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Degree of approximation of the conjugate of signals (functions) belonging to Lip(α, r)-class by $(C,1)(E,q)$ means of conjugate trigonometric Fourier series

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Abstract

In this paper, we determine the degree of approximation of the conjugate of 2π-periodic signals (functions) belonging to Lip(α, r) ($0 < \alpha \leq 1$, $r \geq 1$)-class by using Cesàro-Euler $(C,1)(E,q)$ means of their conjugate trigonometric Fourier series. Our result generalizes the result of Lal and Singh (Tamkang J. Math. 33(3):269-274, 2002).

MSC: 41A10

Keywords: conjugate Fourier series; Lip(α, r)-class; $(C,1)(E,q)$ means

1 Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with $\{s_n\}$, the sequence of its nth partial sum. The sequence-to-sequence transform

$$C_n^1 = \frac{1}{n+1}\sum_{k=0}^{n} s_k, \quad n = 0,1,2,\ldots, \tag{1.1}$$

defines the Cesàro means of order one of $\{s_n\}$. The series $\sum_{n=0}^{\infty} u_n$ is said to be $(C,1)$ summable to s, if $\lim_{n\to\infty} C_n^1 = s$. The sequence-to-sequence transform

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} s_k, \quad q > 0, n = 0,1,2,\ldots, \tag{1.2}$$

defines the Euler means of order $q > 0$ of $\{s_n\}$. By super imposing the $(C,1)$ means on (E,q) means of $\{s_n\}$, we get $(C,1)(E,q)$ means of $\{s_n\}$ denoted by $C_n^1 E_n^q$ and defined by

$$C_n^1 E_n^q = \frac{1}{n+1}\sum_{k=0}^{n} E_k^q = \frac{1}{n+1}\sum_{k=0}^{n} (q+1)^{-k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu} s_\nu. \tag{1.3}$$

The series $\sum_{n=0}^{\infty} u_n$ is said to be $(C,1)(E,q)$ summable to s, if $\lim_{n\to\infty} C_n^1 E_n^q = s$.

For a given 2π-periodic Lebesgue integrable signal (function), let

$$s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \tag{1.4}$$

denote the $(n+1)$th partial sum, called trigonometric polynomial of degree n (or order n), of the Fourier series of $f \in L_1[-\pi,\pi]$.

The conjugate of Fourier series of f is defined by

$$\sum_{k=1}^{\infty}(b_k \cos kx - a_k \sin kx), \tag{1.5}$$

and its nth partial sum is defined as

$$\widetilde{s_n}(f;x) = \sum_{k=1}^{n}(b_k \cos kx - a_k \sin kx) \tag{1.6}$$

The conjugate of f denoted by $\widetilde{f}$ is defined by

$$2\pi\widetilde{f}(x) = -\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi}\psi(t)\cot(t/2)\,dt,$$

where $\psi(t) = f(x+t) - f(x-t)$ [1, p.131].

A function $f \in \operatorname{Lip}\alpha$, if

$$|f(x+t) - f(x)| = O(t^{\alpha}),$$

and $f \in \operatorname{Lip}(\alpha, r)$ if

$$\left(\int_0^{2\pi}|f(x+t) - f(x)|^r\,dx\right)^{1/r} = O(t^{\alpha}), \quad 0 < \alpha \le 1, r \ge 1.$$

The L_r-norm for $f \in L_r[-\pi,\pi]$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi}|f(x)|^r\,dx\right)^{1/r}, \quad r \ge 1.$$

The L_∞-norm is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}.$$

The degree of approximation $E_n(f)$ of a function $f \in \operatorname{Lip}(\alpha, r)$ by trigonometric polynomials $T_n(x)$ of degree n is given by

$$E_n(f) = \min_{T_n}\|f - T_n\|_r.$$

This method of approximation is called trigonometric Fourier approximation (tfa). We also write

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1/2)t}{\sin(t/2)}$$

and $\tau = [1/t]$, the integral part of $1/t$.

2 Known result

Various investigators such as Dhakal [2], Lal and Singh [3], Mittal *et al.* [4, 5], Nigam [6], Qureshi [7, 8] have studied the degree of approximation in various function spaces such as $\operatorname{Lip}\alpha$, $\operatorname{Lip}(\alpha, r)$, $\operatorname{Lip}(\xi(t), r)$ and weighted $(L_r, \xi(t))$ by using triangular matrix summability and product summability $(C,1)(E,1)$, $(N,p_n)(E,1)$. Recently, Lal and Singh [3] have determined the degree of approximation of the conjugate of $f \in \operatorname{Lip}(\alpha, r)$ by $(C,1)(E,1)$ means of conjugate Fourier series. Lal and Singh [3] have proved the following.

Theorem 1 [3] *If $f : R \to R$ is a 2π-periodic and $\operatorname{Lip}(\alpha, r)$ function, then the degree of approximation of its conjugate function $\widetilde{f}(x)$ by the $(C,1)(E,1)$ product means of conjugate series of Fourier series of f satisfies, for $n = 0, 1, 2, \ldots$,*

$$M_n(\widetilde{f}) = \operatorname{Min}\left\|(CE)_n^1 - \widetilde{f}\right\|_r = O\left(n^{-\alpha+1/r}\right), \tag{2.1}$$

where

$$(CE)_n^1 = \frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}S_i\right),$$

is $(C,1)(E,1)$ means of series (1.5).

3 Main result

Recently, Nigam and Sharma [9] have studied the degree of approximation of functions belonging to $\operatorname{Lip}(\xi(t), r)$-class through $(C,1)(E,q)$ means of their Fourier series. In this paper, we use the $(C,1)(E,q)$ means of conjugate Fourier series of $f \in \operatorname{Lip}(\alpha, r)$ to determine the degree of approximation of the conjugate of f, which in turn generalizes the result of Lal and Singh [3]. More precisely we prove

Theorem 2 *Let $f(x)$ be a 2π-periodic, Lebesgue integrable function and belong to the $\operatorname{Lip}(\alpha, r)$-class with $r \geq 1$ and $\alpha r \geq 1$. Then the degree of approximation of $\widetilde{f}(x)$, the conjugate of $f(x)$ by $(C,1)(E,q)$ means of its conjugate Fourier series is given by*

$$\left\|C_n^1 E_n^q - \widetilde{f}\right\|_r = O\left(n^{-\alpha+1/r}\right), \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

provided

$$\left(\int_0^{\pi/(n+1)}\left(\left|\psi(t)\right|/t^\alpha\right)^r dt\right)^{1/r} = O\left((n+1)^{-1}\right), \tag{3.2}$$

$$\left(\int_{\pi/(n+1)}^{\pi}\left(t^{-\delta}\left|\psi(t)\right|/t^\alpha\right)^r dt\right)^{1/r} = O\left((n+1)^{\delta}\right), \tag{3.3}$$

where δ is an arbitrary number such that $(\alpha + \delta)s + 1 < 0$ and $1/r + 1/s = 1$ for $r > 1$.

Remark 1 The authors have used conditions $(\int_0^{\pi/(n+1)} |\frac{t\psi(t)}{t^\alpha}|^r\,dt)^{1/r} = O(1)$ implied by (3.2) and (3.3), but not mentioned in the statement of Theorem 1 [3, pp.271-272].

4 Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 $|K_n(t)| = O(1/t) + O((n+1)t)$ *for* $0 \le t \le \pi/(n+1) \le \pi/(\nu+1)$.

Proof

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1/2)t}{\sin(t/2)}\right| \\
&= \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1-1/2)t}{\sin(t/2)}\right| \\
&\le \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\left|\frac{\cos(\nu+1)t\cos(t/2)+\sin(\nu+1)t\sin(t/2)}{\sin(t/2)}\right| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\big[O(1/t)+O\big(\sin(\nu+1)t\big)\big] \\
&= O\left[\frac{1}{(n+1)t}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right] \\
&\quad + O\left[\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}(\nu+1)t\right] \\
&= O\left[\frac{1}{(n+1)t}(n+1)\right] + O\left[\frac{1}{(n+1)}(n+1)(n+1)t\right] \\
&= O(1/t) + O\big((n+1)t\big),
\end{aligned}$$

in view of $\sin(\nu+1)t \le (\nu+1)t$ for $0 \le t < \pi/(\nu+1)$ and $(\sin(t/2))^{-1} < \pi/t$ for $0 < t \le \pi$ [10, p.247]. □

Lemma 2 $|K_n(t)| = O(1/t) + O(1)$ *for* $\pi/(\nu+1) \le t \le \pi$.

Proof

$$\begin{aligned}
|K_n(t)| &\le \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1/2)t}{\sin(t/2)}\right| \\
&= \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos(\nu+1-1/2)t}{\sin(t/2)}\right| \\
&\le \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\left|\frac{\cos(\nu+1)t\cos(t/2)+\sin(\nu+1)t\sin(t/2)}{\sin(t/2)}\right| \\
&= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\big[O(1/t)+O(1)\big] \\
&= O\left[\frac{1}{(n+1)t}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right]
\end{aligned}$$

$$\begin{aligned}
&+ O\left[\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\right] \\
&= O\left[\frac{1}{(n+1)t}(n+1)\right] + O\left[\frac{1}{(n+1)}(n+1)\right] \\
&= O(1/t) + O(1),
\end{aligned}$$

in view of $|\sin(\nu+1)t| \leq 1$ and $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$ [10, p.247]. □

5 Proof of Theorem 2

The integral representation of $\widetilde{s_n}(f;x)$ is given by

$$\widetilde{s_n}(f;x) = -\frac{1}{\pi}\int_0^{\pi}\psi(t)\frac{\cos(t/2)-\cos(n+1/2)t}{2\sin(t/2)}\,dt.$$

Therefore, we have

$$\widetilde{s_n}(f;x) - \widetilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+1/2)t}{\sin(t/2)}\,dt.$$

Now, denoting $(C,1)(E,q)$ transform of $\widetilde{s_n}(f;x)$ by $C_n^1E_n^q$, we write

$$\begin{aligned}
C_n^1E_n^q - \widetilde{f} &= \frac{1}{2\pi(n+1)}\left[\sum_{k=0}^{n}\frac{1}{(1+q)^k}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\cos(\nu+1/2)t\,dt\right] \\
&= \left[\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right]\psi(t)K_n(t)\,dt = I_1 + I_2, \quad \text{say.} \qquad (5.1)
\end{aligned}$$

Using Lemma 1, Hölder's inequality, condition (3.2) and Minkowiski's inequality, we have

$$\begin{aligned}
|I_1| &= \int_0^{\pi/(n+1)}\left|\psi(t)\right|\left|K_n(t)\right|dt \\
&\leq \left[\int_0^{\pi/(n+1)}\left(\left|\psi(t)\right|/t^{\alpha}\right)^r dt\right]^{1/r}\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left(t^{\alpha}\left|K_n(t)\right|\right)^s dt\right]^{1/s} \\
&= O\left((n+1)^{-1}\right)\left[\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}\left(t^{\alpha-1}+(n+1)t^{\alpha+1}\right)^s dt\right]^{1/s} \\
&= O\left((n+1)^{-1}\right)\left[\left(\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}t^{(\alpha-1)s}\,dt\right)^{1/s} + \left(\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)}(n+1)t^{(\alpha+1)s}\,dt\right)^{1/s}\right] \\
&= O\left((n+1)^{-1}\right)\left[(n+1)^{-\alpha+1-1/s} + (n+1)(n+1)^{-\alpha-1-1/s}\right] \\
&= O\left((n+1)^{-1}\right)\left[(n+1)^{-\alpha+1/r} + (n+1)(n+1)^{-\alpha-1-1+1/r}\right] \\
&= O\left[(n+1)^{-\alpha+1/r-1} + (n+1)^{-\alpha-2+1/r}\right] \\
&= O\left((n+1)^{-\alpha-1+1/r}\right). \qquad (5.2)
\end{aligned}$$

Now, we consider

$$|I_2| \leq \int_{\pi/(n+1)}^{\pi}\left|\psi(t)\right|\left|K_n(t)\right|dt.$$

Using Lemma 2, Hölder's inequality, condition (3.3) and Minkowiski's inequality, we have

$$\begin{aligned}
|I_2| &\le \left[\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{t^{\alpha}}\right)^r dt\right]^{1/r}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{\alpha}|K_n(t)|}{t^{-\delta}}\right)^s dt\right]^{1/s} \\
&= O\big((n+1)^{\delta}\big)\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{\alpha}}{t^{-\delta}}\big(O(1/t)+O(1)\big)\right)^s dt\right]^{1/s} \\
&= O\big((n+1)^{\delta}\big)\left[\int_{\pi/(n+1)}^{\pi}\big(t^{\alpha+\delta-1}+t^{\alpha+\delta}\big)^s dt\right]^{1/s} \\
&= O\big((n+1)^{\delta}\big)\left[\left(\int_{\pi/(n+1)}^{\pi} t^{(\alpha+\delta-1)s}\,dt\right)^{1/s}+\left(\int_{\pi/(n+1)}^{\pi} t^{(\alpha+\delta)s}\,dt\right)^{1/s}\right] \\
&= O\big((n+1)^{\delta}\big)\big[(n+1)^{(-\alpha-\delta+1)-1/s}+(n+1)^{(-\alpha-\delta)-1/s}\big] \quad \big((\alpha+\delta)s+1<0\big) \\
&= O\big[(n+1)^{-\alpha+1-1/s}+(n+1)^{-\alpha-1/s}\big] \\
&= O\big[(n+1)^{-\alpha+1/r}+(n+1)^{-\alpha-1+1/r}\big] = O\big[(n+1)^{-\alpha+1/r}\big(1+(n+1)^{-1}\big)\big] \\
&= O\big((n+1)^{-\alpha+1/r}\big).
\end{aligned} \tag{5.3}$$

Combining (5.1)-(5.3), we have

$$\big|C_n^1E_n^q-\widetilde{f}\big| = O\big((n+1)^{-\alpha+1/r}\big).$$

Hence,

$$\big\|C_n^1E_n^q-\widetilde{f}\big\|_r = \left(\int_0^{2\pi}\big|C_n^1E_n^q-\widetilde{f}(x)\big|^r dx\right)^{1/r} = O\big(n^{-\alpha+1/r}\big).$$

This completes the proof of Theorem 2.

Remark 2 The proof of Theorem 2 for $r = 1$, *i.e.*, $s = \infty$, can be written by using sup norm while using Hölder's inequality.

6 Corollaries

Corollary 1 *When* $q = 1$, *then* $(C,1)(E,q)$ *means reduces to* $(C,1)(E,1)$ *means.*
Hence, Theorem 2 *reduces to Theorem* 1.

Corollary 2 *If* $f : R \to R$ *is a* 2π*-periodic, Lebesgue integrable and belonging to the* $\mathrm{Lip}\,\alpha$ $(0 < \alpha \le 1)$ *class, then the degree of approximation of* $\widetilde{f}(x)$, *the conjugate of* $f(x) \in \mathrm{Lip}\,\alpha$, *with* $0 < \alpha \le 1$ *by* $(C,1)(E,q)$ *means of its Fourier series is given by*

$$\big\|C_n^1E_n^q-\widetilde{f}\big\|_\infty = O\big(n^{-\alpha}\big) \quad \textit{for } n = 0,1,2,\ldots.$$

Proof If $r \to \infty$ in Theorem 2, then for $0 < \alpha < 1$,

$$\big\|C_n^1E_n^q-\widetilde{f}(x)\big\|_\infty = O\big(n^{-\alpha}\big).$$

For $\alpha = 1$, we can write an independent proof by using $\alpha = 1$ and $\psi(t) = O(t)$ in I_1 and I_2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SS has identified the problem of this paper and US has suggested the solution and corrected the manuscript written by SS.

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