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A new proof of fractional Hu-Meyer formula and its applications

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Abstract
This paper is concerned with the Hu-Meyer formula for fractional Brownian motion with the Hurst parameter less than 1/2. By the mollifier approximation, the Hu-Meyer formula is explicitly obtained based on the multiple Stratonovich integral, and the proof is different from the known methods. Moreover, the obtained Hu-Meyer formula can be applied to derive the convergence rate of the multiple fractional Stratonovich integral.
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1 Introduction

It is well known that Hu and Meyer [1] introduced a new multiple stochastic integral with respect to a Wiener process, called a multiple Stratonovich integral, which is in general different from the usually studied multiple Wiener-Itô integral. The authors also proposed a formula (called Hu-Meyer formula) that gives the relationship of the Stratonovich integral with the Itô integrals of some functions called the traces that involve integrals of f on the diagonals.

An increasing interest is visible in the last decade in modeling long dependence phenomena in the fields of dynamical system, economics, hydrology, telecommunication network by using fractional Brownian motion (fBm for short). The fBm is a suitable generalization of standard Brownian motion which exhibits long-range dependence.

Recently, many authors have considered an integral with respect to fBm. Duncan *et al.* [2] employed the Wick products to define a fractional stochastic integral whose mean is zero. This property is very convenient for both theoretical development and practical applications. For more details, one can see [3] and the references therein.

Bardina *et al.* [4] constructed a multiple Stratonovich integral with respect to fBm with the Hurst parameter $H < 1/2$ under some conditions. They defined the traces to obtain the Hu-Meyer formula that gives the Stratonovich integral as a sum of Itô integrals of these traces.

In this paper, we consider a similar problem for the multiple Stratonovich integral. Inspired by [5], we define the integral of Stratonovich type in the mollifier approximation sense. Unlike our construction, in the paper [4], the Stratonovich integral is defined in the Solé-Utzet sense (see [6]). Our aim here is to present a new proof of the Hu-Meyer formula for fBm. We also do not make use of the integral representation of fBm in terms of ordinary Brownian motion as in [7], where the hypothesis involves the transferring operator which is difficult to verify.

We have organized the paper as follows. Section 2 recalls some results from [3] on the multiple Stratonovich integral, which will be used in the remainder of the paper. Section 3 gives the Hu-Meyer formula and its proof. As an application, the fourth section is devoted to the convergence rate of the multiple fractional Stratonovich integral.

2 Multiple Stratonovich integral

In this paper, we denote by $(\Omega, \mathcal{F}, P)$ the basic probability space. The expectation on this basic probability space is denoted by $\mathbb{E}$. The fBm $(B_t^H, t \geq 0)$ of the Hurst parameter H is a Gaussian process with mean 0 and covariance given by

$$\mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad 0 \leq s, t < \infty.$$

Throughout this paper, we assume $H < 1/2$.

For a fixed positive integer n and a suitable (deterministic) function $f(t_1, \ldots, t_n)$ of n variables the multiple Itô integral

$$I_n(f) = \int_{0 \leq t_1, \ldots, t_n \leq T} f(t_1, \ldots, t_n)\, dB_{t_1}^H \cdots dB_{t_n}^H$$

and the multiple Stratonovich integral

$$S_n(f) = \int_{0 \leq t_1, \ldots, t_n \leq T} f(t_1, \ldots, t_n)\, d^\circ B_{t_1}^H \cdots d^\circ B_{t_n}^H \tag{2.1}$$

are well defined (see [3, 8] and the references therein).

Following the notations in [9], we define

$$\bar{f}(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n) & \text{if } (x_1, \ldots, x_n) \in [0, T]^n, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

with

$$V_y f(x) = f(y).$$

This implies $V_y f(x)$ is obtained by using a variable y instead of x.

For a continuous function of n variables $f(t_1, \ldots, t_n)$, we define

$$V_{k,s} f = V_{k,s} f(t_1, \ldots, t_n) = f(t_1, \ldots, t_{k-1}, s, t_{k+1}, \ldots, t_n).$$

This means $V_{k,s} f$ is obtained from f by replacing the kth variable t_k by s.

3 Hu-Meyer formula

Now, the Wick product $\diamond$ of two functionals is introduced. To extend the theory of stochastic calculus from Brownian motions to fBms, the Wick calculus for Gaussian processes (or Gaussian measures) is used. The Wick product of two exponential functions (see [2]) $\varepsilon(f)$

and $\varepsilon(g)$ is defined as

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f+g), \tag{3.1}$$

where

$$\varepsilon(f) := \exp\left\{\int_0^\infty f_t\, dB_t^H - \frac{1}{2}|f|_\phi^2\right\}.$$

Using the linear property, we can generalize the Wick product to the linear combination of exponential functionals. Then the Wick product can be extended to a general random variable by taking limit.

Proposition 1 *Let X and Y be two random variables. Then we have*

$$X \diamond Y = X \cdot Y - E(XY).$$

Proof By the definition of an exponential function,

$$\varepsilon(X) := e^{tX-\frac{1}{2}t^2E(X^2)}, \qquad \varepsilon(Y) := e^{sY-\frac{1}{2}s^2E(Y^2)},$$

using the expression (3.1)

$$\varepsilon(X) \diamond \varepsilon(Y) = e^{tX-\frac{1}{2}t^2E(X^2)} \diamond e^{sY-\frac{1}{2}s^2E(Y^2)} = e^{tX+sY-\frac{1}{2}E(tX+sY)^2},$$

we will compare the coefficients of the term $s \cdot t$ in the two sides of the above equality. Observe that the coefficient of $s \cdot t$ in the left is $X \diamond Y$ and the one in the right is $X \cdot Y - E(XY)$. This fact implies the truth of the proposition. □

As in [5], for $\varphi_\varepsilon(t,s)$ and fixed t, as ε tends to zero, $\varphi_\varepsilon(t,\cdot)$ tends to the Dirac function at t, $\delta(t-s)$. Define

$$\dot{B}_t^\varepsilon = \int_0^T \varphi_\varepsilon(t,s)\, dB_s^H = \int_0^T \varphi_\varepsilon(t,s)\dot{B}_s^H\, ds. \tag{3.2}$$

Obviously,

$$\dot{B}_t^\varepsilon \to \int_0^T \delta(t-s)\dot{B}_s^H\, ds = \dot{B}_t^H$$

when ε tends to zero. Then $\dot{B}_t^\varepsilon$ is a Gaussian random variable (see [10]). Furthermore, from [4], we have

$$\begin{aligned}\lim_{\varepsilon\to 0} E\left(\dot{B}_{t_1}^\varepsilon \dot{B}_{t_2}^\varepsilon\right) &= \lim_{\varepsilon\to 0}\left\langle \varphi_\varepsilon(t_1,s), \varphi_\varepsilon(t_2,s)\right\rangle_H \\ &= \lim_{\varepsilon\to 0}\frac{1}{2}H(1-2H)\int_{\mathbb{R}^2}\frac{(I-V_{s_2})\varphi_\varepsilon(t_1,s_1)(I-V_{s_2})\varphi_\varepsilon(t_2,s_1)}{|s_1-s_2|^{2-2H}}\, ds_1\, ds_2 \\ &= \frac{1}{2}H(1-2H)\frac{I-V_{t_2}}{|t_1-t_2|^{2-2H}}.\end{aligned} \tag{3.3}$$

Lemma 1 *For $\dot{B}^{\varepsilon}_{t_i}$ defined by* (3.2), $i = 1, \dots, n$, *we have*

$$\dot{B}^{\varepsilon}_{t_1} \cdots \dot{B}^{\varepsilon}_{t_n} = \sum_{i_1,\dots,i_n} \frac{1}{2^k \cdot k!} \dot{B}^{\varepsilon}_{t_{i_n}} \diamond \cdots \diamond \dot{B}^{\varepsilon}_{t_{i_{n-2k-1}}} E\bigl(\dot{B}^{\varepsilon}_{t_{i_1}} \dot{B}^{\varepsilon}_{t_{i_2}}\bigr) \cdots E\bigl(\dot{B}^{\varepsilon}_{t_{i_{2k-1}}} \dot{B}^{\varepsilon}_{t_{i_{2k}}}\bigr), \tag{3.4}$$

where $i_1, \dots, i_n$ run over all permutations of $\{1, \dots, n\}$ *and* $t_j \in [0, T]$, $j = 1, \dots, n$.

Proof Let $X_i = \dot{B}^{\varepsilon}_{t_i} = \int_0^T \varphi_\varepsilon(t_i, s_i) \dot{B}^H_{s_i} \, ds_i$ $(i = 1, \dots, n)$, by (3.1) we obtain

$$\varepsilon(t_1 X_1) \diamond \cdots \diamond \varepsilon(t_n X_n) = \varepsilon(t_1 X_1 + \cdots + t_n X_n).$$

Then, by the definition of an exponential function,

$$e^{t_1 X_1 + \cdots + t_n X_n} = e^{\frac{1}{2} E(t_1 X_1 + \cdots + t_n X_n)^2} \varepsilon(t_1 X_1) \diamond \cdots \diamond \varepsilon(t_n X_n). \tag{3.5}$$

Next we will compare the coefficients of $t_1 \cdots t_n$. On the one hand, it is obvious that the left-hand side of (3.5) is equal to

$$\sum_{i_1,\dots,i_n} \frac{t_1^{i_1} \cdots t_n^{i_n}}{i_1! \cdots i_n!} X_1^{i_1} \cdots X_n^{i_n},$$

therefore, the coefficient of the term $t_1 \cdots t_n$ on the left-hand side is $X_1 \cdots X_n$.

On the other hand, the right-hand side of (3.5) coincides with

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} \bigl(E(t_1 X_1 + \cdots + t_n X_n)^2\bigr)^k \varepsilon(t_1 X_1) \diamond \cdots \diamond \varepsilon(t_n X_n) \\
&\quad = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} \bigl(E(t_1 X_1 + \cdots + t_n X_n)^2\bigr)^k \sum_{k=0}^{\infty} \frac{t_1^k X_1^{\diamond k}}{k!} \cdots \sum_{k=0}^{\infty} \frac{t_n^k X_n^{\diamond k}}{k!} \\
&\quad = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!} \Bigl(\sum_{1 \le i \le j \le n} t_i t_j E(X_i X_j) \Bigr)^k \sum_{k=0}^{\infty} \frac{t_1^k X_1^{\diamond k}}{k!} \cdots \sum_{k=0}^{\infty} \frac{t_n^k X_n^{\diamond k}}{k!}.
\end{aligned}$$

Notice that the coefficient of $t_1 \cdots t_n$ on the right-hand side is

$$\sum_{\sigma} \frac{1}{2^k \cdot k!} X_{i_n} \diamond \cdots \diamond X_{i_{n-2k-1}} E(X_{i_1} X_{i_2}) \cdots E(X_{i_{2k-1}} X_{i_{2k}}),$$

where σ are the permutations of $\{1, 2, \dots, n\}$. This completes the proof. □

Let us state the main result of this section. The following explains the relations between the multiple Itô integral and the Stratonovich integral.

Theorem 1 (Hu-Meyer formula) *Let $f \in L^2_s([0, T]^n)$. There exists the limit in $L^2(\Omega)$ of*

$$S_n(f^\varepsilon) = \int_{[0,T]^n} f(t_1, \dots, t_n) \dot{B}^{\varepsilon}_{t_1} \cdots \dot{B}^{\varepsilon}_{t_n} \, dt_1 \cdots dt_n,$$

and the limit is given by the extended Hu-Meyer formula

$$S_n(f) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)!k!2^k} I_{n-2k}\big(Tr^k f\big),$$

where

$$Tr^k(f) := \big(H(1-2H)\big)^k \int_{\mathbb{R}^{2k}} \frac{\prod_{i=1}^k (I - V_{2i,t_{2i-1}})\bar{f}(t_1,\ldots,t_{2k},\cdot)}{\prod_{i=1}^k |t_{2i-1}-t_{2i}|^{2-2H}}\, dt_1 \cdots dt_{2k},$$

with the convention that $Tr^0 f = f$.

Remark 1 This result is not the same as Theorem 4.4 in [4], where the traces that appear are defined by a limit procedure, not in the way stated here.

Proof Using Lemma 1 and the property of the Wick product, we have that

$$\begin{aligned}
S_n(f^\varepsilon) &= \int_{[0,T]^n} f(t_1,\ldots,t_n)\dot{B}^\varepsilon_{t_1}\cdots\dot{B}^\varepsilon_{t_n}\, dt_1\cdots dt_n \\
&= \sum_\sigma \frac{1}{2^k\cdot k!}\int_{[0,T]^n} f(t_1,\ldots,t_n)\dot{B}^\varepsilon_{t_{n-2k-1}}\diamond\cdots\diamond\dot{B}^\varepsilon_{t_n} E\big(\dot{B}^\varepsilon_{t_1}\dot{B}^\varepsilon_{t_2}\big) \\
&\quad \cdots E\big(\dot{B}^\varepsilon_{t_{2k-1}}\dot{B}^\varepsilon_{t_{2k}}\big)\, dt_1\cdots dt_n \\
&= \sum_\sigma \frac{1}{2^k\cdot k!}\int_{[0,T]^n} f(t_1,\ldots,t_n)\dot{B}^\varepsilon_{t_{n-2k-1}}\diamond\cdots\diamond\dot{B}^\varepsilon_{t_n}\big\langle\varphi_\varepsilon(t_1,\cdot),\varphi_\varepsilon(t_2,\cdot)\big\rangle_H \\
&\quad \cdots\big\langle\varphi_\varepsilon(t_{2k-1},\cdot),\varphi_\varepsilon(t_{2k},\cdot)\big\rangle_H\, dt_1\cdots dt_n \\
&= \sum_\sigma \frac{1}{2^k\cdot k!}\int_{[0,T]^{n-2k}} g^\varepsilon(t_{2k+1},\ldots,t_n)\dot{B}^\varepsilon_{t_{n-2k-1}}\diamond\cdots\diamond\dot{B}^\varepsilon_{t_n}\, dt_{n-2k-1}\cdots dt_n \\
&= \sum_\sigma \frac{1}{2^k\cdot k!}\int_{[0,T]^{n-2k}} g^\varepsilon(t_{2k+1},\ldots,t_n)\dot{B}^\varepsilon_{t_{2k+1}}\diamond\cdots\diamond\dot{B}^\varepsilon_{t_n}\, dt_{2k+1}\cdots dt_n,
\end{aligned}$$

where

$$\begin{aligned}
&g^\varepsilon(t_{2k+1},\ldots,t_n) \\
&\quad = \int_{[0,T]^{2k}} f(t_1,\ldots,t_n)\big\langle\varphi_\varepsilon(t_1,\cdot),\varphi_\varepsilon(t_2,\cdot)\big\rangle_H\cdots\big\langle\varphi_\varepsilon(t_{2k-1},\cdot),\varphi_\varepsilon(t_{2k},\cdot)\big\rangle_H\, dt_1\cdots dt_{2k}.
\end{aligned}$$

Submitting (3.3) to the above expression,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} g^\varepsilon(t_{2k+1},\ldots,t_n) \\
&\quad = \big(H(1-2H)\big)^k \int_{\mathbb{R}^{2k}} \frac{\prod_{i=1}^k (I - V_{2i,t_{2i-1}})\bar{f}(t_1,\ldots,t_{2k},\cdot)}{\prod_{i=1}^k |t_{2i-1}-t_{2i}|^{2-2H}}\, dt_1\cdots dt_{2k} \\
&\quad = Tr^k f.
\end{aligned}$$

By the continuity of the multiple Itô-type integrals on the $(\mathcal{L}_T^H)^{\otimes(n-2k)}$ spaces [11], it follows that

$$\begin{aligned}
&\int_{[0,T]^{n-2k}} g^{\varepsilon}(t_{2k+1},\ldots,t_n)\dot{B}^{\varepsilon}_{t_{2k+1}} \diamond \cdots \diamond \dot{B}^{\varepsilon}_{t_n}\, dt_{2k+1}\cdots dt_n \\
&\quad = \int_{[0,T]^{n-2k}} g^{\varepsilon}(t_{2k+1},\ldots,t_n)\int_0^T \varphi_\varepsilon(t_{2k+1},s_{2k+1})\, dB^H_{s_{2k+1}} \\
&\qquad \diamond\cdots\diamond \int_0^T \varphi_\varepsilon(t_n,s_n)\, dB^H_{s_n} dt_{2k+1}\cdots dt_n \\
&\quad \to I_{n-2k}\bigl(Tr^k f\bigr),
\end{aligned}$$

which is in the $L^2(\Omega)$ sense as $\varepsilon \to 0$.

Denote

$$\int_{[0,T]^k} g^{\varepsilon}(t_1,\ldots,t_k)\dot{B}^{\varepsilon}_{t_1} \diamond\cdots\diamond \dot{B}^{\varepsilon}_{t_k}\, dt_1\cdots dt_k = \int_{[0,T]^k} h^{\varepsilon}(s_1,\ldots,s_k)\, dB^H_{s_1}\diamond\cdots\diamond dB^H_{s_k},$$

where

$$h^{\varepsilon}(s_1,\ldots,s_k) = \int_{[0,T]^k} g^{\varepsilon}(t_1,\ldots,t_k)\varphi_\varepsilon(t_1,s_1)\cdots\varphi_\varepsilon(t_k,s_k)\, dt_1\cdots dt_k.$$

It is easy to prove that $h^{\varepsilon}(s_1,\ldots,s_k)$ converge to $h(s_1,\ldots,s_k)$ in the same way as in [4]. Since σ are the permutations of $\{1,2,\ldots,n\}$, we get the desired result. □

4 Applications to the convergence rate of the multiple Stratonovich integral

To complement the paper, we introduce some notations. Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0,T]$. Denote

$$\Delta_i = t_{i+1} - t_i, \qquad \Delta = \max_i \Delta_i.$$

Without ambiguity, we will also denote the interval $(t_i, t_{i+1}]$ by Δ_i. We also consider a class of partitions Π such that

$$C_\Pi = \sup_{\pi\in\Pi}\sup_{i,j}\frac{\Delta_i^\pi}{\Delta_j^\pi} < \infty. \tag{4.1}$$

Let $B_t^{H,\pi}$ be the interpolation approximation of B_t^H,

$$B_t^{H,\pi} = B^H_{t_i} + \frac{\Delta B^H_{t_i}}{\Delta_i}(t-t_i), \quad \text{when } t\in\Delta_i,$$

where

$$\Delta B^H_{t_i} = B^H_{t_{i+1}} - B^H_{t_i}.$$

Consider the approximation of the multiple stochastic integral

$$S_n^\pi(f) = \int_{0\le t_1,\ldots,t_n\le T} f(t_1,\ldots,t_n)\, dB^{H,\pi}_{t_1}\cdots dB^{H,\pi}_{t_n}. \tag{4.2}$$

It is proved in [4] that, under some mild conditions, $S_n^\pi(f)$ converges to $S_n(f)$ in the mean square sense. Then the natural question is: what is the precise asymptotic, *i.e.*, convergence rate?

Our main result in this section is stated as follows.

Theorem 2 *Suppose that $f \in C^{n+1}([0,T]^n)$. Given a sequence partition π of the interval $[0,T]$ satisfying (4.1), there is a random variable $S_n(f)$ such that $S_n^\pi(f)$ converges to $S_n(f)$ in the mean square sense. Moreover, there is a constant C, independent of partition π, such that*

$$\mathbb{E}\left|S_n^\pi(f) - S_n(f)\right|^2 \le C\Delta^{4H}. \tag{4.3}$$

We must point out that the Hu-Meyer formula will be the key tool used in order to obtain the convergence rate of the interpolation approximation for general n considered in the section. In order to prove the above theorem, we also need the following results.

Proposition 2 *Assume $t_1, x_1 \in \Delta_{i_1}$, $t_2, x_2 \in \Delta_{j_1}$ and $s_1, y_1 \in \Delta_{i_2}$, $s_2, y_2 \in \Delta_{j_2}$. Let f continuously bound first and second derivatives on $[0,T]^2$. Then we have*

$$\begin{aligned}&\left(\prod_{k=1}^{2}(I - V_{k,s_k})f(t_1,t_2) - \prod_{k=1}^{2}(I - V_{k,y_k})f(x_1,x_2)\right)^2\\ &\quad\le C|x_1 - y_1|^2\Delta^2 + C|x_2 - y_2|^2\Delta^2 + C\Delta^4.\end{aligned}$$

Proof Notice that

$$(t_1 - s_1)(t_2 - s_2) = (\delta_1 + x_1 - y_1)(\delta_2 + x_2 - y_2),$$

where

$$\begin{aligned}\delta_1 &= t_1 - x_1 + y_1 - s_1,\\ \delta_2 &= t_2 - x_2 + y_2 - s_2.\end{aligned}$$

Since $t_1, x_1 \in \Delta_{i_1}$, $y_1, s_1 \in \Delta_{j_1}$, we get

$$\begin{aligned}|\delta_1| &= |t_1 - x_1 + y_1 - s_1| \le |t_1 - x_1| + |y_1 - s_1|\\ &\le \Delta_{i_1} + \Delta_{j_1} \le 2\Delta.\end{aligned}$$

Similarly, we also have $|\delta_2| \le 2\Delta$.

We denote $f_1 = \frac{\partial f}{\partial x_1}$, $f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}$, a simple computation implies that

$$\begin{aligned}&\prod_{k=1}^{2}(I - V_{k,s_k})f(t_1,t_2)\\ &\quad= \int_0^1\int_0^1 f_{12}\big(s_1 + \theta_1(t_1 - s_1), s_2 + \theta_2(t_2 - s_2)\big)\,d\theta_1\,d\theta_2(t_1 - s_1)(t_2 - s_2),\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\prod_{k=1}^{2}(I - V_{k,y_k})f(x_1,x_2) \\
&\quad= \int_0^1\int_0^1 f_{12}\big(y_1+\theta_1(x_1-y_1), y_2+\theta_2(x_2-y_2)\big)\,d\theta_1\,d\theta_2(x_1-y_1)(x_2-y_2) \\
&\quad= A(x_1-x_2)(y_1-y_2),
\end{aligned} \tag{4.5}$$

where $A = \int_0^1\int_0^1 f_{12}(y_1+\theta_1(x_1-y_1), y_2+\theta_2(x_2-y_2))\,d\theta_1\,d\theta_2$ is a bounded constant.

Denote

$$\begin{aligned}
\delta_3 &= f_{12}\big(s_1+\theta_1(t_1-s_1), s_2+\theta_2(t_2-s_2)\big) \\
&\quad - f_{12}\big(y_1+\theta_1(x_1-y_1), y_2+\theta_2(x_2-y_2)\big),
\end{aligned}$$

obviously, $|\delta_3| \le C|\Delta|$. According to (4.4) and (4.5), we have

$$\begin{aligned}
&\left|\prod_{k=1}^{2}(I-V_{k,s_k})f(t_1,t_2) - \prod_{k=1}^{2}(I-V_{k,y_k})f(x_1,x_2)\right| \\
&\quad= \big|(A+\delta_3)(\delta_1+x_1-y_1)(\delta_2+x_2-y_2) - A(x_1-y_1)(x_2-y_2)\big| \\
&\quad= \big|A\delta_1\delta_2 + A(x_2-y_2)\delta_1 + A(x_1-y_1)\delta_2 + \delta_1\delta_2\delta_3 + (x_2-y_2)\delta_1\delta_3 \\
&\qquad + (x_1-y_1)\delta_2\delta_3 + (x_1-y_1)(x_2-y_2)\delta_3\big| \\
&\quad\le |A\delta_1\delta_2| + \big|A(x_2-y_2)\delta_1\big| + \big|A(x_1-y_1)\delta_2\big| + |\delta_1\delta_2\delta_3| + \big|(x_2-y_2)\delta_1\delta_3\big| \\
&\qquad + \big|(x_1-y_2)\delta_2\delta_3\big| + \big|(x_1-y_1)(x_2-y_2)\delta_3\big| \\
&\quad\le C\Delta^2 + C|x_2-y_2|\Delta + C|x_1-y_1|\Delta,
\end{aligned}$$

the proof is complete. □

It is easy to obtain the following result by calculation.

Lemma 2 *Let* $f \in C^{n+1}([0,T]^n)$. *If we denote*

$$f_{2k+1} = \frac{\partial f}{\partial x_{2k+1}}, \qquad \ldots, \qquad f_{2k+1\cdots n} = \frac{\partial^{n-2k} f}{\partial x_{2k+1}\cdots\partial x_n},$$

then we have

$$\begin{aligned}
&\big|Tr^k f^{\pi}_{2k+1\cdots n}(x_1,\ldots,x_{2k},x_{2k+1},\ldots,x_n) - Tr^k f_{2k+1\cdots n}(x_1,\ldots,x_{2k},x_{2k+1},\ldots,x_n)\big| \\
&\quad\le C\Delta^{2H}.
\end{aligned}$$

Lemma 3 *Suppose that* $f \in C^{n+1}([0,T]^n)$. *If we denote*

$$f_1 = \frac{\partial f}{\partial x_1}, \qquad f_{12} = \frac{\partial^2 f}{\partial x_1\partial x_2}, \qquad \ldots, \qquad f_{12\cdots n} = \frac{\partial^n f}{\partial x_1\cdots\partial x_n},$$

we have

$$\prod_{k=1}^{n}(I-V_{k,y_k})f(x_1,\dots,x_n)$$
$$=\int_0^1\cdots\int_0^1 f_{12\cdots n}\bigl(y_1+\theta_1(x_1-y_1),\dots,y_n+\theta_n(x_n-y_n)\bigr)\,d\theta_1\cdots d\theta_n\prod_{k=1}^{n}(x_k-y_k),$$

and

$$\left|\prod_{k=1}^{n}(I-V_{k,y_k})f(x_1,\dots,x_n)\right|\le C\prod_{k=1}^{n}|x_k-y_k|.$$

Proof It follows easily by induction. □

Lemma 4 *Suppose that* $f\in\mathcal{C}^{n+1}([0,T]^n)$. *Then, for* $t_l,x_l\in\Delta_{i_l}$, $s_l,y_l\in\Delta_{j_l}$, $l=1,\dots,n$,

$$\left(\prod_{k=1}^{n}(I-V_{k,s_k})f(t_1,\dots,t_n)-\prod_{k=1}^{n}(I-V_{k,y_k})f(x_1,\dots,x_n)\right)^2$$
$$\le C\sum_{k=1}^{n-1}\left(\sum_{1\le i_1<i_2<\cdots<i_k\le n}(x_{i_1}-y_{i_1})^2\cdots(x_{i_k}-y_{i_k})^2\right)\Delta^{2(n-k)}+C\Delta^{2n}.$$

Proof Observe that for $t_l,x_l\in\Delta_{i_l}$, $s_l,y_l\in\Delta_{j_l}$, $l=1,\dots,n$,

$$t_l-s_l=t_l-x_l+x_l-y_l+y_l-s_l=:\delta_l+x_l-y_l,$$

where

$$\delta_l=t_l-x_l+y_l-s_l.$$

It is obvious that $|\delta_l|\le|t_l-x_l|+|y_l-s_l|\le\Delta_{i_l}+\Delta_{j_l}\le 2\Delta$.

Set

$$f_{12\cdots n}\bigl(s_1+\theta_1(t_1-s_1),\dots,s_n+\theta_n(t_n-s_n)\bigr)$$
$$=:f_{12\cdots n}\bigl(y_1+\theta_1(x_1-y_1),\dots,y_n+\theta_n(x_n-y_n)\bigr)+\delta_{n+1},\tag{4.6}$$

where

$$\delta_{n+1}=f_{12\cdots n}\bigl(s_1+\theta_1(t_1-s_1),\dots,s_n+\theta_n(t_n-s_n)\bigr)$$
$$-f_{12\cdots n}\bigl(y_1+\theta_1(x_1-y_1),\dots,y_n+\theta_n(x_n-y_n)\bigr).\tag{4.7}$$

Moreover, $|\delta_{n+1}|\le C|\Delta|$.

Without loss of generality, we can write

$$A=\int_0^1\cdots\int_0^1 f_{12\cdots n}\bigl(y_1+\theta_1(x_1-y_1),\dots,y_n+\theta_n(x_n-y_n)\bigr)\,d\theta_1\cdots d\theta_n.$$

Clearly, A is a bounded constant.

Putting together (4.6) and (4.7) and using Lemma 3, we get that

$$\begin{aligned}
&\left|\prod_{k=1}^{n}(I-V_{k,s_k})f(t_1,\dots,t_n)-\prod_{k=1}^{n}(I-V_{k,y_k})f(x_1,\dots,x_n)\right| \\
&\quad=\left|\int_0^1\cdots\int_0^1\bigl[f_{12\cdots n}\bigl(y_1+\theta_1(x_1-y_1),\dots,y_n+\theta_n(x_n-y_n)\bigr)+\delta_{n+1}\bigr]\,d\theta_1\cdots d\theta_k\right. \\
&\qquad\left.\times(\delta_1+x_1-y_1)\cdots(\delta_n+x_n-y_n)-A(x_1-y_1)\cdots(x_n-y_n)\right| \\
&\quad=\bigl|(A+\delta_{n+1})(\delta_1+x_1-y_1)\cdots(\delta_n+x_n-y_n)-A(x_1-y_1)\cdots(x_n-y_n)\bigr| \\
&\quad\le C\sum_{k=1}^{n-1}\Biggl(\sum_{1\le i_1<i_2<\cdots<i_k\le n}|x_{i_1}-y_{i_1}|\cdots|x_{i_k}-y_{i_k}|\Biggr)\Delta^{n-k}+C\Delta^{n}.
\end{aligned}$$

□

Proof of Theorem 2 If we take $\varphi_\varepsilon(x,y)=\sum_i\frac{1}{\Delta_i}1_{\Delta_i}(x)1_{\Delta_i}(y)$, then we have the polygonal approximation (4.2). By using Theorem 1, it is easy to see that

$$\begin{aligned}
S_n^\pi(f)-S_n(f)&=\sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\bigl(Tr^k f^\pi\bigr) \\
&\quad-\sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\bigl(Tr^k f\bigr).
\end{aligned}$$

Set

$$g^\pi(x_1,\dots,x_n)=f^\pi(x_1,\dots,x_n)-f(x_1,\dots,x_n).$$

Then

$$\begin{aligned}
S_n^\pi(f)-S_n(f)&=\sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\bigl(Tr^k f^\pi\bigr) \\
&\quad-\sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\bigl(Tr^k f\bigr) \\
&=\sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\bigl(Tr^k g^\pi\bigr).
\end{aligned}$$

Using the properties of multiple Wiener-Itô integrals (see [11]), we derive the following:

$$\begin{aligned}
\mathbb{E}\bigl|S_n^\pi(f)-S_n(f)\bigr|^2&=\sum_{k\le[n/2]}\frac{(n!)^2}{2^{2k}(k!)^2[(n-2k)!]^2}\mathbb{E}\bigl|I_{n-2k}\bigl(Tr^k g^\pi\bigr)\bigr|^2 \\
&=\sum_{k\le[n/2]}\frac{(n!)^2}{2^{2k}(k!)^2[(n-2k)!]^2}\bigl\|Tr^k g^\pi\bigr\|^2_{\mathcal{L}^{H,n-2k}}.
\end{aligned}$$

In order to prove (4.3), we will check $\|Tr^k g^\pi\|^2_{\mathcal{L}^{H,n-2k}}$ for all $0\le k\le[n/2]$.

For $k=0$, we get

$$\bigl\|Tr^k g^\pi\bigr\|^2_{\mathcal{L}^{H,n-2k}}=\bigl\|g^\pi\bigr\|^2_{\mathcal{L}^{H,n}}\le C\Delta^{2H+1}.$$

For $1 \leq k \leq [n/2]$, we also write

$$\begin{aligned}
&\left\| Tr^k f^\pi - Tr^k f \right\|^2_{\mathcal{L}^{H,n-2k}} \\
&= C \int_{\mathbb{R}^{2(n-2k)}} \frac{(\prod_{i=2k+1}^n (I - V_{i,y_i}) \bar{F^\pi}(x_{2k+1}, \ldots, x_n))^2}{\prod_{i=2k+1}^n |x_i - y_i|^{2-2H}} \, dx_{2k+1} \cdots dx_n \, dy_{2k+1} \cdots dy_n \\
&= CF^\pi_{2(n-2k)} + \cdots + CF^\pi_{2(n-2k)-k'} + \cdots + CF^\pi_{n-2k} \quad \left(1 \leq k' < n - 2k\right),
\end{aligned}$$

where $F^\pi = Tr^k f^\pi - Tr^k f$.

By some elementary calculations, we know that the main terms which determine the convergence rate are $F^\pi_{2(n-2k)}$ and F^π_{n-2k}, whose expressions are similar to the correspondence terms F^π_{2n} and F^π_n respectively.

On the one hand, by Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
&\left(\prod_{i=2k+1}^n (I - V_{i,y_i}) F^\pi (x_{2k+1}, \ldots, x_n) \right)^2 \\
&= \left| \int_0^1 \cdots \int_0^1 F^\pi_{2k+1 \cdots n}\big(y_{2k+1} + \theta_{2k+1}(x_{2k+1} - y_{2k+1}), \ldots, \right. \\
&\quad \left. y_n + \theta_n (x_n - y_n)\big) \, d\theta_{2k+1} \cdots d\theta_n (x_{2k+1} - y_{2k+1}) \cdots (x_n - y_n) \right|^2 \\
&\leq \Delta^{4H} (x_{2k+1} - y_{2k+1})^2 \cdots (x_n - y_n)^2.
\end{aligned}$$

Note that

$$\begin{aligned}
F^\pi_{2(n-2k)} &= \int_{I_T{}^{2(n-2k)}} \frac{(\prod_{i=2k+1}^n (I - V_{i,y_i}) F^\pi (x_{2k+1}, \ldots, x_n))^2}{\prod_{i=2k+1}^n |x_i - y_i|^{2-2H}} \, dx_{2k+1} \cdots dx_n \, dy_{2k+1} \cdots dy_n \\
&\leq \Delta^{4H} \int_{I_T{}^{2(n-2k)}} \prod_{i=2k+1}^n |x_i - y_i|^{2H} \, dx_{2k+1} \cdots dx_n \, dy_{2k+1} \cdots dy_n \\
&\leq C \Delta^{4H}.
\end{aligned}$$

On the other hand, clearly, we have

$$\begin{aligned}
F^\pi_{n-2k} &= \int_{I_T{}^{n-2k}} \left(F^\pi (x_{2k+1}, \ldots, x_n) \right)^2 \prod_{i=2k+1}^n \mathcal{H}(x_i) \, dx_{2k+1} \cdots dx_n \\
&\leq C \Delta^2.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left\| Tr^k f^\pi - Tr^k f \right\|^2_{\mathcal{L}^{H,n-2k}} &\leq C \Delta^{2H+1} + C \Delta^{4H} + C \Delta^2 \\
&\leq C \Delta^{4H},
\end{aligned}$$

and the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BW proposed the problem and finished the proof. TH gave BW some useful advice to improve the convergence rate. All authors read and approved the final manuscript.

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