delta_1(u,t,\lambda)\,dt, \frac{1}{T}\int_0^T \Delta_2(u,t,\lambda)\,dt, \frac{1}{T}\int_0^T \Delta_3(u,t,\lambda)\,dt\right)^T,$$
$$K_p(I-Q)N(u,\lambda) = \left(\Phi_1(u,t,\lambda), \Phi_2(u,t,\lambda), \Phi_3(u,t,\lambda)\right)^T,$$

where

$$\Phi_j(u,t,\lambda) = \int_0^t \Delta_j(u,s,\lambda)\,ds - \frac{1}{T}\int_0^T\int_0^t \Delta_j(u,s,\lambda)\,ds\,dt - \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T \Delta_j(u,s,\lambda)\,ds, \quad j = 1,2,3.$$

Obviously, QN and $K_p(I-Q)N$ are continuous. By the Arzela-Ascoli theorem, it is not difficult to show that $\overline{K_p(I-Q)N(\overline{\Omega}\times[0,1])}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega}\times[0,1])$ is bounded. Thus, N is L-compact on $\bar{\Omega}\times[0,1]$ with any open bounded set $\Omega \subset X$.

In order to apply Lemma 2.1, we need to find eight appropriate open, bounded subsets Ω_i ($i = 1,2,\ldots,8$) in X.

Corresponding to the operator equation $Lu = \lambda N(u,\lambda)$, $\lambda \in (0,1)$, we have

$$u_1'(t) = \lambda\left[\frac{k_1(t) + (1-\lambda)c_1(t)e^{u_1(t)}}{k_1(t) + c_1(t)e^{u_1(t)}}\right]\left[\frac{a_1(t)}{k_1(t)} - \frac{a_{11}(t)e^{\theta_1 u_1(t)}}{k_1(t)} - \frac{\lambda a_{13}(t)e^{\theta_3 u_3(t)}}{k_1(t)}\right] + \lambda^2 D_1(t)\left[\frac{e^{u_2(t)}}{e^{u_1(t)}} - 1\right] - \frac{\lambda H_1(t)}{e^{u_1(t)}}, \tag{2.3}$$

$$u_2'(t) = \lambda\left[\frac{k_2(t) + (1-\lambda)c_2(t)e^{u_2(t)}}{k_2(t) + c_2(t)e^{u_2(t)}}\right]\left[\frac{a_2(t)}{k_2(t)} - \frac{a_{22}(t)e^{\theta_2 u_2(t)}}{k_2(t)}\right] + \lambda^2 D_2(t)\left[\frac{e^{u_1(t)}}{e^{u_2(t)}} - 1\right] - \frac{\lambda H_2(t)}{e^{u_2(t)}}, \tag{2.4}$$

$$u_3'(t) = \lambda\left[\frac{k_3(t) + (1-\lambda)c_3(t)e^{u_3(t)}}{k_3(t) + c_3(t)e^{u_3(t)}}\right]\left[\frac{a_3(t)}{k_3(t)} - \frac{a_{33}(t)e^{\theta_3 u_3(t)}}{k_3(t)} - \frac{\lambda a_{31}(t)e^{\theta_1 u_1(t)}}{k_3(t)}\right] - \frac{\lambda H_3(t)}{e^{u_3(t)}}. \tag{2.5}$$

Suppose that $(u_1(t), u_2(t), u_3(t))^T$ is a T-periodic solution of (2.3), (2.4) and (2.5) for some $\lambda \in (0,1)$.

Choose $t_i^M, t_i^m \in [0,T]$, $i = 1,2,3$, such that

$$u_i(t_i^M) = \max_{t\in[0,T]} u_i(t), \qquad u_i(t_i^m) = \min_{t\in[0,T]} u_i(t), \quad i = 1,2,3.$$

Then it is clear that

$$u_i'(t_i^M) = 0, \qquad u_i'(t_i^m) = 0, \quad i = 1,2,3.$$

From this and (2.3), (2.4), (2.5), we obtain that

$$\begin{aligned} 0 &= \left[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\right] \\ &\quad \times \left[\frac{a_1(t_1^M) - a_{11}(t_1^M)e^{\theta_1 u_1(t_1^M)}}{k_1(t_1^M)} - \frac{\lambda a_{13}(t_1^M)e^{\theta_3 u_3(t_1^M)}}{k_1(t_1^M)}\right] \\ &\quad + \lambda D_1(t_1^M)\left[\frac{e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}} - 1\right] - \frac{H_1(t_1^M)}{e^{u_1(t_1^M)}}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} 0 &= \left[\frac{k_2(t_2^M) + (1-\lambda)c_2(t_2^M)e^{u_2(t_2^M)}}{k_2(t_2^M) + c_2(t_2^M)e^{u_2(t_2^M)}}\right]\left[\frac{a_2(t_2^M)}{k_2(t_2^M)} - \frac{a_{22}(t_2^M)e^{\theta_2 u_2(t_2^M)}}{k_2(t_2^M)}\right] \\ &\quad + \lambda D_2(t_2^M)\left[\frac{e^{u_1(t_2^M)}}{e^{u_2(t_2^M)}} - 1\right] - \frac{H_2(t_2^M)}{e^{u_2(t_2^M)}}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} 0 &= \left[\frac{k_3(t_3^M) + (1-\lambda)c_3(t_3^M)e^{u_3(t_3^M)}}{k_3(t_3^M) + c_3(t_3^M)e^{u_3(t_3^M)}}\right]\left[\frac{a_3(t_3^M)}{k_3(t_3^M)} - \frac{a_{33}(t_3^M)e^{\theta_3 u_3(t_3^M)}}{k_3(t_3^M)} - \frac{\lambda a_{31}(t_3^M)e^{\theta_1 u_1(t_3^M)}}{k_3(t_3^M)}\right] \\ &\quad - \frac{H_3(t_3^M)}{e^{u_3(t_3^M)}}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} 0 &= \left[\frac{k_1(t_1^m) + (1-\lambda)c_1(t_1^m)e^{u_1(t_1^m)}}{k_1(t_1^m) + c_1(t_1^m)e^{u_1(t_1^m)}}\right] \\ &\quad \times \left[\frac{a_1(t_1^m) - a_{11}(t_1^m)e^{\theta_1 u_1(t_1^m)}}{k_1(t_1^m)} - \frac{\lambda a_{13}(t_1^m)e^{\theta_3 u_3(t_1^m)}}{k(t_1^m)}\right] \\ &\quad + \lambda D_1(t_1^m)\left[\frac{e^{u_2(t_1^m)}}{e^{u_1(t_1^m)}} - 1\right] - \frac{H_1(t_1^m)}{e^{u_1(t_1^m)}}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} 0 &= \left[\frac{k_2(t_2^m) + (1-\lambda)c_2(t_2^m)e^{u_2(t_2^m)}}{k_2(t_2^m) + c_2(t_2^m)e^{u_2(t_2^m)}}\right]\left[\frac{a_2(t_2^m)}{k_2(t_2^m)} - \frac{a_{22}(t_2^m)e^{\theta_2 u_2(t_2^m)}}{k_2(t_2^m)}\right] \\ &\quad + \lambda D_2(t_2^m)\left[\frac{e^{u_1(t_2^m)}}{e^{u_2(t_2^m)}} - 1\right] - \frac{H_2(t_2^m)}{e^{u_2(t_2^m)}}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} 0 &= \left[\frac{k_3(t_3^m) + (1-\lambda)c_3(t_3^m)e^{u_3(t_3^m)}}{k_3(t_3^m) + c_3(t_3^m)e^{u_3(t_3^m)}}\right]\left[\frac{a_3(t_3^m)}{k_3(t_3^m)} - \frac{a_{33}(t_3^m)e^{\theta_3 u_3(t_3^m)}}{k_3(t_3^m)} - \frac{\lambda a_{31}(t_3^m)e^{\theta_1 u_1(t_3^m)}}{k_3(t_3^m)}\right] \\ &\quad - \frac{H_3(t_3^m)}{e^{u_3(t_3^m)}}. \end{aligned} \tag{2.11}$$

Claim A.

$$\max\{u_1(t_1^M), u_2(t_2^M)\} < \ln N_1,$$

and

$$u_3(t_3^M) < \frac{1}{\theta_3}\ln\left(\frac{a_3}{a_{33}}\right)^u = \ln N_2.$$

For $u(t_i^M)$ $(i = 1, 2)$, there are two cases to consider.

Case 1. Assume that $u_1(t_1^M) \geq u_2(t_2^M)$, then $u_1(t_1^M) \geq u_2(t_1^M)$.
From this and (2.6), we have

$$a_1(t_1^M) - a_{11}(t_1^M)e^{\theta_1 u_1(t_1^M)} > 0,$$

which implies

$$e^{\theta_1 u_1(t_1^M)} < \frac{a_1(t_1^M)}{a_{11}(t_1^M)} \leq \left(\frac{a_1}{a_{11}}\right)^u.$$

That is,

$$u_2(t_2^M) \leq u_1(t_1^M) < \frac{1}{\theta_1} \ln\left(\frac{a_1}{a_{11}}\right)^u \leq \ln N_1.$$

Case 2. Assume that $u_1(t_1^M) < u_2(t_2^M)$, then $u_2(t_2^M) > u_1(t_2^M)$.
From this and (2.7), we have

$$a_2(t_2^M) - a_{22}(t_2^M)e^{\theta_2 u_2(t_2^M)} > 0,$$

which implies

$$e^{\theta_2 u_2(t_2^M)} < \frac{a_2(t_2^M)}{a_{22}(t_2^M)} \leq \left(\frac{a_2}{a_{22}}\right)^u.$$

That is,

$$u_1(t_1^M) < u_2(t_2^M) < \frac{1}{\theta_2} \ln\left(\frac{a_2}{a_{22}}\right)^u \leq \ln N_1.$$

Therefore,

$$\max\{u_1(t_1^M), u_2(t_2^M)\} < \ln N_1. \tag{2.12}$$

For $u_3(t_3^M)$, it follows from (2.8) that

$$a_3(t_3^M) - a_{33}(t_3^M)e^{\theta_3 u_3(t_3^M)} > 0,$$

which implies

$$u_3(t_3^M) < \frac{1}{\theta_3} \ln\left(\frac{a_3}{a_{33}}\right)^u = \ln N_2. \tag{2.13}$$

Claim B.

$$u_i(t_i^M) > \ln u_i^+ \quad \text{or} \quad u_i(t_i^M) < \ln u_i^-, \quad i = 1, 2, 3$$

and

$$u_i(t_i^m) > \ln u_i^+ \quad \text{or} \quad u_i(t_i^m) < \ln u_i^-, \quad i = 1, 2, 3.$$

It follows from (2.6) that

$$\begin{aligned}&\left[\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\right]\left[\frac{a_1(t_1^M)}{k_1(t_1^M)}-\frac{a_{11}(t_1^M)e^{\theta_1 u_1(t_1^M)}}{k_1(t_1^M)}\right]-\frac{H_1(t_1^M)}{e^{u_1(t_1^M)}}\\&\quad=\lambda\left[\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\right]\frac{a_{13}(t_1^M)e^{\theta_3 u_3(t_1^M)}}{k_1(t_1^M)}\\&\qquad-\lambda D_1\bigl(t_1^M\bigr)\frac{e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}}+\lambda D_1\bigl(t_1^M\bigr).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left[\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\right]\left[\left(\frac{a_1}{k_1}\right)^l-\left(\frac{a_{11}}{k_1}\right)^u e^{\theta_1 u_1(t_1^M)}\right]-\frac{H_1^u}{e^{u_1(t_1^M)}}\\&\quad<\left[\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\right]\left(\frac{a_{13}}{k_1}\right)^u\left(\frac{a_3}{a_{33}}\right)^u+D_1^u.\end{aligned}$$

From this and noticing that

$$\frac{k_1(t_1^M)}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\le\frac{k_1(t_1^M)+(1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\le 1,$$

we have

$$\begin{aligned}&\frac{k_1(t_1^M)}{k_1(t_1^M)+c_1(t_1^M)e^{u_1(t_1^M)}}\left(\frac{a_1}{k_1}\right)^l-\left(\frac{a_{11}}{k_1}\right)^u e^{\theta_1 u_1(t_1^M)}-\frac{H_1^u}{e^{u_1(t_1^M)}}\\&\quad<\left(\frac{a_{13}}{k_1}\right)^u\left(\frac{a_3}{a_{33}}\right)^u+D_1^u,\end{aligned}$$

which implies

$$\left[\frac{k_1^l}{k_1^l+c_1^uN_1}\left(\frac{a_1}{k_1}\right)^l-\left(\frac{a_{13}}{k_1}\right)^u\left(\frac{a_3}{a_{33}}\right)^u-D_1^u\right]-\left(\frac{a_{11}}{k_1}\right)^u e^{\theta_1 u_1(t_1^M)}-\frac{H_1^u}{e^{u_1(t_1^M)}}<0.$$

From the assertion (1) of Lemma 2.3 and the above inequality, we have

$$u_1\bigl(t_1^M\bigr)>\ln u_1^+\quad\text{or}\quad u_1\bigl(t_1^M\bigr)<\ln u_1^-. \tag{2.14}$$

Similarly, from (2.9), we obtain

$$u_1\bigl(t_1^m\bigr)>\ln u_1^+\quad\text{or}\quad u_1\bigl(t_1^m\bigr)<\ln u_1^-. \tag{2.15}$$

By a similar argument, it follows from (2.7) that

$$\left[\frac{k_2^l}{k_2^l+c_2^uN_1}\left(\frac{a_2}{k_2}\right)^l-D_2^u\right]-\left(\frac{a_{22}}{k_2}\right)^u e^{\theta_2 u_2(t_2^M)}-\frac{H_2^u}{e^{u_2(t_2^M)}}<0.$$

From the assertion (1) of Lemma 2.3 and the above inequality, we have

$$u_2(t_2^M) > \ln u_2^+ \quad \text{or} \quad u_2(t_2^M) < \ln u_2^-. \tag{2.16}$$

Similarly, from (2.10), we obtain

$$u_2(t_2^m) > \ln u_2^+ \quad \text{or} \quad u_2(t_2^m) < \ln u_2^-. \tag{2.17}$$

By a similar argument, it follows from (2.8) and (2.12) that

$$\left[\frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_3}{k_3}\right)^l - \left(\frac{a_{31}}{k_3}\right)^u N_1^{\theta_1}\right] - \left(\frac{a_{33}}{k_3}\right)^u e^{\theta_3 u_3(t_3^M)} - \frac{H_3^u}{e^{u_3(t_3^M)}} < 0.$$

From the assertion (1) of Lemma 2.3 and the above inequality, we have

$$u_3(t_3^M) > \ln u_3^+ \quad \text{or} \quad u_3(t_3^M) < \ln u_3^-. \tag{2.18}$$

Similarly, from (2.11), we obtain

$$u_3(t_3^m) > \ln u_3^+ \quad \text{or} \quad u_3(t_3^m) < \ln u_3^-. \tag{2.19}$$

Claim C.

$$\ln l_i^- < u_i(t_i^M) < \ln l_i^+, \quad i = 1, 2, 3,$$

and

$$\ln l_i^- < u_i(t_i^m) < \ln l_i^+, \quad i = 1, 2, 3.$$

It follows from (2.6) that

$$\left[\frac{k_1(t_1^M) + (1-\lambda)c_1(t_1^M)e^{u_1(t_1^M)}}{k_1(t_1^M) + c_1(t_1^M)e^{u_1(t_1^M)}}\right]\left[\frac{a_1(t_1^M)}{k_1(t_1^M)} - \frac{a_{11}(t_1^M)e^{\theta_1 u_1(t_1^M)}}{k_1(t_1^M)}\right] > \frac{H_1(t_1^M) - D_1(t_1^M)e^{u_2(t_1^M)}}{e^{u_1(t_1^M)}}.$$

Hence, we have

$$\left(\frac{a_1}{k_1}\right)^u - \frac{k_1^l}{k_1^l + c_1^u N_1}\left(\frac{a_{11}}{k_1}\right)^l e^{\theta_1 u_1(t_1^M)} - \frac{H_1^l - D_1^u N_1}{e^{u_1(t_1^M)}} > 0.$$

From the assertion (3) of Lemma 2.3 and the above inequality, we have

$$\ln l_1^- < u_1(t_1^M) < \ln l_1^+. \tag{2.20}$$

Similarly, from (2.9), we obtain

$$\ln l_1^- < u_1(t_1^m) < \ln l_1^+. \tag{2.21}$$

By a similar argument, it follows from (2.7) that

$$\left(\frac{a_2}{k_2}\right)^u - \frac{k_2^l}{k_2^l + c_2^u N_1}\left(\frac{a_{22}}{k_2}\right)^l e^{\theta_2 u_2(t_2^M)} - \frac{H_2^l - D_2^u N_1}{e^{u_2(t_2^M)}} > 0.$$

From the assertion (3) of Lemma 2.3 and the above inequality, we have

$$\ln l_2^- < u_2\left(t_2^M\right) < \ln l_2^+. \tag{2.22}$$

Similarly, from (2.10), we obtain

$$\ln l_2^- < u_2\left(t_2^m\right) < \ln l_2^+. \tag{2.23}$$

By a similar argument, it follows from (2.8) that

$$\left(\frac{a_3}{k_3}\right)^u - \frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_{33}}{k_3}\right)^l e^{\theta_3 u_3(t_3^M)} - \frac{H_3^l}{e^{u_3(t_3^M)}} > 0.$$

From the assertion (3) of Lemma 2.3 and the above inequality, we have

$$\ln l_3^- < u_3\left(t_3^M\right) < \ln l_3^+. \tag{2.24}$$

Similarly, from (2.11), we obtain

$$\ln l_3^- < u_3\left(t_3^m\right) < \ln l_3^+. \tag{2.25}$$

It follows from (2.14), (2.15), (2.20), (2.21) that

$$u_1\left(t_1^M\right) \in \left(\ln l_1^-, \ln u_1^-\right) \cup \left(\ln u_1^+, \ln l_1^+\right), \tag{2.26}$$

$$u_1\left(t_1^m\right) \in \left(\ln l_1^-, \ln u_1^-\right) \cup \left(\ln u_1^+, \ln l_1^+\right). \tag{2.27}$$

It follows from (2.16), (2.17), (2.22), (2.23) that

$$u_2\left(t_2^M\right) \in \left(\ln l_2^-, \ln u_2^-\right) \cup \left(\ln u_2^+, \ln l_2^+\right), \tag{2.28}$$

$$u_2\left(t_2^m\right) \in \left(\ln l_2^-, \ln u_2^-\right) \cup \left(\ln u_2^+, \ln l_2^+\right). \tag{2.29}$$

It follows from (2.18), (2.19), (2.24), (2.25) that

$$u_3\left(t_3^M\right) \in \left(\ln l_3^-, \ln u_3^-\right) \cup \left(\ln u_3^+, \ln l_3^+\right), \tag{2.30}$$

$$u_3\left(t_3^m\right) \in \left(\ln l_3^-, \ln u_3^-\right) \cup \left(\ln u_3^+, \ln l_3^+\right). \tag{2.31}$$

Clearly, $l_i^\pm$, $u_i^\pm$ $(i = 1,2,3)$ are independent of λ. Now, let us consider $QN(u,0)$ with $u = (u_1,u_2,u_3)^T \in R^3$. Note that

$$QN(u,0) = \begin{pmatrix} \overline{(\frac{a_1}{k_1})} - \overline{(\frac{a_{11}}{k_1})} e^{\theta_1 u_1} - \frac{\bar{H}_1}{e^{u_1}} \\ \overline{(\frac{a_2}{k_2})} - \overline{(\frac{a_{22}}{k_2})} e^{\theta_2 u_2} - \frac{\bar{H}_2}{e^{u_2}} \\ \overline{(\frac{a_3}{k_3})} - \overline{(\frac{a_{33}}{k_3})} e^{\theta_3 u_3} - \frac{\bar{H}_3}{e^{u_3}} \end{pmatrix}.$$

Letting $QN(u,0) = 0$, we have

$$\overline{\left(\frac{a_1}{k_1}\right)} - \overline{\left(\frac{a_{11}}{k_1}\right)} e^{\theta_1 u_1} - \frac{\bar{H}_1}{e^{u_1}} = 0, \tag{2.32}$$

$$\overline{\left(\frac{a_2}{k_2}\right)} - \overline{\left(\frac{a_{22}}{k_2}\right)} e^{\theta_2 u_2} - \frac{\bar{H}_2}{e^{u_2}} = 0, \tag{2.33}$$

$$\overline{\left(\frac{a_3}{k_3}\right)} - \overline{\left(\frac{a_{33}}{k_3}\right)} e^{\theta_3 u_3} - \frac{\bar{H}_3}{e^{u_3}} = 0. \tag{2.34}$$

Therefore, it follows from the assertion (2) of Lemma 2.3 that $QN(u,0) = 0$ has eight distinct solutions:

$$\tilde{u}_1 = \left(\ln x_1^+, \ln x_2^+, \ln x_3^+\right)^T, \qquad \tilde{u}_2 = \left(\ln x_1^+, \ln x_2^-, \ln x_3^+\right)^T, \tag{2.35}$$

$$\tilde{u}_3 = \left(\ln x_1^-, \ln x_2^+, \ln x_3^+\right)^T, \qquad \tilde{u}_4 = \left(\ln x_1^-, \ln x_2^-, \ln x_3^+\right)^T, \tag{2.36}$$

$$\tilde{u}_5 = \left(\ln x_1^+, \ln x_2^+, \ln x_3^-\right)^T, \qquad \tilde{u}_6 = \left(\ln x_1^+, \ln x_2^-, \ln x_3^-\right)^T, \tag{2.37}$$

$$\tilde{u}_7 = \left(\ln x_1^-, \ln x_2^+, \ln x_3^-\right)^T, \qquad \tilde{u}_8 = \left(\ln x_1^-, \ln x_2^-, \ln x_3^-\right)^T. \tag{2.38}$$

Let

$$\Omega_1 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \max_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+), \\ \min_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+). \end{array} \right\},$$

$$\Omega_2 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \max_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+), \\ \min_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+). \end{array} \right\},$$

$$\Omega_3 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \max_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+), \\ \min_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+). \end{array} \right\},$$

$$\Omega_4 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \max_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+), \\ \min_{t\in[0,T]} u_3(t) \in (\ln u_3^+, \ln l_3^+). \end{array} \right\},$$

$$\Omega_5 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \max_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-), \\ \min_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-). \end{array} \right\},$$

$$\Omega_6 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \min_{t\in[0,T]} u_1(t) \in (\ln u_1^+, \ln l_1^+), \\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \max_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-), \\ \min_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-). \end{array} \right\},$$

$$\Omega_7 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \max_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \min_{t\in[0,T]} u_2(t) \in (\ln u_2^+, \ln l_2^+), \\ \max_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-), \\ \min_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-). \end{array} \right\},$$

$$\Omega_8 = \left\{ u = (u_1, u_2, u_3)^T \in X \,\middle|\, \begin{array}{l} \max_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \min_{t\in[0,T]} u_1(t) \in (\ln l_1^-, \ln u_1^-), \\ \max_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \min_{t\in[0,T]} u_2(t) \in (\ln l_2^-, \ln u_2^-), \\ \max_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-), \\ \min_{t\in[0,T]} u_3(t) \in (\ln l_3^-, \ln u_3^-). \end{array} \right\}.$$

Then $\Omega_1, \Omega_2, \dots, \Omega_8$ are bounded open subsets of X. It follows from (2.1) and (2.35)-(2.38) that $\tilde{u}_i \in \Omega_i$ $(i = 1, 2, \dots, 8)$. From (2.1), (2.26)-(2.31), it is easy to see that $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ $(i, j = 1, 2, \dots, 8,\ i \neq j)$ and Ω_i satisfies (a) in Lemma 2.1 for $i = 1, 2, \dots, 8$. Moreover, $QN(u, 0) \neq 0$ for $u \in \partial\Omega_i \cap \operatorname{Ker} L$. By Lemma 2.2, a direct computation gives

$$\deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_1 \cap \operatorname{Ker}L, 0\} = -1, \qquad \deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_2 \cap \operatorname{Ker}L, 0\} = 1,$$
$$\deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_3 \cap \operatorname{Ker}L, 0\} = 1, \qquad \deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_4 \cap \operatorname{Ker}L, 0\} = -1,$$
$$\deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_5 \cap \operatorname{Ker}L, 0\} = 1, \qquad \deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_6 \cap \operatorname{Ker}L, 0\} = -1,$$
$$\deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_7 \cap \operatorname{Ker}L, 0\} = -1, \qquad \deg_B\{JQN(\cdot,0)|_{\operatorname{Ker}L}, \Omega_8 \cap \operatorname{Ker}L, 0\} = 1.$$

Here, J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 2.1. Hence, (2.2) has at least eight T-periodic solutions $(u_1^i(t), u_2^i(t), u_3^i(t))^T$ $(i = 1, 2, \dots, 8)$ and $(u_1^i, u_2^i, u_3^i)^T \in \operatorname{dom} L \cap \bar{\Omega}_i$. Obviously, $(u_1^i, u_2^i, u_3^i)^T$ $(i = 1, 2, \dots, 8)$ are different. Let $x_j^i(t) = e^{u_j^i(t)}$ $(j = 1, 2)$, $y^i(t) = e^{u_3^i(t)}$ $(i = 1, 2, \dots, 8)$.

Then $(x_1^i(t), x_2^i(t), y^i(t))^T$ $(i = 1, 2, \ldots, 8)$ are eight different positive T-periodic solutions of (1.1). The proof is complete. □

Corollary 2.1 *In addition to* (H_1), (H_5), *assume further that the following conditions hold*:

$(\mathrm{H}_2)^*$ $\frac{k_1^l}{k_1^l + c_1^u N_1}(\frac{a_1}{k_1})^l > (\frac{a_{13}}{k_1})^u (\frac{a_3}{a_{33}})^u + D_1^u + (\frac{a_{11}}{k_1})^u + H_1^u$.

$(\mathrm{H}_3)^*$ $\frac{k_2^l}{k_2^l + c_2^u N_1}(\frac{a_2}{k_2})^l > D_2^u + (\frac{a_{22}}{k_2})^u + H_2^u$.

$(\mathrm{H}_4)^*$ $\frac{k_3^l}{k_3^l + c_3^u N_2}(\frac{a_3}{k_3})^l > (\frac{a_{31}}{k_3})^u N_1^{\theta_1} + (\frac{a_{33}}{k_3})^u + H_3^u$.

Then system (1.1) *has at least eight positive* T*-periodic solutions*.

Proof By Lemma 2.4, we have

$$(1 + \theta_i)\left[\left(\frac{a_{ii}}{k_i}\right)^u\right]^{\frac{1}{1+\theta_i}} \left[\frac{H_i^u}{\theta_i}\right]^{\frac{\theta_i}{1+\theta_i}} \le \left(\frac{a_{ii}}{k_i}\right)^u + H_i^u, \quad i = 1, 2, 3.$$

Therefore, the conditions in Theorem 2.1 are satisfied. □

Example 2.2 In (1.1), take

$$T = 4, \qquad \theta_1 = \theta_2 = \theta_3 = 0.5,$$

$$k_1(t) = 4 + \sin(0.5\pi t), \qquad k_2(t) = 3 + \sin(0.5\pi t), \qquad k_3(t) = 1.5 + 0.5\sin(0.5\pi t),$$

$$c_i(t) = 0.02 + 0.02\sin(0.5\pi t) \quad (i = 1, 2, 3), \qquad D_1(t) = D_2(t) = \frac{1 + \sin^2(0.5\pi t)}{2000},$$

$$H_1(t) = \frac{1 + \sin^2(0.5\pi t)}{24}, \qquad H_2(t) = \frac{1 + \sin^2(0.5\pi t)}{15}, \qquad H_3(t) = \frac{1 + \sin^2(0.5\pi t)}{200},$$

$$a_1(t) = \left[4 + \sin(0.5\pi t)\right]^2, \qquad a_{11}(t) = \frac{[4 + \sin(0.5\pi t)]^2}{5},$$

$$a_{13}(t) = \frac{[4 + \sin(0.5\pi t)]^2}{40},$$

$$a_2(t) = \left[3 + \sin(0.5\pi t)\right]^2, \qquad a_{22}(t) = \frac{[3 + \sin(0.5\pi t)]^2}{5},$$

$$a_3(t) = \left[1.5 + 0.5\sin(0.5\pi t)\right]^2, \qquad a_{33}(t) = \frac{[1.5 + 0.5\sin(0.5\pi t)]^2}{5},$$

$$a_{31}(t) = \frac{[1.5 + 0.5\sin(0.5\pi t)]^2}{200}.$$

Then we have

$$k_1^l = 3, \qquad k_2^l = 2, \qquad k_3^l = 1, \qquad c_i^u = 0.04 \quad (i = 1, 2, 3), \qquad D_1^u = D_2^u = \frac{1}{1000},$$

$$H_1^u = \frac{1}{12}, \qquad H_2^u = \frac{2}{15}, \qquad H_3^u = \frac{1}{100}, \qquad H_1^l = \frac{1}{24}, \qquad H_2^l = \frac{1}{15},$$

$$N_1 = 25, \qquad N_2 = 25,$$

$$\left(\frac{a_1}{k_1}\right)^l = 3, \qquad \left(\frac{a_{11}}{k_1}\right)^u = 1, \qquad \left(\frac{a_{13}}{k_1}\right)^u = \frac{1}{8},$$

$$\left(\frac{a_2}{k_2}\right)^l = 2, \qquad \left(\frac{a_{22}}{k_2}\right)^u = \frac{4}{5},$$
$$\left(\frac{a_3}{k_3}\right)^l = 1, \qquad \left(\frac{a_{33}}{k_3}\right)^u = \frac{2}{5}, \qquad \left(\frac{a_{31}}{k_3}\right)^u = 0.01, \qquad \left(\frac{a_3}{a_{33}}\right)^u = 5.$$

Therefore,

$$\frac{k_1^l}{k_1^l + c_1^u N_1}\left(\frac{a_1}{k_1}\right)^l = \frac{9}{4} > 2, \qquad \left(\frac{a_{13}}{k_1}\right)^u\left(\frac{a_3}{a_{33}}\right)^u + D_1^u + \left(\frac{a_{11}}{k_1}\right)^u + H_1^u = \frac{41}{24} + \frac{1}{1000} < 2,$$
$$\frac{k_2^l}{k_2^l + c_2^u N_1}\left(\frac{a_2}{k_2}\right)^l = \frac{4}{3} > 1, \qquad D_2^u + \left(\frac{a_{22}}{k_2}\right)^u + H_2^u = \frac{14}{15} + \frac{1}{1000} < 1,$$
$$\frac{k_3^l}{k_3^l + c_3^u N_2}\left(\frac{a_3}{k_3}\right)^l = \frac{1}{2}, \qquad \left(\frac{a_{31}}{k_3}\right)^u N_1^{\theta_1} + \left(\frac{a_{33}}{k_3}\right)^u + H_3^u = 0.46,$$
$$D_i^u N_1 = \frac{1}{40}, \quad i = 1, 2.$$

Hence, the conditions in Corollary 2.1 are satisfied. By Corollary 2.1, system (1.1) has at least eight positive four-periodic solutions.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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