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# Harmonic function for which the second dilatation is $\alpha$-spiral

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**Abstract**

Let $f = h + \overline{g}$ be a harmonic function in the unit disc $\mathbb{D}$. We will give some properties of $f$ under the condition the second dilatation is $\alpha$-spiral.

**MSC:** 30C45; 30C55

**Keywords:** Harmonic functions; growth theorem; distortion theorem; coefficient inequality

## 1 Introduction

A planar harmonic mapping in the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is a complex-valued harmonic function $f$ which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected, the mapping $f$ has a canonical decomposition $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$. As usual, we call $h$ the analytic part of $f$ and $g$ the co-analytic part of $f$. An elegant and complete account of the theory of planar harmonic mapping is given in Duren's monograph [1].

Lewy [2] proved in 1936 that the harmonic function $f$ is locally univalent in a simply connected domain $\mathbb{D}_1$ if and only if its Jacobian

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 > 0$$

is different from zero in $\mathbb{D}_1$. In view of this result, locally univalent harmonic mappings in the unit disc are either sense-reversing if

$$\left|g'(z)\right| > \left|h'(z)\right|$$

in $\mathbb{D}_1$ or sense-preserving if

$$\left|g'(z)\right| < \left|h'(z)\right|$$

in $\mathbb{D}_1$. Throughout this paper, we will restrict ourselves to the study of sense-preserving harmonic mappings. However, since $f$ is sense-preserving if and only if $\overline{f}$ is sense-reserving, all the results obtained in this article regarding sense-preserving harmonic mappings can be adapted to sense-reversing ones. Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc and the second-complex dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ in $\mathbb{D}$; therefore, we can take $h(z) = z + a_2z^2 + \cdots$,

$g(z) = b_1 z + b_2 z^2 + \cdots$. Thus, the class of all harmonic mappings being sense-preserving in the unit disc can be defined by

$$S_H = \{f = h(z) + \overline{g(z)} | h(z) = z + a_2 z^2 + \cdots,$$
$$g(z) = b_1 z + b_2 z^2 + \cdots, f \text{ sense-preserving}\}.$$

Let $\Omega$ be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by $P$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ which are regular in $\mathbb{D}$ such that

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$ for all $z \in \mathbb{D}$.

Next, let $S^*$ denote the family of functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ which are regular in $\mathbb{D}$ such that

$$z \frac{s'(z)}{s(z)} = p(z) \tag{1.2}$$

for some $p(z) \in P$ for all $z \in \mathbb{D}$.

Let $s_1(z) = z + \alpha_2 z^2 + \alpha_3 z_3 + \cdots$ and $s_2(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.

Now, we consider the following class of harmonic mappings in the plane:

$$S^*_{\mathrm{HPST}}(\alpha) = \Bigg\{f = h(z) + \overline{g(z)} | f \in S_H, h(z) \in S^*,$$
$$\operatorname{Re}\big(e^{i\alpha} w(z)\big) = \operatorname{Re}\left(e^{i\alpha} \frac{g'(z)}{h'(z)}\right) > 0, |\alpha| < \frac{\pi}{2}\Bigg\}. \tag{1.3}$$

In the present paper, we will investigate the class $S^*_{\mathrm{HPST}}(\alpha)$.

We will need the following lemma and theorem in the sequel.

**Theorem 1.1** ([3, 4]) *Let $h(z)$ be an element of $S^*$, then*

$$\frac{r}{(1+r)^2} \le |h(z)| \le \frac{r}{(1-r)^2},$$

*for all $|z| = r < 1$.*

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^3}.$$

*These inequalities are sharp because the extremal function is $h(z) = \frac{z}{(1-z)^2}$.*

**Lemma 1.2** ([2, 5]) *Let $h(z)$ and $g(z)$ be regular in $\mathbb{D}$, $h(z)$ map $|z| < 1$ onto a many-sheeted starlike region,* $\operatorname{Re}(e^{i\alpha} \frac{g'(z)}{h'(z)}) > 0$, $|\alpha| < \frac{\pi}{2}$ *for $|z| < 1$. $h(0) = g(0) = 0$. Then* $\operatorname{Re}(e^{i\alpha} \frac{g(z)}{h(z)}) > 0$ *for $|z| < 1$.*

## 2 Main results

**Lemma 2.1** *Let $f = h(z) + \overline{g(z)}$ be an element of $S^*_{\mathrm{HPST}}(\alpha)$ then*

$$\frac{|b_1| - r}{1 - |b_1|r} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1| + r}{1 + |b_1|r} \tag{2.1}$$

*for all $|z| = r < 1$. This inequality is sharp because the extremal function is*

$$e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{z + b}{1 + \overline{b}z},$$

*where $b = e^{i\alpha}b_1$.*

*Proof* Since

$$w(z) = \frac{g'(z)}{h'(z)} = \frac{(b_1 z + b_2 z^2 + \cdots)'}{(z + a_2 z^2 + \cdots)'} = \frac{b_1 + 2b_2 z + \cdots}{1 + 2a_2 z + \cdots},$$

$$W(z) = e^{i\alpha}w(z) = e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{e^{i\alpha}b_1 + 2e^{i\alpha}b_2 z + \cdots}{1 + 2a_2 z + \cdots} \quad \Rightarrow \quad W(0) = e^{i\alpha}b_1 = b,$$

$$\left|W(z)\right| = \left|e^{i\alpha}w(z)\right| = \left|e^{i\alpha}\right|\left|w(z)\right| = \left|w(z)\right| < 1,$$

then the function

$$\phi(z) = \frac{W(z) - W(0)}{1 - \overline{W(0)}W(z)} = \frac{W(0) - b}{1 - \overline{b}W(0)} = \frac{b - b}{1 - b^2} = 0$$

satisfies the condition of the Schwarz lemma. Using the definition of subordination, we have

$$W(z) = e^{i\alpha}w(z) = e^{i\alpha}\frac{g'(z)}{h'(z)} = \frac{b + \phi(z)}{1 + \overline{b}\phi(z)} \quad \Leftrightarrow \quad e^{i\alpha}\frac{g'(z)}{h'(z)} \prec \frac{b + z}{1 + \overline{b}z}.$$

On the other hand, the transformation $(\frac{b+z}{1+\overline{b}z})$ maps $|z| < 1$ onto the disc with the center

$$C(r) = \left(\frac{\alpha_1(1 - r^2)}{1 - |b_1|^2 r^2}, \frac{\alpha_2(1 - r^2)}{1 - |b_1|^2 r^2}\right), \quad b = \alpha_1 + i\alpha_2$$

and the radius

$$\rho(r) = \frac{(1 - |b_1|^2)r}{1 - |b_1|^2 r^2}.$$

Therefore, we can write

$$\left|e^{i\alpha}\frac{g'(z)}{h'(z)} - \frac{b_1(1 - r^2)}{1 - |b_1|^2 r^2}\right| \leq \frac{(1 - |b_1|^2)r}{1 - |b_1|^2 r^2} \tag{2.2}$$

which gives (2.1). □

**Corollary 2.2** *Let $f \in S^*_{\mathrm{HPST}}(\alpha)$, then*

$$\frac{r(|b_1|-r)}{(1+r)^2(1-|b_1|r)} \leq \left|g(z)\right| \leq \frac{r(|b_1|+r)}{(1-r)^2(1+|b_1|r)}, \tag{2.3}$$

$$\frac{(1-r)(|b_1|-r)}{(1+r)^3(1-|b_1|r)} \leq \left|g'(z)\right| \leq \frac{(1+r)(|b_1|+r)}{(1-r)^3(1+|b_1|r)} \tag{2.4}$$

*for all $|z| = r < 1$.*

*Proof* Using Lemma 1.2 and Lemma 2.1, then we can write

$$\left|h(z)\right|\frac{|b_1|-r}{1-|b_1|r} \leq \left|g(z)\right| \leq \left|h(z)\right|\frac{|b_1|+r}{1+|b_1|r}, \tag{2.5}$$

$$\left|h'(z)\right|\frac{|b_1|-r}{1-|b_1|r} \leq \left|g'(z)\right| \leq \left|h'(z)\right|\frac{|b_1|+r}{1+|b_1|r}. \tag{2.6}$$

If we use Theorem 1.1 in the inequalities (2.5) and (2.6), we get (2.3) and (2.4). □

**Corollary 2.3** *Let $f = h(z) + \overline{g(z)}$ be an element of $S^*_{HPTS}(\alpha)$, then*

$$\frac{(1-|b_1|^2)(1-r)^3}{(1+r)^5(1+|b_1|r)^2} \leq J_{f(z)} \leq \frac{(1-|b_1|^2)(1+r)^3}{(1-r)^5(1+|b_1|r)^2} \tag{2.7}$$

*for all $|z| = r < 1$.*

*Proof* Since

$$J_{f(z)} = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 = \left|h'(z)\right|^2\left(1-\left|w(z)\right|^2\right), \tag{2.8}$$

using Lemma 2.1 and Theorem 1.1 in the equality (2.8) and after simple calculations, we get (2.7). □

**Corollary 2.4** *If $f = h(z) + \overline{g(z)}$ is an element of $S^*_{HPTS}(\alpha)$, then*

$$\begin{aligned}
&\frac{1}{(1+a)^3(-1+r)^2}\left(2(1+a)\left(a(-2+r)-r\right)r + (-1+a)^2(-1+r)^2\log(1-r)\right. \\
&\quad \left. - (-1+a)^2(-1+r)^2\log(1+ar)\right) \leq \left|f(z)\right| \leq \frac{1}{(1+a)^2(-1+r^2)^2} \\
&\quad \left(-2r\left(-1+r+4ar+r^2+r^3-a^2\left(-1+r\left(-3+r+r^2\right)\right)\right)\right. \\
&\quad \left. + (-1+a)^2\left(-1+r^2\right)^2\log(1-r) - (-1+a)^2\left(-1+r^2\right)^2\log(1+ar)\right),
\end{aligned} \tag{2.9}$$

*where $a = |b_1|$ for all $|z| = r < 1$.*

*Proof* Using Corollary 2.2 and Theorem 1.1, we obtain

$$\left(\left|h'(z)\right| - \left|g'(z)\right|\right) \geq \frac{(1-r^4)(1+|b_1|r) - (1+r^4)(|b_1|+r)}{(1-r)^3(1+r)^3(1+|b_1|r)},$$

and

$$\left(\left|h'(z)\right| + \left|g'(z)\right|\right) \leq \frac{(1+r)^2(1+|b_1|)}{(1-r)^3(1+|b_1|r)}.$$

Therefore, we have

$$\begin{aligned}
&\left(\left|h'(z)\right| - \left|g'(z)\right|\right)|dz| \leq |df| \leq \left(\left|h'(z)\right| + \left|g'(z)\right|\right)|dz| \\
&\Rightarrow \quad \frac{(1-r)^4(1+|b_1|r) - (1+r)^4(|b_1|+r)}{(1-r)^3(1+r)^3(1+|b_1|r)}\,dr \leq |df| \leq \frac{(1+r)^2(1+|b_1|)}{(1-r)^3(1+|b_1|r)}\,dr. \qquad (2.10)
\end{aligned}$$

Integrating the last inequality (2.10), we get (2.9). □

**Theorem 2.5** *Let $f = h(z) + \overline{g(z)}$ be an element of $S^*_{HPTS}(\alpha)$, then*

$$\sum_{k=1}^{n} |A_k|^2 \leq |t+1|^2 + \sum_{k=1}^{n} |B_k|^2 \qquad (2.11)$$

*where $A_k = (k+1)(\frac{b_{k+1}}{b_1} - a_{k+1})$; $B_k = (k+1)(\frac{b_{k+1}}{b_1} + ta_{k+1})$; $a_k$ and $b_k$ are the coefficients of the functions $h(z)$ and $g(z)$; $k = 1, 2, 3, \ldots, n$; $t = 2s - 1$; $s = e^{-i\alpha}\cos\alpha$.*

*Proof* Since

$$g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots \quad \Rightarrow \quad g'(z) = b_1 + 2b_2 z + 3b_3 z^2 + \cdots.$$

We denote by $G(z) = \frac{1}{b_1} g(z)$

$$\begin{aligned}
&G'(z) = \frac{1}{b_1} g'(z) = 1 + 2\frac{b_2}{b_1} z + 3\frac{b_3}{b_1} z^2 + \cdots, \qquad h(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \\
&h'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots,
\end{aligned}$$

then we have

$$\begin{cases} \frac{1}{\cos\alpha}\left(e^{i\alpha}\frac{\frac{1}{b_1}g'(z)}{h'(z)} - i\sin\alpha\right) = p(z) \quad \Leftrightarrow \quad e^{i\alpha}\frac{\frac{1}{b_1}g'(z)}{h'(z)} = \cos\alpha\, p(z) + i\sin\alpha, \\ \quad \Leftrightarrow \quad \frac{\frac{1}{b_1}g'(z)}{h'(z)} = 1 + e^{-i\alpha}\cos\alpha(p(z) - 1). \end{cases} \qquad (2.12)$$

Since $p(z)$ is in $P$, there is a function $\phi(z)$ satisfying the conditions of the Schwarz lemma such that

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} \quad \Leftrightarrow \quad p(z) - 1 = \frac{2\phi(z)}{1-\phi(z)}. \qquad (2.13)$$

Using this equation in (2.12) and after the following calculations given above

$$\frac{\frac{1}{b_1}g'(z)}{h'(z)} = 1 + e^{-i\alpha}\cos\alpha\left(p(z) - 1\right) = 1 + s\left(\frac{2\phi(z)}{1-\phi(z)}\right) \quad \Rightarrow,$$

we get the following equality:

$$\frac{1}{b_1}g'(z) - h'(z) = \left(th'(z) + \frac{1}{b_1}g'(z)\right). \tag{2.14}$$

If $\phi(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ , we have

$$\sum_{k=1}^{n} A_k z^k + \sum_{k=n+1}^{\infty} D_k z^k = \left[(1+t) + \sum_{k=1}^{n} B_k z^k\right]\phi(z), \tag{2.15}$$

where

$$\sum_{k=n+1}^{\infty} D_k z^k = \sum_{k=n+1}^{\infty} A_k z^k - \left(c_1 B_n z^{n+1} + c_1 B_{n+1} z^{n+2} + \cdots\right).$$

Therefore, the equality (2.15) can be considered in the following form:

$$F(z) = G(z)\phi(z). \tag{2.16}$$

Using the Clunie method [6], then we can write

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(re^{i\theta}\right)\right|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|G\left(re^{i\theta}\right)\right|^2 d\theta,$$

which gives

$$\sum_{k=1}^{n} |A_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |D_k|^2 r^{2k} \le \left(|t+1|^2 + \sum_{k=1}^{n} |B_k|^2 r^{2k}\right). \tag{2.17}$$

Eventually, we will let $r \to 1^-$, then we have

$$\sum_{k=1}^{n} |A_k|^2 \le |t+1|^2 + \sum_{k=1}^{n} |B_k|^2. \qquad \square$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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