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Volume-preserving diffeomorphisms with inverse shadowing

Manseob Lee*

*Correspondence:
lmsds@mokwon.ac.kr
Department of Mathematics,
Mokwon University, Daejeon,
302-729, Korea

Abstract

Let f be a volume-preserving diffeomorphism of a closed C^∞ n -dimensional Riemannian manifold M . In this paper, we prove the equivalence between the following conditions:

- f belongs to the C^1 -interior of the set of volume-preserving diffeomorphisms which satisfy the inverse shadowing property with respect to the continuous methods,
- f belongs to the C^1 -interior of the set of volume-preserving diffeomorphisms which satisfy the weak inverse shadowing property with respect to the continuous methods,
- f belongs to the C^1 -interior of the set of volume-preserving diffeomorphisms which satisfy the orbital inverse shadowing property with respect to the continuous methods,
- f is Anosov.

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1 Introduction

Let M be a closed C^∞ n -dimensional Riemannian manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 -topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM . Let $f : M \rightarrow M$ be a diffeomorphism, and let $\Lambda \subset M$ be a closed f -invariant set.

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ ($-\infty \leq a < b \leq \infty$) in M is called a δ -pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$. We say that f has the *shadowing property* on Λ if for any $\epsilon > 0$, there is $\delta > 0$ such that for any δ -pseudo orbit $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ of f , there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for $i \in \mathbb{Z}$. Note that in this definition, the shadowing point $y \in M$ is not necessarily contained in Λ . We say that f belongs to the *C^1 -interior shadowing property* if there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property.

The shadowing property usually plays an important role in the investigation of stability theory and ergodic theory [1].

Now, we introduce the notion of the inverse shadowing property which is a 'dual' notion of the shadowing property. The inverse shadowing property was introduced by Corless and Pilyugin in [2], and the qualitative theory of dynamical systems with the property was

developed by various authors (see [2–7]). In this paper, we introduce the various inverse shadowing properties.

Let $M^{\mathbb{Z}}$ be the space of all two-sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in M$, endowed with the product topology. For any $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ -pseudo orbits of f . A mapping $\varphi : M \rightarrow \Phi_f(\delta) \subset M^{\mathbb{Z}}$ is said to be a δ -method for f if $\varphi(x)_0 = x$, and each $\varphi(x)$ is a δ -pseudo orbit of f through x , where $\varphi(x)_0$ denotes the 0th component of $\varphi(x)$. For convenience, write $\varphi(x)$ for $\{\varphi(x)_k\}_{k \in \mathbb{Z}}$. The set of all δ -methods for f will be denoted by $\mathcal{T}_0(f, \delta)$. We say that φ is a *continuous δ -method* for f if φ is continuous. The set of all continuous δ -methods for f will be denoted by $\mathcal{T}_c(f, \delta)$. If $g : M \rightarrow M$ is a homeomorphism with $d_0(f, g) < \delta$, then g induces a continuous δ -method φ_g for f by defining

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

Let $\mathcal{T}_h(f, \delta)$ denote the set of all continuous δ -methods φ_g for f which are induced by a homeomorphism $g : M \rightarrow M$ with $d_0(f, g) < \delta$, where d_0 is the usual C^0 -metric. Let $\mathcal{T}_d(f, \delta)$ denote the set of all continuous δ -methods φ_g for f which are induced by $g \in \text{Diff}(M)$ with $d_1(f, g) < \delta$. Then, clearly, we know that

$$\mathcal{T}_d(f) \subset \mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f),$$

$\mathcal{T}_\alpha(f) = \bigcup_{\delta > 0} \mathcal{T}_\alpha(f, \delta)$, $\alpha = 0, c, h, d$. We say that f has the *inverse shadowing property* with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0, c, h, d$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ -method $\varphi \in \mathcal{T}_\alpha(f, \delta)$ and any point $x \in M$, there is a point $y \in M$, such that

$$d(f^k(x), \varphi(y)_k) < \epsilon, \quad k \in \mathbb{Z}.$$

We say that f has the *weak inverse shadowing property* with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0, c, h, d$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ -method $\varphi \in \mathcal{T}_\alpha(f, \delta)$ and any point $x \in M$, there is a point $y \in M$ such that

$$\varphi(y) \subset B_\epsilon(\mathcal{O}_f(x)),$$

where $B_\epsilon(A) = \{x \in M : d(x, A) \leq \epsilon\}$. Note that if $f \in \text{Diff}(M)$ has the inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ ($\alpha = 0, c, h, d$), then by the definition, it clearly has the weak inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ ($\alpha = 0, c, h, d$). We say that f has the *orbital inverse shadowing property* with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0, c, h, d$, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any δ -method $\varphi \in \mathcal{T}_\alpha(f, \delta)$ and any point $x \in M$, there is a point $y \in M$ such that

$$\mathcal{O}_f(x) \subset B_\epsilon(\varphi(y)) \quad \text{and} \quad \varphi(y) \subset B_\epsilon(\mathcal{O}_f(x)).$$

Note that if f has the inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ ($\alpha = 0, c, h, d$), then it has the orbital inverse shadowing property with respect to the class $\mathcal{T}_\alpha(f)$ ($\alpha = 0, c, h, d$). But the converse does not hold. Indeed, an irrational rotation on the unit circle has the orbital inverse shadowing property but does not have the inverse shadowing

property with respect to the class $\mathcal{T}_d(f)$. We say that f belongs to the C^1 -interior inverse (weak inverse or orbital inverse) shadowing property with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0, c, h, d$ if there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the inverse (weak inverse or orbital inverse) shadowing property with respect to the class $\mathcal{T}_\alpha(f)$, $\alpha = 0, c, h, d$.

Lee [6], showed that a diffeomorphism belongs to the C^1 -interior inverse shadowing property with respect to the class $\mathcal{T}_d(f)$ if and only if it is structurally stable. And Pilyugin [7] proved that a diffeomorphism belongs to the C^1 -interior inverse shadowing property with respect to the class $\mathcal{T}_c(f)$ if and only if it is structurally stable. Thus, we can restate the above facts as follows.

Theorem 1.1 *Let $f \in \text{Diff}(M)$. A diffeomorphism f belongs to the C^1 -interior inverse shadowing property with respect to the class $\mathcal{T}_d(f)$ (resp. $\mathcal{T}_c(f)$) if and only if it is structurally stable.*

In [3] Choi, Lee and Zhang showed that a diffeomorphism belongs to the C^1 -interior weak inverse shadowing property with respect to the class $\mathcal{T}_d(f)$ if and only if it satisfies both Axiom A and the no-cycle condition. Moreover, they proved that a diffeomorphism belongs to the C^1 -interior orbital inverse shadowing property with respect to the class $\mathcal{T}_d(f)$ if and only if it satisfies both Axiom A and the strong transversal condition. From the above facts, we get the following.

Theorem 1.2 *Let $f \in \text{Diff}(M)$. If a diffeomorphism f belongs to the C^1 -interior weak inverse shadowing property with respect to the class $\mathcal{T}_d(f)$, then f satisfies both Axiom A and the no-cycle condition. Moreover, if f belongs to the C^1 -interior orbital inverse shadowing property with respect to the class $\mathcal{T}_d(f)$, then it is structurally stable.*

By the theorem, even though a diffeomorphism is contained in the C^1 -interior of the set of diffeomorphisms possessing the weak inverse shadowing property with respect to the class $\mathcal{T}_d(f)$, it does not necessarily satisfy the strong transversality condition.

A periodic point p of f is *hyperbolic* if $Df^{\pi(p)}$ has eigenvalues with absolute values different from the one, where $\pi(p)$ is the period of p . Denote by $\mathcal{F}(M)$ the set of $f \in \text{Diff}(M)$ such that there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, every $p \in P(g)$ is hyperbolic, where $P(g)$ is the set of periodic points of g . It is proved by Hayashi [8] that $f \in \mathcal{F}(M)$ if and only if f satisfies both Axiom A and the no-cycle condition.

Let Λ be a closed $f \in \text{Diff}(M)$ -invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df -invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then we say that f is an *Anosov diffeomorphism*.

2 Statement of the results

A fundamental problem in differentiable dynamical systems is to understand how a robust dynamic property on the underlying manifold would influence the behavior of the tangent map on the tangent bundle. For instance, in [9], Mañé proved that any C^1 structurally

stable diffeomorphism is an Axiom A diffeomorphism. And in [10], Palis extended this result to Ω -stable diffeomorphisms.

Let M be a compact C^∞ n -dimensional Riemannian manifold endowed with a volume form ω . Let μ denote the measure associated with ω that we call the Lebesgue measure, and let d denote the metric induced by the Riemannian structure. Denote by $\text{Diff}_\mu(M)$ the set of diffeomorphisms which preserves the Lebesgue measure μ endowed with the C^1 -topology. In the volume-preserving case, the Axiom A condition is equivalent to the diffeomorphism being Anosov, since $\Omega(f) = M$ by the Poincaré recurrence theorem.

We define the set $\mathcal{F}_\mu(M)$ as the set of diffeomorphisms $f \in \text{Diff}_\mu(M)$ which has a C^1 -neighborhood $\mathcal{U}(f) \subset \text{Diff}_\mu(M)$ such that for any $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic. Note that $\mathcal{F}_\mu(M) \subset \mathcal{F}(M)$ (see [11, Corollary 1.2]).

Very recently, Arbieto and Catalan [11] proved that if a volume-preserving diffeomorphism is contained in $\mathcal{F}_\mu(M)$, then it is Anosov. Indeed, at first they used the Mañé results ([9, Proposition II.1]). Then they showed that $\overline{P(f)}$ is hyperbolic, where $P(f)$ is the set of periodic points of f . And they proved that the nonwandering set $\Omega(f) = \overline{P(f)}$ by Pugh's closing lemma. Finally, by the Poincaré recurrence theorem, $\Omega(f) = M$. From the above facts, we can restate the theorem as follows.

Theorem 2.1 *Any diffeomorphism in $\mathcal{F}_\mu(M)$ is Anosov.*

In [12], Lee showed that if a volume-preserving diffeomorphism belongs to the C^1 -interior expansive or C^1 -interior shadowing property, then it is Anosov. And [13] proved that if a volume-preserving diffeomorphism belongs to the C^1 -interior weak shadowing property or C^1 -interior weak limit shadowing property, then it is Anosov. From these results, we study the cases when a volume-preserving diffeomorphism f is in C^1 -interior various inverse shadowing property with respect to the class $\mathcal{T}_d(f)$, then it is Anosov. Let $\text{int } \mathcal{IS}_\mu(M)$ denote the set of volume-preserving diffeomorphisms in $\text{Diff}_\mu(M)$ satisfying the inverse shadowing property with respect to the class \mathcal{T}_d , and let $\text{int } \mathcal{WIS}_\mu(M)$ (respect., $\text{int } \mathcal{OIS}_\mu(M)$) denote the set of volume-preserving diffeomorphisms in $\text{Diff}_\mu(M)$ satisfying the weak inverse shadowing property with respect to the class \mathcal{T}_d (respect., the orbital inverse shadowing property with respect to the class \mathcal{T}_d). From now, we only consider the class \mathcal{T}_d when we mention the inverse shadowing property; that is, the 'inverse shadowing property' implies the 'inverse shadowing property with respect to the class \mathcal{T}_d '. Now we are in a position to state the theorem of our paper.

Theorem 2.2 *Let $f \in \text{Diff}_\mu(M)$. We have*

$$\text{int } \mathcal{IS}_\mu(M) = \text{int } \mathcal{OIS}_\mu(M) = \text{int } \mathcal{WIS}_\mu(M) = \mathcal{AN}_\mu(M),$$

where $\mathcal{AN}_\mu(M)$ is the set of Anosov volume-preserving diffeomorphisms in $\text{Diff}_\mu(M)$.

3 Proof of Theorem 2.2

Let M be a compact C^∞ n -dimensional Riemannian manifold endowed with a volume form ω , and let $f \in \text{Diff}_\mu(M)$. To prove the results, we will give the following well-known Franks lemma for the conservative case, stated and proved in [14, Proposition 7.4].

Lemma 3.1 *Let $f \in \text{Diff}_\mu^1(M)$ and \mathcal{U} be a C^1 -neighborhood of f in $\text{Diff}_\mu^1(M)$. Then there exist a C^1 -neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\epsilon > 0$ such that if $g \in \mathcal{U}_0$, for any finite f -invariant set $E = \{x_1, \dots, x_m\}$, any neighborhood U of E and any volume-preserving linear maps $L_j : T_{x_j}M \rightarrow T_{g(x_j)}M$ with $\|L_j - D_{x_j}g\| \leq \epsilon$ for all $j = 1, \dots, m$, there is a conservative diffeomorphism $g_1 \in \mathcal{U}$ coinciding with f on E and out of U , and $D_{x_j}g_1 = L_j$ for all $j = 1, \dots, m$.*

Remark 3.2 From the Moser theorem (see [15]), there is a smooth conservative change of coordinates $\varphi_x : U(x) \rightarrow T_xM$ such that $\varphi_x(x) = \vec{0}$, where $U(x)$ is a small neighborhood of $x \in M$.

Proposition 3.3 *If $f \in \text{int } \mathcal{IS}_\mu(M)$, then every periodic point of f is hyperbolic.*

Proof Take $f \in \text{int } \mathcal{IS}_\mu(M)$ and $\mathcal{U}(f)$ is a C^1 -neighborhood of $f \in \text{int } \mathcal{IS}_\mu(M)$. Let $\epsilon > 0$ and $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a corresponding number and a C^1 -neighborhood given by Lemma 3.1. Suppose that there exists a nonhyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{V}(f)$. To simplify the notation of the proof, we may assume that $g(p) = p$. Then there is at least one eigenvalue λ of $D_p g$ such that $|\lambda| = 1$. Denote by E_p^c the eigenspace corresponding to λ . Then we see that if $\lambda \in \mathbb{R}$, then $\dim E_p^c = 1$, and if $\lambda \in \mathbb{C}$, then $\dim E_p^c = 2$.

First, we consider $\dim E_p^c = 1$. For simplicity, we may assume that $\lambda = 1$ (the other case is similar). By making use of Lemma 3.1, we linearize g at p with respect to the Moser theorem; that is, by choosing $\alpha > 0$ sufficiently small, we construct g_1 C^1 -nearby g such that

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

Then $g_1(p) = g(p) = p$. Since the eigenvalue λ of $D_p g_1$ is one, $D_p g_1(v) = v$ for any $v \in E_p^c(\alpha)$. Take $v_0 \in E_p^c(\alpha)$ such that $\|v_0\| = \alpha/4$. Then we set

$$\mathcal{I}_{v_0} = \{t \cdot v_0 : -1 \leq t \leq 1\} \subset \varphi_p(B_\alpha(p)).$$

Take $\epsilon_1 = \alpha/8$. Let $0 < \delta < \epsilon_1$ be the number of the inverse shadowing property of g_1 . Then by our construction of g_1 , $\varphi_p^{-1}(\mathcal{I}_{v_0}) \subset B_\alpha(p)$. Put $\mathcal{J}_p = \varphi_p^{-1}(\mathcal{I}_{v_0})$. Then we see that $g_1(\mathcal{J}_p) = \mathcal{J}_p$ and it is the identity map. For the above $\delta > 0$, we can define $\mathcal{T}_d(g_1)$ -method as follows. For any $x \in M$, we set $\varphi_p(x) = (v_1, \dots, v_n)$ and $D_p g(v_1, \dots, v_n) = (v_1, A(v_2, \dots, v_n))$. Here A is corresponding to $|\lambda| \neq 1$. Then we define

$$h(x) = (\varphi_p^{-1}(v_1) + \delta/2, \varphi_p^{-1}(A(v_2, \dots, v_n))) = (x_1 + \delta/2, x'),$$

where $\varphi_p^{-1}(A(v_2, \dots, v_n)) = x' = (x_2, \dots, x_n)$. Clearly, $d_1(g_1, h) < \delta$, and $h \in \mathcal{T}_d(g_1)$. Let p be identified with $\mathbf{0} = (0, \dots, 0)$. Then choose $x = (x_1, 0, \dots, 0) \in \mathcal{J}_p$ such that $d(\mathbf{0}, x) = 2\epsilon_1$. Then

$$d(h^i(\mathbf{0}), g_1^i(x)) = d(\mathbf{0}, x) = 2\epsilon_1.$$

Thus, g_1 does not have the inverse shadowing property.

We take a point $y = (y_1, 0, \dots, 0) \in \mathcal{J}_p$ such that $d(y, x) < \epsilon_1$ and $h^i(y) = (\varphi_p^{-1}(w_1) + i\delta/2, \varphi_p^{-1}(A^i(w_2, \dots, w_n))) = (y_1 + i\delta/2, y^i)$, where $\varphi_p(y) = (w_1, \dots, w_n)$, $y' = \varphi_p^{-1}(A(w_2, \dots, w_n))$ and A corresponding to $|\lambda| \neq 1$. Then $d(g^i(x), h^i(y)) = d(x, y_1 + i\delta/2)$ and for some $k \in \mathbb{Z}$,

$$d(x, y_1 + k\delta/2) > \alpha/8 = \epsilon_1.$$

This is a contradiction.

Therefore, we can choose a point $y \in M \setminus \mathcal{J}_p$ such that $d(x, y) < \epsilon_1$. Since $d(g_1^i(x), h^i(y)) = d((x_1, 0, \dots, 0), (y_1 + i\delta/2, y^i))$, we can find $k \in \mathbb{Z}$ such that

$$d((x_1, 0, \dots, 0), (y_1 + k\delta/2, y^k)) > \epsilon_1,$$

where $y^k = \varphi_p^{-1}(A^k(w_1, \dots, w_n))$. This is a contradiction since $f \in \text{int } \mathcal{IS}_\mu(M)$.

Finally, if $\lambda \in \mathbb{C}$, then $\dim E_p^c = 2$. To avoid the notational complexity, we may assume that $g(p) = p$. As in the first case, by Lemma 3.1, there are $\alpha > 0$ and $g_1 \in \mathcal{V}(f)$ such that $g_1(p) = g(p) = p$ and

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}$$

With a C^1 -small modification of the map $D_p g$, we may suppose that there is $l > 0$ (the minimum number) such that $D_p g^l(v) = v$ for any $v \in \varphi_p(B_\alpha(p)) \subset T_p M$. Take $v_0 \in \varphi_p(B_\alpha(p))$ such that $\|v_0\| = \alpha/4$, and set

$$\mathcal{L}_p = \varphi_p^{-1}(\{t \cdot v_0 : 1 \leq t \leq 1 + \alpha/4\}).$$

Then \mathcal{L}_p is an arc such that

- $g_1^i(\mathcal{L}_p) \cap g_1^j(\mathcal{L}_p) = \emptyset$ for $0 \leq i \neq j \leq l-1$,
- $g_1^l(\mathcal{L}_p) = \mathcal{L}_p$, and
- $g_1^l|_{\mathcal{L}_p}$ is the identity map.

Note that g_1 has the inverse shadowing property if and only if g_1^k has the inverse shadowing property for all $k \in \mathbb{Z}$ (see [6]). As in the first case, we can show that g_1 does not have the inverse shadowing property, which contradicts the fact that $g_1 \in \mathcal{U}(f)$. Thus, every periodic point of $f \in \text{int } \mathcal{IS}_\mu(M)$ is hyperbolic. \square

Proposition 3.4 *If $f \in \text{int } \mathcal{WIS}_\mu(M)$, then every periodic point of f is hyperbolic.*

Proof Take $f \in \text{int } \mathcal{WIS}_\mu(M)$, and $\mathcal{U}(f)$ is a C^1 -neighborhood of $f \in \text{int } \mathcal{WIS}_\mu(M)$. Let $\epsilon > 0$ and $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a corresponding number and a C^1 -neighborhood given by Lemma 3.1. Suppose that there exists a nonhyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{V}(f)$. To simplify the notation of the proof, we may assume that $g(p) = p$. Then, as in the proof of Proposition 3.3, we can take $\alpha > 0$ sufficiently small and a smooth map $\varphi_p : B_\alpha(p) \rightarrow T_p M$. Then we can make an arc $\mathcal{J}_p \subset B_\alpha(p)$ and for some $g_1 \in \mathcal{V}(f)$. Take $\epsilon_1 = (\text{length } \mathcal{J}_p)/4$. Let $0 < \delta < \epsilon_1$ be the number of the weak inverse shadowing property for some g_1 . Then we can construct a map $h \in \text{Diff}(M)$ as in the proof of Proposition 3.3.

Let p be identified with $\mathbf{0} = (0, \dots, 0)$. Then choose a point $x = (x_1, 0, \dots, 0) \in \mathcal{J}_p$ such that $d(\mathbf{0}, x) = 2\epsilon_1$. Since g_1 has the weak inverse shadowing property,

$$\mathcal{O}_h(\mathbf{0}) \subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)).$$

However, for any $y \in \mathcal{J}_p$,

$$g_1^i(y) = y \quad \text{and} \quad h^i(y) = (y_1 + i\delta/2, y^i),$$

where $y^i = \varphi_p^{-1}(A^i(w_2, \dots, w_n))$ is as in the proof of Proposition 3.3. Thus, it is easily seen that

$$\mathcal{O}_h(\mathbf{0}) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)).$$

If $y = (y_1, 0, \dots, 0) \in \mathcal{J}_p \setminus \{p\}$, then

$$h^k(y) = (y_1 + k\delta/2, y^k) = (y_1 + k\delta/2, 0),$$

where $y^k = \varphi_p^{-1}(A^k(w_2, \dots, w_n))$. Thus, we know that

$$\mathcal{O}_h(y) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)).$$

This is a contradiction.

Finally, we can choose a point $y \in M \setminus \mathcal{J}_p$ such that $d(x, y) < \epsilon_1$. Then we know that

$$h^k(y) = (y_1 + k\delta/2, y^k),$$

where $y^k = \varphi_p^{-1}(A^k(w_2, \dots, w_n))$. Therefore,

$$h^l(y) \notin B_{\epsilon_1}(\mathcal{J}_p),$$

for some $l \in \mathbb{Z}$. Then

$$\mathcal{O}_h(y) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)) = B_{\epsilon_1}(x).$$

Thus, g_1 does not have the weak inverse shadowing property. This is a contradiction.

If $\lambda \in \mathbb{C}$, then as in the proof of Proposition 3.3, for $g_1 \in \mathcal{V}(f)$, we can take $l > 0$ such that $D_p g_1^l(v) = v$ for any $v \in \varphi_p(B_\alpha(p)) \subset T_p M$. As in the previous argument, we get a contradiction. Thus, every periodic point of $f \in \text{int } \mathcal{WIS}_\mu(M)$ is hyperbolic. Consequently, if $f \in \text{int } \mathcal{WIS}_\mu(M)$, then $f \in \mathcal{F}_\mu(M)$. \square

Proposition 3.5 *If $f \in \text{int } \mathcal{OIS}_\mu(M)$, then every periodic point of f is hyperbolic.*

Proof The proof is almost the same as that of Proposition 3.4. Indeed, let $f \in \text{int } \mathcal{OIS}_\mu(M)$ and $\mathcal{U}(f)$ be a C^1 -neighborhood of $f \in \text{int } \mathcal{OIS}_\mu(M)$. Let $\epsilon > 0$ and $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ be a corresponding number and a C^1 -neighborhood given by Lemma 3.1. Suppose that there exists

a nonhyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{V}(f)$. To simplify the notation of the proof, we may assume that $g(p) = p$. Then, as in the proof of Proposition 3.4, we can take $\alpha > 0$ sufficiently small and a smooth map $\varphi_p : B_\alpha(p) \rightarrow T_pM$. Then we can make an arc $\mathcal{J}_p \subset B_\alpha(p)$ and for some $g_1 \in \mathcal{V}(f)$. Take $\epsilon_1 = (\text{length} \mathcal{J}_p)/4$. Let $0 < \delta < \epsilon_1$ be the number of the orbital inverse shadowing property for some g_1 . Then we can construct a map $h \in \text{Diff}(M)$ as in the proof of Proposition 3.4. Let p be identified with $\mathbf{0} = (0, \dots, 0)$. Then choose a point $x = (x_1, 0, \dots, 0) \in \mathcal{J}_p$ such that $d(\mathbf{0}, x) = 2\epsilon_1$. Since g_1 has the orbital inverse shadowing property,

$$\mathcal{O}_h(\mathbf{0}) \subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)) \quad \text{and} \quad \mathcal{O}_{g_1}(x) \subset B_{\epsilon_1}(\mathcal{O}_h(\mathbf{0})).$$

However, for any $y \in \mathcal{J}_p$,

$$g_1^i(y) = y \quad \text{and} \quad h^i(y) = (y_1 + i\delta/2, y^i),$$

where $y^i = \varphi_p^{-1}(A^i(w_2, \dots, w_n)) = (y_2, \dots, y_n)$ is as in the proof of Proposition 3.3. Thus, it is easily seen that

$$\mathcal{O}_h(\mathbf{0}) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)).$$

This is a contradiction since $f \in \text{int } \mathcal{OIS}_\mu(M)$.

If $y = (y_1, 0, \dots, 0) \in \mathcal{J}_p \setminus \{p\}$, then

$$h^k(y) = (y_1 + k\delta/2, y^k) = (y_1 + k\delta/2, 0),$$

where $y^k = \varphi_p^{-1}(A^k(w_2, \dots, w_n))$. Thus, we know that

$$\mathcal{O}_h(y) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)).$$

This is a contradiction.

Finally, we can choose a point $y \in M \setminus \mathcal{J}_p$ such that $d(x, y) < \epsilon_1$. Then we know that

$$h^k(y) = (y_1 + k\delta/2, y^k),$$

where $y^k = \varphi_p^{-1}(A^k(w_2, \dots, w_n))$. Therefore,

$$h^l(y) \notin B_{\epsilon_1}(\mathcal{J}_p),$$

for some $l \in \mathbb{Z}$. Then

$$\mathcal{O}_h(y) \not\subset B_{\epsilon_1}(\mathcal{O}_{g_1}(x)) = B_{\epsilon_1}(x).$$

Thus, g_1 does not have the orbital inverse shadowing property. This is a contradiction.

If $\lambda \in \mathbb{C}$, then as in the proof of Proposition 3.4, for $g_1 \in \mathcal{V}(f)$, we can take $l > 0$ such that $D_p g_1^l(v) = v$ for any $v \in \varphi_p(B_\alpha(p)) \subset T_pM$. As in the previous argument, in order to reach the same contradiction. Thus, every periodic point of $f \in \text{int } \mathcal{OIS}_\mu(M)$ is hyperbolic. \square

Competing interests

The author declares that they have no competing interests.

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