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A new more accurate half-discrete Hilbert-type inequality

Aizhen Wang* and Bicheng Yang

*Correspondence:
zhenmaths@gdei.edu.cn
Department of Math., Guangdong University of Education Guangzhou, Guangdong, 510303, P.R. China

Abstract

By using the way of weight functions and the idea of introducing parameters and by means of Hadamard's inequality, we give a more accurate half-discrete Hilbert-type inequality with a best constant factor. We also consider its best extension with parameters, equivalent forms, operator expressions as well as some reverses.

MSC: 26D15; 47A07

Keywords: weight function; parameter; reverse; equivalent form; Hadamard's inequality; Hilbert-type inequality

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following famous Hilbert-type integral inequality (*cf.* [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\ln(m/n)}{m-n} a_m b_n < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{1}$$

where the constant factor $[\pi/\sin(\pi/p)]^2$ is the best possible. The integral analogue of inequality (1) is given as follows (*cf.* [1]). If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$ and $g(x)$ are non-negative real functions such that $0 < \int_0^{\infty} f^p(x)\,dx < \infty$ and $0 < \int_0^{\infty} g^q(x)\,dx < \infty$, then

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty} \frac{\ln(x/y)}{x-y} f(x)g(y)\,dx\,dy \\ &\quad < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left\{\int_0^{\infty} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\int_0^{\infty} g^q(x)\,dx\right\}^{\frac{1}{q}}, \end{aligned} \tag{2}$$

where the constant factor $[\pi/\sin(\pi/p)]^2$ is the best possible. We named inequality (2) Hilbert-type integral inequality. In 2007, Yang proved the following more accurate Hilbert-type inequality (*cf.* [2]). If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2} \leq \alpha \leq 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} a_m b_n < \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left\{\sum_{n=0}^{\infty} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{3}$$

where the constant factor $[\pi/\sin(\pi/p)]^2$ is still the best possible. Inequalities (1)-(3) are important in mathematical analysis and its applications [3]. There are lots of improve-

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ments, generalizations, and applications of inequalities (1)-(3); for more details, refer to [4–17].

At present, the research into half-discrete Hilbert-type inequalities is a new direction and has gradually heated up. We find a few results on the half-discrete Hilbert-type inequalities with the non-homogeneous kernel, which were published earlier (*cf.* [1], Th. 351 and [18]). Recently, Yang has given some half-discrete Hilbert-type inequalities (*cf.* [19–25]). Zhong proved a half-discrete Hilbert-type inequality with the non-homogeneous kernel as follows (*cf.* [26]). If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 2$, $a_n, f(x) \geq 0$, $f(x)$ is a measurable function in $(0, \infty)$ such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty$ and $0 < \int_0^{\infty} x^{q(1-\frac{\lambda}{2})-1} f^q(x)\,dx < \infty$, then

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\ln(nx)}{(nx)^{\lambda} - 1} f(x) a_n \,dx < \left(\frac{\pi}{\lambda}\right)^2 \left\{\sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p\right\}^{\frac{1}{p}} \left\{\int_0^{\infty} x^{q(1-\frac{\lambda}{2})-1} f^q(x)\,dx\right\}^{\frac{1}{q}}, \tag{4}$$

where the constant factor $(\frac{\pi}{\lambda})^2$ is the best possible.

In this paper, by using the way of weight functions and the idea of introducing parameters and by means of Hadamard's inequality, we give a half-discrete Hilbert-type inequality with a best constant factor as follows:

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{\ln(\frac{x}{n})}{x - n} f(x)\,dx < \pi^2 \left(\sum_{n=1}^{\infty} a_n^2 \int_0^{\infty} f^2(x)\,dx\right)^{\frac{1}{2}}. \tag{5}$$

The main objective of this paper is to consider its more accurate extension with parameters, equivalent forms, operator expressions as well as some reverses.

2 Some lemmas

Lemma 1 *If $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, define the following beta function* (*cf.* [1]):

$$\int_0^{\infty} \frac{\ln u}{u - 1} u^{\frac{1}{s}-1}\,du = \left[B\left(\frac{1}{s}, \frac{1}{r}\right)\right]^2 = \left(\frac{\pi}{\sin\frac{\pi}{s}}\right)^2. \tag{6}$$

Lemma 2 *Suppose that $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 \leq \beta \leq \frac{1}{2}$, $\nu \in (-\infty, \infty)$, $0 < \lambda \leq 1$. Define the weight functions $\omega(n)$ and $\tilde{\omega}(x)$ as follows*:

$$\omega_\lambda(n) := (n-\beta)^{\frac{\lambda}{r}} \int_\nu^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} (x-\nu)^{\frac{\lambda}{s}-1}\,dx \quad (n \in \mathbf{N}), \tag{7}$$

$$\tilde{\omega}_\lambda(x) := (x-\nu)^{\frac{\lambda}{s}} \sum_{n=1}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} (n-\beta)^{\frac{\lambda}{r}-1} \quad \bigl(x \in (\nu, \infty)\bigr). \tag{8}$$

Setting $k_\lambda(r) := [\frac{\pi}{\lambda \sin(\frac{\pi}{r})}]^2$, we have the following inequalities:

$$0 < k_\lambda(r)\bigl(1 - \theta_\lambda(x)\bigr) < \tilde{\omega}_\lambda(x) < \omega_\lambda(n) = k_\lambda(r), \tag{9}$$

$$0 < \theta_\lambda(x) := \left[\frac{\sin(\frac{\pi}{r})}{\pi}\right]^2 \int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}} \frac{\ln v}{v - 1} v^{\frac{1}{r}-1}\,dv = O\left(\frac{1}{(x-\nu)^{\lambda/2r}}\right) \quad \bigl(x \in (\nu, \infty)\bigr). \tag{10}$$

Proof Putting $u = (\frac{x-\nu}{n-\beta})^{\lambda}$ in (8), we have

$$\begin{aligned}\omega_{\lambda}(n) &= \frac{1}{\lambda^2}\int_0^{\infty}\frac{\ln u}{u-1}u^{\frac{1}{s}-1}\,du = \frac{1}{\lambda^2}\left[B\left(\frac{1}{s},\frac{1}{r}\right)\right]^2 \\ &= \frac{1}{\lambda^2}\left[B\left(\frac{1}{r},\frac{1}{s}\right)\right]^2 = \left[\frac{\pi}{\lambda\sin(\frac{\pi}{r})}\right]^2 = k_{\lambda}(r).\end{aligned} \tag{11}$$

For fixed $x \in (\nu,\infty)$, setting

$$f(t) := \frac{(x-\nu)^{\frac{\lambda}{s}}\ln(\frac{x-\nu}{t-\beta})}{(x-\nu)^{\lambda}-(t-\beta)^{\lambda}}(t-\beta)^{\frac{\lambda}{r}-1} \quad \bigl(t \in (\beta,\infty)\bigr), \tag{12}$$

in view of the conditions, we find $f'(t) < 0$ and $f''(t) > 0$ (*cf.* [27]). By Hadamard's inequality (cf. [28]),

$$f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)\,dt \quad (n \in \mathbf{N}), \tag{13}$$

and putting $v = (\frac{t-\beta}{x-\nu})^{\lambda}$, we obtain

$$\begin{aligned}\tilde{\omega}_{\lambda}(x) &= \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^{\infty}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t)\,dt = \int_{\frac{1}{2}}^{\infty} f(t)\,dt \\ &\le \int_{\beta}^{\infty} f(t)\,dt = \frac{1}{\lambda^2}\int_0^{\infty}\frac{\ln v}{v-1}v^{\frac{1}{r}-1}\,du = k_{\lambda}(r), \\ \tilde{\omega}_{\lambda}(x) &= \sum_{n=1}^{\infty} f(n) > \int_1^{\infty} f(t)\,dt = \int_{\beta}^{\infty} f(t)\,dt - \int_{\beta}^{1} f(t)\,dt \\ &= k_{\lambda}(r) - \frac{1}{\lambda^2}\int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}}\frac{\ln v}{v-1}v^{\frac{1}{r}-1}\,dv = k_{\lambda}(r)\bigl(1-\theta_{\lambda}(x)\bigr) > 0,\end{aligned}$$

where,

$$0 < \theta_{\lambda}(x) := \left[\frac{\sin(\frac{\pi}{r})}{\pi}\right]^2\int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}}\frac{\ln v}{v-1}v^{\frac{1}{r}-1}\,dv \quad \bigl(x \in (\nu,\infty)\bigr).$$

Since $\lim_{v\to 0^+}\frac{\ln v}{v-1}v^{\frac{1}{2r}} = \lim_{v\to\infty}\frac{\ln v}{v-1}v^{\frac{1}{2r}} = 0$ and $\frac{\ln v}{v-1}v^{\frac{1}{2r}}|_{v=1} = 1$, in view of the bounded properties of a continuous function, there exists $M > 0$ such that $0 < \frac{\ln v}{v-1}v^{\frac{1}{2r}} \le M$ $(v \in (0,\infty))$. For $x \in (\nu,\infty)$, we have

$$\begin{aligned}0 &< \int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}}\frac{\ln v}{v-1}v^{\frac{1}{r}-1}\,dv = \int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}}\frac{\ln v}{v-1}v^{\frac{1}{2r}}\cdot v^{\frac{1}{2r}-1}\,dv \\ &\le M\int_0^{(\frac{1-\beta}{x-\nu})^{\lambda}} v^{\frac{1}{2r}-1}\,dv = \frac{2Mr(1-\beta)^{\lambda/2r}}{(x-\nu)^{\lambda/2r}}.\end{aligned} \tag{14}$$

Hence, we proved that (9) and (10) are valid. □

Lemma 3 *Suppose that* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\frac{1}{p} + \frac{1}{q} = 1$ $(p \neq 0, 1)$, $0 \leq \beta \leq \frac{1}{2}$, $\nu \in (-\infty, \infty)$, $0 < \lambda \leq 1$, $a_n \geq 0$, *and* $f(x)$ *is a non-negative real measurable function in* (ν, ∞). *Then*

(i) *for* $p > 1$, *we have the following inequalities*:

$$J := \left\{ \sum_{n=1}^{\infty} (n-\beta)^{\frac{p\lambda}{r}-1} \left[\int_{\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} f(x)\, dx \right]^{p} \right\}^{\frac{1}{p}}$$

$$\leq \left[k_{\lambda}(r)\right]^{\frac{1}{q}} \left\{ \int_{\nu}^{\infty} \tilde{\omega}_{\lambda}(x)(x-\nu)^{p(1-\frac{\lambda}{s})-1} f^{p}(x)\, dx \right\}^{\frac{1}{p}}, \tag{15}$$

$$L_1 := \left\{ \int_{\nu}^{\infty} \frac{(x-\nu)^{\frac{q\lambda}{s}-1}}{\tilde{\omega}_{\lambda}^{q-1}(x)} \left[\sum_{n=1}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} a_n \right]^{q} dx \right\}^{\frac{1}{q}}$$

$$\leq \left\{ k_{\lambda}(r) \sum_{n=1}^{\infty} (n-\beta)^{q(1-\frac{\lambda}{r})-1} a_n^{q} \right\}^{\frac{1}{q}}, \tag{16}$$

where $\omega_{\lambda}(n)$ *and* $\tilde{\omega}_{\lambda}(x)$ *are defined by* (7) *and* (8).

(ii) *for* $p < 1$ $(p \neq 0)$, *we have the reverses of* (15) *and* (16).

Proof (i) By (7)-(10) and Hölder's inequality [28], we have

$$\left[\int_{\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} f(x)\, dx \right]^{p}$$

$$= \left\{ \int_{\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \left[\frac{(x-\nu)^{(1-\frac{\lambda}{s})/q}}{(n-\beta)^{(1-\frac{\lambda}{r})/p}} f(x) \right] \left[\frac{(n-\beta)^{(1-\frac{\lambda}{r})/p}}{(x-\nu)^{(1-\frac{\lambda}{s})/q}} \right] dx \right\}^{p}$$

$$\leq \int_{\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \frac{(x-\nu)^{(1-\frac{\lambda}{s})(p-1)}}{(n-\beta)^{1-\frac{\lambda}{r}}} f^{p}(x)\, dx$$

$$\times \left[\int_{\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \frac{(n-\beta)^{(1-\frac{\lambda}{r})(q-1)}}{(x-\nu)^{1-\frac{\lambda}{s}}}\, dx \right]^{p-1}$$

$$= \int_{\nu}^{\infty} \frac{f^{p}(x) \ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \frac{(x-\nu)^{(1-\frac{\lambda}{s})(p-1)}}{(n-\beta)^{1-\frac{\lambda}{r}}}\, dx \left[(n-\beta)^{q(1-\frac{\lambda}{r})-1} \omega_{\lambda}(n) \right]^{p-1}$$

$$= (n-\beta)^{1-\frac{p\lambda}{r}} k_{\lambda}^{p-1}(r) \int_{\nu}^{\infty} \frac{f^{p}(x) \ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \frac{(x-\nu)^{(1-\frac{\lambda}{s})(p-1)}}{(n-\beta)^{1-\frac{\lambda}{r}}}\, dx. \tag{17}$$

By the Lebesgue term-by-term integration theorem [29] and (9), we obtain

$$J^{p} \leq k_{\lambda}^{p-1}(r) \sum_{n=1}^{\infty} \int_{\nu}^{\infty} \frac{f^{p}(x) \ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} \frac{(x-\nu)^{(1-\frac{\lambda}{s})(p-1)}}{(n-\beta)^{1-\frac{\lambda}{r}}}\, dx$$

$$= k_{\lambda}^{p-1}(r) \int_{\nu}^{\infty} \sum_{n=1}^{\infty} \frac{(n-\beta)^{\frac{\lambda}{r}-1} \ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} (x-\nu)^{\frac{\lambda}{s}+p(1-\frac{\lambda}{s})-1} f^{p}(x)\, dx$$

$$= k_{\lambda}^{p-1}(r) \int_{\nu}^{\infty} \tilde{\omega}_{\lambda}(x)(x-\nu)^{p(1-\frac{\lambda}{s})-1} f^{p}(x)\, dx. \tag{18}$$

Hence, (15) is valid. Using Hölder's inequality, the Lebesgue term-by-term integration theorem, and (9) again, we have

$$\begin{aligned}
&\left[\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}a_n\right]^q \\
&\quad=\left\{\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\left[\frac{(x-\nu)^{(1-\frac{\lambda}{s})/q}}{(n-\beta)^{(1-\frac{\lambda}{r})/p}}\right]\left[\frac{(n-\beta)^{(1-\frac{\lambda}{r})/p}}{(x-\nu)^{(1-\frac{\lambda}{s})/q}}a_n\right]\right\}^q \\
&\quad\le\left[\tilde{\omega}_{\lambda}(x)(x-\nu)^{p(1-\frac{\lambda}{s})-1}\right]^{q-1}\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\frac{(n-\beta)^{(1-\frac{\lambda}{r})(q-1)}a_n^q}{(x-\nu)^{1-\frac{\lambda}{s}}} \\
&\quad=\tilde{\omega}_{\lambda}^{q-1}(x)(x-\nu)^{1-\frac{q\lambda}{s}}\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\frac{(n-\beta)^{(1-\frac{\lambda}{r})(q-1)}}{(x-\nu)^{1-\frac{\lambda}{s}}}a_n^q, \qquad (19)
\end{aligned}$$

$$\begin{aligned}
L_1^q&\le\int_{\nu}^{\infty}\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\frac{(n-\beta)^{(1-\frac{\lambda}{r})(q-1)}}{(x-\nu)^{1-\frac{\lambda}{s}}}a_n^q\,dx \\
&=\sum_{n=1}^{\infty}\left[(n-\beta)^{\frac{\lambda}{r}}\int_{\nu}^{\infty}\frac{(x-\nu)^{\frac{\lambda}{s}-1}\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\,dx\right](n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q \\
&=\sum_{n=1}^{\infty}\omega_{\lambda}(n)(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q=k_{\lambda}(r)\sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q. \qquad (20)
\end{aligned}$$

Hence, (16) is valid.

(ii) For $0 < p < 1$ ($q < 0$) or $p < 0$ ($0 < q < 1$), using the reverse Hölder inequality, in the same way, we obtain the reverses of (15) and (16). □

Lemma 4 *By the assumptions of Lemma* 2 *and Lemma* 3, *we set* $\phi(x) := (x-\nu)^{p(1-\frac{\lambda}{s})-1}$, $\widetilde{\phi}(x) := (1-\theta_{\lambda}(x))\phi(x)$, $\psi(n) := (n-\beta)^{q(1-\frac{\lambda}{r})-1}$,

$$\begin{aligned}
L_{p,\phi}(\nu,\infty)&:=\left\{f;\|f\|_{p,\phi}=\left[\int_{\nu}^{\infty}\phi(x)\left|f(x)\right|^p\,dx\right]^{1/p}<\infty\right\}, \\
l_{q,\psi}&:=\left\{a=\{a_n\};\|a\|_{q,\psi}=\left[\sum_{n=1}^{\infty}\psi(n)|a_n|^q\right]^{1/q}<\infty\right\}.
\end{aligned}$$

(Note *If* $p > 1$, *then* $L_{p,\phi}(\nu,\infty)$ *and* $l_{q,\psi}$ *are normal spaces*; *if* $0 < p < 1$ *or* $p < 0$, *then both* $L_{p,\phi}(\nu,\infty)$ *and* $l_{q,\psi}$ *are not normal spaces, but we still use the formal symbols in the following*.)

For $0 < \varepsilon < \frac{|p|\lambda}{s}$, *setting* $\widetilde{a} = \{\widetilde{a}_n\}_{n=1}^{\infty}$, *and* $\widetilde{f}(x)$ *as follows*:

$$\widetilde{a}_n=(n-\beta)^{\frac{\lambda}{r}-\frac{\varepsilon}{q}-1};\qquad \widetilde{f}(x)=\begin{cases}0, & x\in(\nu,1+\nu),\\ (x-\nu)^{\frac{\lambda}{s}-\frac{\varepsilon}{p}-1}, & x\in[1+\nu,\infty),\end{cases} \qquad (21)$$

(i) *if* $p > 1$, *there exists a constant* $k > 0$ *such that*

$$\widetilde{I}:=\sum_{n=1}^{\infty}\widetilde{a}_n\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\widetilde{f}(x)\,dx<k\|\widetilde{f}\|_{p,\phi}\|\widetilde{a}\|_{q,\psi}, \qquad (22)$$

then it follows

$$k\left[\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right]^{1/q} > \frac{1}{\lambda^2(1-\beta)^{\varepsilon}}\int_0^{(1-\beta)^\lambda}\frac{\ln t}{t-1}t^{\frac{1}{r}+\frac{\varepsilon}{p\lambda}-1}\,dt + \frac{1}{\lambda^2}\int_{(1-\beta)^\lambda}^{\infty}\frac{\ln t}{t-1}t^{\frac{1}{r}-\frac{\varepsilon}{q\lambda}-1}\,dt; \tag{23}$$

(ii) *if* $0 < p < 1$, *there exists a constant* $k > 0$ *such that*

$$\widetilde{I} = \sum_{n=1}^{\infty}\widetilde{a}_n\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}\widetilde{f}(x)\,dx > k\|\widetilde{f}\|_{p,\widetilde{\phi}}\|\widetilde{a}\|_{q,\psi}, \tag{24}$$

then it follows

$$k\big(1-\varepsilon O(1)\big)^{1/p} < \frac{1}{\lambda^2}\left[\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right]^{1/p}\left[B\left(\frac{1}{s}-\frac{\varepsilon}{p\lambda},\frac{1}{r}+\frac{\varepsilon}{p\lambda}\right)\right]^2; \tag{25}$$

(iii) *if* $p < 0$, *there exists a constant* $k > 0$ *such that*

$$\widetilde{I} = \sum_{n=1}^{\infty}\widetilde{a}_n\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}\widetilde{f}(x)\,dx > k\|\widetilde{f}\|_{p,\phi}\|\widetilde{a}\|_{q,\psi}, \tag{26}$$

then it follows

$$k\left[\frac{1}{(1-\beta)^{\varepsilon}}\right]^{1/q} < \frac{\varepsilon+1-\beta}{\lambda^2(1-\beta)^{\varepsilon+1}}\left[B\left(\frac{1}{s}-\frac{\varepsilon}{p\lambda},\frac{1}{r}+\frac{\varepsilon}{p\lambda}\right)\right]^2. \tag{27}$$

Proof We can obtain

$$\|\widetilde{f}\|_{p,\phi} = \left\{\int_{\nu}^{\infty}(x-\nu)^{p(1-\frac{\lambda}{s})-1}\widetilde{f}^p(x)\,dx\right\}^{1/p} = \left\{\int_{1+\nu}^{\infty}(x-\nu)^{-1-\varepsilon}\,dx\right\}^{1/p} = \left(\frac{1}{\varepsilon}\right)^{1/p}, \tag{28}$$

$$\|\widetilde{a}\|_{q,\psi}^q = \sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}\widetilde{a}_n^q = \sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon} < (1-\beta)^{-1-\varepsilon} + \int_1^{\infty}(x-\beta)^{-1-\varepsilon}\,dx = \frac{\varepsilon+1-\beta}{\varepsilon(1-\beta)^{\varepsilon+1}}, \tag{29}$$

$$\|\widetilde{a}\|_{q,\psi}^q = \sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon} > \int_1^{\infty}(x-\beta)^{-1-\varepsilon}\,dx = \frac{1}{\varepsilon(1-\beta)^{\varepsilon}}. \tag{30}$$

(i) For $p > 1$, then $q > 1$, $\frac{\lambda}{r}-\frac{\varepsilon}{q}-1 < 0$, by (22), (28), and (29), we find

$$\widetilde{I} < k\left(\frac{1}{\varepsilon}\right)^{1/p}\left[\frac{\varepsilon+1-\beta}{\varepsilon(1-\beta)^{\varepsilon+1}}\right]^{1/q} = \frac{k}{\varepsilon}\left[\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right]^{1/q}, \tag{31}$$

$$\widetilde{I} = \int_{1+\nu}^{\infty}(x-\nu)^{\frac{\lambda}{s}-\frac{\varepsilon}{p}-1}\left[\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}(n-\beta)^{\frac{\lambda}{r}-\frac{\varepsilon}{q}-1}\right]dx \geq \int_{1+\nu}^{\infty}(x-\nu)^{\frac{\lambda}{s}-\frac{\varepsilon}{p}-1}\left[\int_1^{\infty}\frac{\ln(\frac{x-\nu}{y-\beta})}{(x-\nu)^\lambda-(y-\beta)^\lambda}(y-\beta)^{\frac{\lambda}{r}-\frac{\varepsilon}{q}-1}\,dy\right]dx.$$

Setting $t = (\frac{y-\beta}{x-\nu})^{\lambda}$, $z = x - \nu$ in the above integral, we have

$$\widetilde{I} \geq \frac{1}{\lambda^2} \int_1^{\infty} z^{-1-\varepsilon} \left[\int_{(\frac{1-\beta}{z})^{\lambda}}^{\infty} \frac{\ln t}{t-1} t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt \right] dz = I_1 + I_2, \tag{32}$$

and by the Fubini theorem [30], it follows

$$\begin{aligned} I_1 &:= \frac{1}{\lambda^2} \int_1^{\infty} z^{-1-\varepsilon} \left[\int_{(\frac{1-\beta}{z})^{\lambda}}^{(1-\beta)^{\lambda}} \frac{\ln t}{t-1} t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt \right] dz \\ &= \frac{1}{\lambda^2} \int_0^{(1-\beta)^{\lambda}} \frac{\ln t}{t-1} \left[\int_{(1-\beta)t^{-1/\lambda}}^{\infty} z^{-1-\varepsilon} \, dz \right] t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt \\ &= \frac{1}{\varepsilon\lambda^2(1-\beta)^{\varepsilon}} \int_0^{(1-\beta)^{\lambda}} \frac{\ln t}{t-1} t^{\frac{1}{r} + \frac{\varepsilon}{p\lambda} - 1} \, dt, \end{aligned} \tag{33}$$

$$\begin{aligned} I_2 &:= \frac{1}{\lambda^2} \int_1^{\infty} z^{-1-\varepsilon} \left[\int_{(1-\beta)^{\lambda}}^{\infty} \frac{\ln t}{t-1} t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt \right] dz \\ &= \frac{1}{\varepsilon\lambda^2} \int_{(1-\beta)^{\lambda}}^{\infty} \frac{\ln t}{t-1} t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt. \end{aligned} \tag{34}$$

In view of (33) and (34), it follows that

$$\begin{aligned} \widetilde{I} \geq{}& \frac{1}{\varepsilon\lambda^2(1-\beta)^{\varepsilon}} \int_0^{(1-\beta)^{\lambda}} \frac{\ln t}{t-1} t^{\frac{1}{r} + \frac{\varepsilon}{p\lambda} - 1} \, dt \\ &+ \frac{1}{\varepsilon\lambda^2} \int_{(1-\beta)^{\lambda}}^{\infty} \frac{\ln t}{t-1} t^{\frac{1}{r} - \frac{\varepsilon}{q\lambda} - 1} \, dt. \end{aligned} \tag{35}$$

Then by (31) and (35), (23) is valid.

(ii) For $0 < p < 1$, by (24) and (29), we find (notice that $q < 0$)

$$\begin{aligned} \widetilde{I} &> k \|\widetilde{f}\|_{p,\widetilde{\phi}} \|\widetilde{a}_n\|_{q,\psi} = k \left\{ \int_{\nu}^{\infty} \widetilde{\phi}(x) \left|\widetilde{f}(x)\right|^p dx \right\}^{1/p} \|\widetilde{a}\|_{q,\psi} \\ &= k \left\{ \int_{1+\nu}^{\infty} \left[1 - O\left(\frac{1}{(x-\nu)^{\lambda/2r}} \right) \right] (x-\nu)^{-1-\varepsilon} \, dx \right\}^{1/p} \|\widetilde{a}\|_{q,\psi} \\ &= k \left[\frac{1}{\varepsilon} - \int_{1+\nu}^{\infty} O\left(\frac{1}{(x-\nu)^{\frac{\lambda}{2r}+\varepsilon+1}} \right) dx \right]^{1/p} \|\widetilde{a}\|_{q,\psi} \\ &> k \left[\frac{1}{\varepsilon} - O(1) \right]^{1/p} \left[\frac{\varepsilon + 1 - \beta}{\varepsilon(1-\beta)^{\varepsilon+1}} \right]^{1/q} \\ &= \frac{k}{\varepsilon} \left[1 - \varepsilon O(1) \right]^{1/p} \left[\frac{\varepsilon + 1 - \beta}{(1-\beta)^{\varepsilon+1}} \right]^{1/q}. \end{aligned} \tag{36}$$

On the other hand, setting $t = (\frac{x-\nu}{n-\beta})^{\lambda}$ in $\widetilde{I}$, we have

$$\begin{aligned} \widetilde{I} &= \sum_{n=1}^{\infty} (n-\beta)^{\frac{\lambda}{r} - \frac{\varepsilon}{q} - 1} \int_{1+\nu}^{\infty} \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda} - (n-\beta)^{\lambda}} (x-\nu)^{\frac{\lambda}{s} - \frac{\varepsilon}{p} - 1} \, dx \\ &= \frac{1}{\lambda^2} \sum_{n=1}^{\infty} (n-\beta)^{-1-\varepsilon} \int_{\frac{1}{(n-\beta)^{\lambda}}}^{\infty} \frac{\ln t}{t-1} t^{\frac{1}{s} - \frac{\varepsilon}{p\lambda} - 1} \, dt \end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{\lambda^2}\sum_{n=1}^{\infty}(n-\beta)^{-1-\varepsilon}\int_0^{\infty}\frac{\ln t}{t-1}t^{\frac{1}{s}-\frac{\varepsilon}{p\lambda}-1}\,dt \\
&< \frac{\varepsilon+1-\beta}{\lambda^2\varepsilon(1-\beta)^{\varepsilon+1}}\left[B\left(\frac{1}{s}-\frac{\varepsilon}{p\lambda},\frac{1}{r}+\frac{\varepsilon}{p\lambda}\right)\right]^2. \qquad (37)
\end{aligned}
$$

By virtue of (36) and (37), (25) is valid.

(iii) For $p < 0$, then $0 < q < 1$, by (26) and (30), we find

$$
\begin{aligned}
\widetilde{I} &> k\left\{\int_{\nu}^{\infty}\phi(x)\widetilde{f}^p(x)\,dx\right\}^{1/p}\|\widetilde{a}\|_{q,\psi} = k\left\{\int_{1+\nu}^{\infty}(x-\nu)^{-1-\varepsilon}\,dx\right\}^{1/p}\|\widetilde{a}\|_{q,\psi} \\
&> \frac{k}{\varepsilon}\left[\frac{1}{(1-\beta)^{\varepsilon}}\right]^{1/q}. \qquad (38)
\end{aligned}
$$

Then by (37) and (38), (27) is valid. □

3 Main results and applications

Theorem 5 *Suppose that* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, $0 < \lambda \le 1$, $0 \le \beta \le \frac{1}{2}$, $\nu \in (-\infty,+\infty)$, $\phi(x) := (x-\nu)^{p(1-\frac{\lambda}{s})-1}$, $\psi(n) := (n-\beta)^{q(1-\frac{\lambda}{r})-1}$, $f(x), a_n \ge 0$, *satisfying* $f \in L_{p,\phi}(\nu,\infty)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\psi}$, $\|f\|_{p,\phi} > 0$, $\|a\|_{q,\psi} > 0$, *then we have the following equivalent inequalities*:

$$
\begin{aligned}
I &:= \sum_{n=1}^{\infty}a_n\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}f(x)\,dx \\
&= \int_{\nu}^{\infty}f(x)\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}a_n\,dx < k_{\lambda}(r)\|f\|_{p,\phi}\|a\|_{q,\psi}, \qquad (39)
\end{aligned}
$$

$$
J = \left\{\sum_{n=1}^{\infty}(n-\beta)^{\frac{p\lambda}{r}-1}\left[\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})f(x)}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\,dx\right]^p\right\}^{\frac{1}{p}} < k_{\lambda}(r)\|f\|_{p,\phi}, \qquad (40)
$$

$$
L := \left\{\int_{\nu}^{\infty}(x-\nu)^{\frac{q\lambda}{s}-1}\left[\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})a_n}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}} < k_{\lambda}(r)\|a\|_{q,\psi}, \qquad (41)
$$

where the constant factor $k_{\lambda}(r) = [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2$ *is the best possible.*

Proof By the Lebesgue term-by-term integration theorem [29], we find that there are two expressions of I in (39). By (9), (15), and $0 < \|f\|_{p,\phi} < \infty$, we have (40). By Hölder's inequality, we find

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty}\left[(n-\beta)^{\frac{\lambda}{r}-\frac{1}{p}}\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})f(x)}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}\,dx\right]\left[(n-\beta)^{\frac{1}{p}-\frac{\lambda}{r}}a_n\right] \\
&\le J\left\{\sum_{n=1}^{\infty}[(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q\right\}^{1/q} = J\|a\|_{q,\psi}. \qquad (42)
\end{aligned}
$$

Then by (40), (39) is valid. On the other hand, assuming that (39) is valid, set

$$
a_n := (n-\beta)^{\frac{p\lambda}{r}-1}\left[\int_{\nu}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^{\lambda}-(n-\beta)^{\lambda}}f(x)\,dx\right]^{p-1} \quad (n \in \mathbf{N}). \qquad (43)
$$

Then by (39), we have

$$\|a\|_{q,\psi}^q = \sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q = J^p = I \le k_\lambda(r)\|f\|_{p,\phi}\|a\|_{q,\psi}. \tag{44}$$

By (9), (15), and $0 < \|f\|_{p,\phi} < \infty$, it follows that $J < \infty$. If $J = 0$, then (40) is trivially valid; if $J > 0$, then $0 < \|a\|_{q,\psi} = J^{p-1} < \infty$. Thus, the conditions of applying (39) are fulfilled, and by (39), (44) takes a strict sign inequality. Thus, we find

$$J = \|a\|_{q,\psi}^{q-1} < k_\lambda(r)\|f\|_{p,\phi}. \tag{45}$$

Hence, (40) is valid, which is equivalent to (39).

By (9), (16), and $0 < \|a\|_{q,\psi} < \infty$, we obtain (41). By Hölder's inequality again, we have

$$\begin{aligned} I &= \int_\nu^\infty \left[(x-\nu)^{\frac{\lambda}{s}-\frac{1}{q}}\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n\right]\left[(x-\nu)^{\frac{1}{q}-\frac{\lambda}{s}}f(x)\right]dx \\ &\le L\left\{\int_\nu^\infty (x-\nu)^{p(1-\frac{\lambda}{s})-1}f^p(x)\,dx\right\}^{1/p} = L\|f\|_{p,\phi}. \end{aligned} \tag{46}$$

Hence, (39) is valid by using (41). On the other hand, assuming that (39) is valid, set

$$f(x) := (x-\nu)^{\frac{q\lambda}{s}-1}\left[\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n\right]^{q-1} \quad \bigl(x \in (\nu,\infty)\bigr). \tag{47}$$

Then by (39), we find

$$\|f\|_{p,\phi}^p = \int_\nu^\infty (x-\nu)^{p(1-\frac{\lambda}{s})-1}f^p(x)\,dx = L^q = I \le k_\lambda(r)\|f\|_{p,\phi}\|a\|_{q,\psi}. \tag{48}$$

By (9), (16), and $0 < \|a\|_{q,\psi} < \infty$, it follows that $L < \infty$. If $L = 0$, then (41) is trivially valid; if $L > 0$, then $0 < \|f\|_{p,\phi} = L^{q-1} < \infty$, *i.e.*, the conditions of applying (39) are fulfilled and by (48), we still have

$$\begin{aligned} &\|f\|_{p,\phi}^p = L^q = I < k_\lambda(r)\|f\|_{p,\phi}\|a\|_{q,\psi}, \quad \textit{i.e.}, \\ &L = \|f\|_{p,\phi}^{p-1} < k_\lambda(r)\|a\|_{q,\psi}. \end{aligned}$$

Hence, (41) is valid, which is equivalent to (39). It follows that (39), (40), and (41) are equivalent.

If there exits a positive number $k \le k_\lambda(r)$ such that (39) is still valid as we replace $k_\lambda(r)$ by k, then, in particular, (22) is valid ($\widetilde{a}_n, \widetilde{f}(x)$ are taken as (21)). Then we have (23). By (11), the Fatou lemma [30], and (23), we have

$$\begin{aligned} k_\lambda(r) &= \frac{1}{\lambda^2}\int_0^\infty \frac{\ln t}{t-1}t^{\frac{1}{r}-1}\,dt \\ &= \int_0^{(1-\beta)^\lambda} \lim_{\varepsilon\to 0^+}\frac{\ln t}{t-1}t^{\frac{1}{r}+\frac{\varepsilon}{p\lambda}-1}\,dt + \int_{(1-\beta)^\lambda}^\infty \lim_{\varepsilon\to 0^+}\frac{\ln t}{t-1}t^{\frac{1}{r}-\frac{\varepsilon}{q\lambda}-1}\,dt \end{aligned}$$

$$\leq \lim_{\varepsilon\to 0^+}\left[\int_0^{(1-\beta)^\lambda}\frac{\ln t}{t-1}t^{\frac{1}{r}+\frac{\varepsilon}{p\lambda}-1}\,dt+\int_{(1-\beta)^\lambda}^{\infty}\frac{\ln t}{t-1}t^{\frac{1}{r}-\frac{\varepsilon}{q\lambda}-1}\,dt\right]$$
$$\leq \lim_{\varepsilon\to 0^+} k\left[\frac{\varepsilon+1-\beta}{(1-\beta)^{\varepsilon+1}}\right]^{1/q}=k.$$

Hence, $k = k_\lambda(r)$ is the best value of (39). We confirm that the constant factor $k_\lambda(r)$ in (40) ((41)) is the best possible. Otherwise, we can get a contradiction by (42) ((46)) that the constant factor in (39) is not the best possible. □

Remark 6 (i) Define a half-discrete Hilbert operator $T : L_{p,\phi}(\nu,\infty) \to l_{p,\psi^{1-p}}$ as follows. For $f \in L_{p,\phi}(\nu,\infty)$, we define $Tf \in l_{p,\psi^{1-p}}$ satisfying

$$Tf(n)=\int_\nu^\infty \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}f(x)\,dx \quad (n\in\mathbf{N}).$$

Then by (40), it follows $\|Tf\|_{p,\psi^{1-p}} \leq k_\lambda(r)\|f\|_{p,\phi}$, *i.e.*, T is the bounded operator with $\|T\| \leq k_\lambda(r)$. Since the constant factor $k_\lambda(r)$ in (40) is the best possible, we have $\|T\| = k_\lambda(r)$.

(ii) Define a half-discrete Hilbert operator $\widetilde{T} : l_{q,\psi} \to L_{q,\phi^{1-q}}(\nu,\infty)$ in the following way. For $a \in l_{q,\psi}$, we define $\widetilde{T}a \in L_{q,\phi^{1-q}}(\nu,\infty)$ satisfying

$$\widetilde{T}a(x)=\sum_{n=1}^\infty \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n \quad \bigl(x\in(\nu,\infty)\bigr).$$

Then by (41), it follows $\|\widetilde{T}a\|_{q,\phi^{1-q}} \leq k_\lambda(r)\|a\|_{q,\psi}$, *i.e.*, $\widetilde{T}$ is the bounded operator with $\|\widetilde{T}\| \leq k_\lambda(r)$. Since the constant factor $k_\lambda(r)$ in (41) is the best possible, we have $\|\widetilde{T}\| = k_\lambda(r)$.

Theorem 7 *Suppose that* $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>1$, $\frac{1}{r}+\frac{1}{s}=1$, $0<\lambda\leq 1$, $0\leq\beta\leq\frac{1}{2}$, $\nu\in(-\infty,+\infty)$, $\psi(n) := (n-\beta)^{q(1-\frac{\lambda}{r})-1}$, $\widetilde{\phi}(x) = (1-\theta_\lambda(x))(x-\nu)^{p(1-\frac{\lambda}{s})-1}(\theta_\lambda(x) = [\frac{\sin(\frac{\pi}{r})}{\pi}]^2 \times \int_0^{(\frac{1-\beta}{x-\nu})^\lambda}\frac{\ln v}{v-1}v^{\frac{1}{r}-1}\,dv \in (0,1))$, $f(x), a_n \geq 0$, *satisfying* $f \in L_{p,\widetilde{\phi}}(\nu,\infty)$, $a = \{a_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|f\|_{p,\widetilde{\phi}} > 0$, $\|a\|_{q,\psi} > 0$, *then we have the following equivalent inequalities*:

$$I=\sum_{n=1}^\infty a_n\int_\nu^\infty \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}f(x)\,dx$$
$$=\int_\nu^\infty f(x)\sum_{n=1}^\infty \frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n\,dx > k_\lambda(r)\|f\|_{p,\widetilde{\phi}}\|a\|_{q,\psi}, \tag{49}$$

$$J=\left\{\sum_{n=1}^\infty (n-\beta)^{\frac{p\lambda}{r}-1}\left[\int_\nu^\infty \frac{\ln(\frac{x-\nu}{n-\beta})f(x)\,dx}{(x-\nu)^\lambda-(n-\beta)^\lambda}\right]^p\right\}^{\frac{1}{p}} > k_\lambda(r)\|f\|_{p,\widetilde{\phi}}, \tag{50}$$

$$\widetilde{L}:=\left\{\int_\nu^\infty \frac{(x-\nu)^{\frac{q\lambda}{s}-1}}{[(1-\theta_\lambda(x)]^{q-1}}\left[\sum_{n=1}^\infty \frac{\ln(\frac{x-\nu}{n-\beta})a_n}{(x-\nu)^\lambda-(n-\beta)^\lambda}\right]^q dx\right\}^{\frac{1}{q}} > k_\lambda(r)\|a\|_{q,\psi}, \tag{51}$$

where the constant factor $k_\lambda(r) = [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2$ *is the best possible.*

Proof By (9), the reverse of (15), and $0 < \|f\|_{p,\widetilde{\phi}} < \infty$, we have (50). Using the reverse Hölder inequality, we obtain the reverse form of (42) as follows:

$$I \geq J\|a\|_{q,\psi}. \tag{52}$$

Then by (50), (49) is valid.

On the other hand, if (49) is valid, set a_n as (43), then (44) still holds with $0 < p < 1$. By (49), we have

$$\|a\|_{q,\psi}^q = \sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q = J^p = I \geq k_\lambda(r)\|f\|_{p,\widetilde{\phi}}\|a\|_{q,\psi}. \tag{53}$$

Then by (9), the reverse of (18), and $0 < \|f\|_{p,\widetilde{\phi}} < \infty$, it follows that $J = \{\sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q\}^{1/p} > 0$. If $J = \infty$, then (50) is trivially valid; if $J < \infty$, then $0 < \|a\|_{q,\psi} = J^{p-1} < \infty$, *i.e.*, the conditions of applying (49) are fulfilled, and by (53), we still have

$$\|a\|_{q,\psi}^q = J^p = I > k_\lambda(r)\|f\|_{p,\widetilde{\phi}}\|a\|_{q,\psi}, \quad \textit{i.e.}, J = \|a\|_{q,\psi}^{q-1} > k_\lambda(r)\|f\|_{p,\widetilde{\phi}}.$$

Hence, (50) is valid, which is equivalent to (49).

By the reverse of (16), in view of $\tilde{\omega}_\lambda(x) > k_\lambda(r)(1-\theta_\lambda(x))$ and $q < 0$, we have

$$\widetilde{L} > k_\lambda^{\frac{q-1}{q}}(r)L_1 \geq k_\lambda^{\frac{q-1}{q}}(r)\left\{k_\lambda(r)\sum_{n=1}^{\infty}(n-\beta)^{q(1-\frac{\lambda}{r})-1}a_n^q\right\}^{\frac{1}{q}} = k_\lambda(r)\|a\|_{q,\psi}.$$

Then (51) is valid. By the reverse Hölder inequality again, we have

$$\begin{aligned} I &= \int_\nu^\infty \left[\frac{(x-\nu)^{\frac{\lambda}{s}-\frac{1}{q}}}{(1-\theta_\lambda(x))^{\frac{1}{p}}}\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n\right] \\ &\quad \times \left[\left(1-\theta_\lambda(x)\right)^{\frac{1}{p}}(x-\nu)^{\frac{1}{q}-\frac{\lambda}{s}}f(x)\right]dx \geq \widetilde{L}\|f\|_{p,\widetilde{\phi}}. \end{aligned} \tag{54}$$

Hence, (49) is valid by (51). On the other hand, if (49) is valid, set

$$f(x) = \frac{(x-\nu)^{\frac{q\lambda}{s}-1}}{[1-\theta_\lambda(x)]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{\ln(\frac{x-\nu}{n-\beta})}{(x-\nu)^\lambda-(n-\beta)^\lambda}a_n\right]^{q-1} \quad \bigl(x \in (\nu,\infty)\bigr).$$

Then by the reverse of (16) and $0 < \|a\|_{q,\psi} < \infty$, it follows that $\widetilde{L} = \{\int_\nu^\infty[1-\theta_\lambda(x)]^{\frac{1}{p}}(x-\nu)^{p(1-\frac{\lambda}{s})-1}f^p(x)\,dx\}^{\frac{1}{q}} = \|f\|_{p,\widetilde{\phi}}^{p-1} > 0$. If $\widetilde{L} = \infty$, then (51) is trivially valid; if $\widetilde{L} < \infty$, then $0 < \|f\|_{p,\widetilde{\phi}} = \widetilde{L}^{q-1} < \infty$, *i.e.*, the conditions of applying (49) are fulfilled, and we have

$$\|f\|_{p,\widetilde{\phi}}^p = \widetilde{L}^q = I > k_\lambda(r)\|f\|_{p,\widetilde{\phi}}\|a\|_{q,\psi}, \quad \textit{i.e.}, \widetilde{L} = \|f\|_{p,\widetilde{\phi}}^{p-1} > k_\lambda(r)\|a\|_{q,\psi}.$$

Hence, (51) is valid, which is equivalent to (49). It follows that (49), (50), and (51) are equivalent.

If there exists a positive number $k \geq k_\lambda(r)$ such that (49) is still valid as we replace $k_\lambda(r)$ by k, then, in particular, (24) is valid. Hence, we have (25). For $\varepsilon \to 0^+$ in (25), we obtain $k \leq$

$\frac{1}{\lambda^2}[B(\frac{1}{s},\frac{1}{r})]^2 = k_\lambda(r)$. Hence, $k = k_\lambda(r)$ is the best value of (49). We confirm that the constant factor $k_\lambda(r)$ in (50) ((51)) is the best possible. Otherwise, we can get a contradiction by (52) ((54)) that the constant factor in (49) is not the best possible. □

Theorem 8 *If the assumption of* $p > 1$ *in Theorem* 5 *is replaced by* $p < 0$, *then the reverses of* (39), (40), *and* (41) *are valid and equivalent. Moreover, the same constant factor is the best possible.*

Proof In a similar way as in Theorem 7, we can obtain that the reverses of (39), (40), and (41) are valid and equivalent.

If there exists a positive number $k \geq k_\lambda(r)$ such that the reverse of (39) is still valid as we replace $k_\lambda(r)$ by k, then, in particular, (26) is valid. Hence, we have (27). For $\varepsilon \to 0^+$ in (27), we obtain $k \leq \frac{1}{\lambda^2}[B(\frac{1}{s},\frac{1}{r})]^2 = k_\lambda(r)$. Hence, $k = k_\lambda(r)$ is the best value of the reverse of (39). We confirm that the constant factor $k_\lambda(r)$ in the reverse of (40) ((41)) is the best possible. Otherwise, we can get a contradiction by the reverse of (42) ((46)) that the constant factor in the reverse of (39) is not the best possible. □

Remark 9 (i) For $\beta = \nu = 0$ in (39), it follows

$$\sum_{n=1}^{\infty} a_n \int_0^\infty \frac{\ln(\frac{x}{n})}{x^\lambda - n^\lambda} f(x)\,dx < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2 \left\{\int_0^\infty x^{p(1-\frac{\lambda}{s})-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{r})-1} a_n^q\right\}^{\frac{1}{q}}. \tag{55}$$

In particular, for $\lambda = 1$, $p = q = r = s = 2$, (55) reduces to (5).

(ii) For $\nu = \beta$ in (39), we have

$$\sum_{n=1}^{\infty} a_n \int_\beta^\infty \frac{\ln(\frac{x-\beta}{n-\beta})}{x^\lambda - n^\lambda} f(x)\,dx < \left[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\right]^2 \left\{\int_\beta^\infty (x-\beta)^{p(1-\frac{\lambda}{s})-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\beta)^{q(1-\frac{\lambda}{r})-1} a_n^q\right\}^{\frac{1}{q}}, \tag{56}$$

which is more accurate than (55).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AW carried out the study, and wrote the manuscript. BY participated in its design and coordination. All authors read and approved the final manuscript.

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