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Pólya-type polynomial inequalities in Orlicz spaces and best local approximation

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Abstract

We obtain an extension of Pólya-type inequalities for univariate real polynomials in Orlicz spaces. We also give an application to a best local approximation problem.
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1 Introduction

Let X be a bounded open subset of $\mathbb{R}$. Consider the measure space $(X, \mathcal{B}, \mu)$, where μ is the Lebesgue measure, and denote $\mathcal{M} = \mathcal{M}(X)$ the system of all equivalence classes of Lebesgue measurable real valued functions on X. Let Φ be the set of convex functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(x) > 0$ for $x > 0$, and $\varphi(0) = 0$.

Given $\varphi \in \Phi$, we define

$$L^{\phi} = L^{\phi}(X) := \left\{ f \in \mathcal{M} : \int_X \phi\left(\alpha \left|f(x)\right|\right) dx < \infty, \quad \text{for some } \alpha > 0 \right\}.$$

The space L^{φ} is called the Orlicz space determined by φ. This space is endowed with the Luxemburg norm,

$$\|f\|_{\phi,X} = \inf\left\{ \lambda > 0 : \int_X \phi\left(\frac{|f(x)|}{\lambda}\right) \frac{dx}{\mu(X)} \le 1 \right\}.$$

The space L^{φ} with this norm is a Banach space (see [1]). If $E \in \mathcal{B}$ and $\mu(E) > 0$, then $\|\cdot\|_{\varphi,E}$ is a seminorm on $L^{\varphi}(X)$. In the particular case, $\varphi(t) = t^p$, we will use the notation $\|\cdot\|_{p,E}$ instead of $\|\cdot\|_{\varphi,E}$.

Let $\Pi^N \subset \mathcal{M}$, $N \in \mathbb{N}$, be the class of all algebraic polynomials of degree at most N, with real coefficients.

Given $E \in \mathcal{B}$, we recall that a polynomial $g_E \in \Pi^N$ is a best approximation of $f \in L^{\varphi}(X)$ from Π^N respect to $\|\cdot\|_{\varphi,E}$, if

$$\|f - g_E\|_{\phi,E} = \inf\left\{ \|f - P\|_{\phi,E} : \quad P \in \prod^N \right\}.$$

Let x_k, $1 \le k \le n$, be n points in X. We consider a net of measurable sets $\{E\} \subset \mathcal{B}$ such that $E = \bigcup_{k=1}^{n} E_k$, with $\mu(E_k) > 0$ and

$$\sup_{1\le k\le n} \sup_{y\in E_k} |x_k - y| \to 0, \quad \text{as } \mu(E) \to 0.$$

Given $f \in L^{\varphi}(X)$ and Π^N, we consider a net of best approximation functions $\{g_E\}$. If it has a limit in Π^N as $\mu(E) \to 0$, this limit is called the *best local approximation of f from* Π^N on $\{x_1,..., x_n\}$. If the points in our approximation problem have not the same importance the neighborhoods E_k can be adjusted to reflect it. In [2], Chui et al. introduced the balanced neighborhood concept and they studied existence and characterization of best local approximation in L^p-spaces for several points with different size neighborhoods. In [3,4], the last problem was considered for φ-approximation and $\|\cdot\|_{\varphi}$-approximation, respectively, in Orlicz spaces. Other results in these spaces about best local approximation with non balanced neighborhoods were considered in [5].

Polynomial inequalities on measurable sets have been studied extensively in the literature (see [6-8]). In [9], the authors proved the following extension of the Pólya inequality in L^p-spaces, $0 < p \le \infty$.

Theorem 1.1. *Let* $0 < p \le \infty$ *and* $n, N \in \mathbb{N}$. *Let* i_k, $1 \le k \le n$, *be n positive integers such that* $\sum_{k=1}^{n} i_k = N + 1$. *Let* B_k, $1 \le k \le n$, *be disjoint pairwise compact intervals in* $\mathbb{R}$ *with* $0 < \mu(B_k) \le 1$. *Then there exists a constant K depending on p,* i_k *and* B_k, *for* $1 \le k \le n$, *such that*

$$|c_j| \le \frac{K}{\min_{1\le k\le n} \mu(E\cap B_k)^{i_k - 1 + 1/p}} \|P\|_{p,E}, \quad 0 \le j \le N,$$

for all $P(x) = \sum_{j=0}^{N} c_j x^j$, $E \subset \bigcup_{k=1}^{n} B_k$, $\mu(E\cap B_k) > 0$, $1 \le k \le n.$.

They gave an application of this theorem to the existence of the best multipoint local approximation in L^p spaces, with balanced neighborhoods.

In this article, we generalize Theorem 1.1 and the balanced neighborhood concept to L^{φ}. As a consequence of this extension we prove the existence of the best local approximation of a function from Π^N on $\{x_1,..., x_n\}$, with balanced neighborhoods, following the pattern used in [9]. Moreover, we prove that the best local approximation polynomial is the Hermite interpolating polynomial.

We say that a function $\varphi \in \Phi$ satisfies the Δ_2-condition if there exists a constant $k > 0$ such that $\varphi(2x) \le k\varphi(x)$, for $x \ge 0$, and we say that φ satisfies the Δ'-condition if there exists a constant $c > 0$ such that $\varphi(xy) \le c\varphi(x)\varphi(y)$ for $x, y \ge 0$. We point out that the Δ'-condition implies the Δ_2-condition. A detailed treatment about these subjects may be found in [1].

If φ satisfies the Δ'-condition, it is easy to see that there exists a constant $K > 0$ such that

$$\phi^{-1}(x)\phi^{-1}(y) \le K\phi^{-1}(xy), \quad \text{for all } x, y \ge 0. \tag{1}$$

We assume in this article that $\varphi \in \Phi$ and it satisfies the Δ'-condition.

2 Preliminary results

Let $\mathcal{X}_A$ denotes the characteristic function on the measurable set $A \subset X$.

Proposition 2.1. *The family of all seminorms $\| \cdot \|_{\varphi.E}$ with $\mu(E) > 0$, has the following properties:*

(a) $\|\mathcal{X}_E\|_{\phi,E} = \dfrac{1}{\phi^{-1}(1)}$.

(b) if f, g ∈ $L^{\varphi}(X)$ *satisfy* $|f| \leq |g|$ *on E, then* $\|f\|_{\varphi,E} \leq \|g\|_{\varphi,E}$*. The inequality is strict if* $|f| < |g|$ *on some subset of E with positive measure.*

(c) There exists a constant $M > 0$ *such that*

$$\|f\|_{\phi,G} \leq \frac{M}{\phi^{-1}\left(\dfrac{\mu(G)}{\mu(D)}\right)}\|f\|_{\phi,D}, \quad f \in L^{\phi}(X), \tag{2}$$

for all pair of measurable sets G, D, with $G \subset D$ *and* $\mu(G) > 0$.

Proof (a) For $\lambda := 1/\varphi^{-1}(1)$ we have

$$\int_E \phi\left(\frac{|\mathcal{X}_E|}{\lambda}\right)\frac{dx}{\mu(E)} = \int_E \frac{dx}{\mu(E)} = 1.$$

Now, the Δ_2- condition implies $\|\mathcal{X}_E\|_{\phi,E} = 1/\phi^{-1}(1)$.

(b) If $|f| \leq |g|$ on E, then

$$\int_E \phi\left(\frac{|f|}{\lambda}\right)\frac{dx}{\mu(E)} \leq \int_E \phi\left(\frac{|g|}{\lambda}\right)\frac{dx}{\mu(E)}, \quad \lambda > 0,$$

and so $\|f\|_{\varphi,E} \leq \|g\|_{\varphi,E}$. In addition, if $|f| < |g|$ on some subset of E with positive measure, the above inequality is strict. So, the Δ_2-condition implies the assertion.

(c) Given $G \subset D$, $\mu(G) > 0$, and $f \in L^{\varphi}(X)$, for each $\lambda > 0$, we denote

$$\mathfrak{A}(\lambda) := \int_G \phi\left(\frac{|f|}{\lambda}\right)\frac{dx}{\mu(G)} \quad \text{and} \quad \mathfrak{B}(\lambda) := \int_D \phi\left(\frac{|f|}{\lambda}\right)\frac{dx}{\mu(D)}.$$

We consider $\lambda > 0$ such that $\mathfrak{B}(\lambda) \leq 1$. By the Δ'-condition we obtain

$$\mathfrak{A}\left(\frac{\lambda}{\phi^{-1}\left(\dfrac{\mu(G)}{c\,\mu(D)}\right)}\right) \leq \int_D c\frac{\mu(G)}{c\,\mu(D)}\phi\left(\frac{|f|}{\lambda}\right)\frac{dx}{\mu(G)} = \mathfrak{B}(\lambda) \leq 1.$$

Then $\|f\|_{\phi,G} \leq \dfrac{\lambda}{\phi^{-1}\left(\dfrac{\mu(G)}{c\,\mu(D)|}\right)}$, for all $\lambda > 0$ with $\mathfrak{B}(\lambda) \leq 1$. So, the definition of $\|f\|_{\varphi,D}$ and (1) imply $\|f\|_{\phi,G} \leq \dfrac{M}{\phi^{-1}\left(\dfrac{\mu(G)}{\mu(D)}\right)}\|f\|_{\phi,D}$ with $M = \dfrac{K}{\phi^{-1}(c^{-1})}$.

Lemma 2.2. *There exists a constant* $M > 0$ *such that*

$$\left|P^{(j)}(a)\right| \leq \frac{M}{\varepsilon^j}\|P\|_{\phi,[a-\varepsilon,\, a+\varepsilon]},$$

for all $P \in \Pi^N$, $[a - \epsilon, a + \epsilon] \subset X$, *and* $0 \le j \le N$.

Proof. Given $P \in \Pi^N$ and $[a - \epsilon, a + \epsilon] \subset X$, we divide that interval in $2(N + 1)$ close subintervals with the same size. Let J_ϵ be one of them. From Proposition 2.1 (c), we get $\|P\|_{\phi,J_\varepsilon} \le M\|P\|_{\phi,[a-\varepsilon,\ a+\varepsilon]}$, where M is independent on P, a, and ϵ. In addition, there exists $y_\epsilon \in J_\epsilon$ such that $\left|P(y_\varepsilon)\right| \le \phi^{-1}(1)\|P\|_{\phi,J_\varepsilon}$. In fact, if $\phi^{-1}(1)\|P\|_{\phi,J_\varepsilon} < \left|P(y)\right|$, for all $y \in J_\epsilon$, then Proposition 2.1 (a) and (b) yield $\|P\|_{\phi,J_\varepsilon} > \|P\|_{\phi,J_\varepsilon}$. A contradiction.

From the family of intervals J_ϵ, we choose pairwise disjoint $(N + 1)$ intervals, and we denote them with $J_{i,\epsilon}$, $1 \le i \le N + 1$. Let $y_{i,\epsilon} \in J_{i,\epsilon}$ be such that

$$\left|P(y_{i,\varepsilon})\right| \le M\phi^{-1}(1)\|P\|_{\phi,[a-\varepsilon,\ a+\varepsilon],} \quad 1 \le i \le N + 1. \tag{3}$$

If $t_{i,\varepsilon} := \dfrac{y_{i,\varepsilon} - a}{\varepsilon} \in [-1, 1]$, we have

$$P(y_{i,\varepsilon}) = \sum_{j=0}^{N} \frac{P^{(j)}(a)}{j!}(y_{i,\varepsilon} - a)^j = \sum_{j=0}^{N} \frac{P^{(j)}(a)}{j!}\varepsilon^j t_{i,\varepsilon}^j, \quad 1 \le i \le N + 1. \tag{4}$$

The matrix of the linear system (4), $\left(t_{i,\varepsilon}^j\right)$, is a Vandermonde matrix whose determinant has a positive lower bound, because $t_{i,\epsilon} - t_{i',\epsilon} \ge 1/N + 1$ for $i > i'$. Using Cramer's rule and (3), there is a constant which we again denote by M such that

$$\left|P^{(j)}(a)\varepsilon^j\right| \le M\|P\|_{\phi,[a,\varepsilon,\ a+\varepsilon]} \quad 0 \le j \le N.$$

The proof of the following lemma is analogous to the one of Lemma 2.3 in [9], however we give it for sake of completeness.

Lemma 2.3. *Let* $C \subset X$ *be an interval,* $E \subset C$, $\mu(E) > 0$. *For all* $P \in \Pi^N$, *there exists an interval* $F := F(E,P) \subset C$ *such that*

a) $\mu(F) \ge \dfrac{\mu(E)}{2N}$,

b) $\|P\|_{\varphi,F} \le 2N\|P\|_{\varphi,E}$.

Proof. Let $P \in \Pi^N$, $S = 2N$, and let $D_a := \{x \in C : |P(x)| < a\}$. It easy to see that the function $G(a){:} = \mu(D_a)$ is continuous, $G(0) = 0$ and $\lim_{a\to\infty} G(a) = \mu(C)$. Therefore, there exists a constant $a_* \in \mathbb{R}^+$ such that $\mu(D_{a_*}) = \mu(E)/2$. Since $\{x \in C : |P(x)| = a_*\}$ has at most $2N$ elements, there exists k, $1 \le k \le N$, and pairwise disjoint intervals E_j, $1 \le j \le k$, such that $D_{a_*} = \bigcup_{j=1}^{k} E_j$.

We denote $\overline{A} = C\backslash A$, for any set A. Then

$$\mu(E \cap \overline{D}_{a_*}) = \mu(E) - \mu(E \cap D_{a_*}) \ge \mu(E) - \mu(D_{a_*}) = \frac{\mu(E)}{2}. \tag{5}$$

There exists j, $1 \le j \le k$, such that $\mu(E_j) \ge \mu(E)/S$. In fact, if $\mu(E_j) < \mu(E)/S$ for all j, $1 \le j \le k$, we obtain $\mu(D_{a_*}) < k/S\mu(E) \le \mu(E)/2$, which is a contradiction. So, we have proved a) with $F := E_j$.

Using (5), we obtain

$$\mu(E \cap \overline{D}_{a_*}) \geq \frac{\mu(E)}{2} = \mu(D_{a_*}) \geq \mu(F)\mu(F \cap \overline{E}).$$

Therefore

$$\begin{aligned}\int_F \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)} &\leq \int_{F\cap E} \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)} + \phi\left(\frac{a_*}{\lambda}\right)\frac{\mu(E\cap \overline{D}_{a_*})}{\mu(F)}\\ &\leq \int_{E\cap D_{a*}} \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)} + \int_{E\cap \overline{D}_{a*}} \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)}\\ &= \int_E \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)}.\end{aligned}$$

So, (a) implies

$$\mathcal{A}_F(\lambda) := \int_F \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(F)} \leq S\int_E \phi\left(\frac{|P|}{\lambda}\right)\frac{dx}{\mu(E)} =: S\mathcal{A}_E(\lambda).$$

Let λ be such that $\mathcal{A}_E(\lambda) = 1$. The convexity of φ implies $\mathcal{A}_F(S\lambda) \leq 1$. So, $\|P\|_{\varphi,F} \leq S\|P\|_{\varphi,E}$.

3 Pólya inequality

Now, we present the main result concerning to Pólya inequality in L^{φ}.

Theorem 3.1. *Let $\varphi \in \Phi$, and $n, N \in \mathbb{N}$. Let i_k, $1 \leq k \leq n$, be n positive integers such that $\sum_{k=1}^{n} i_k = N + 1$. Let B_k, $1 \leq k \leq n$, be disjoint pairwise compact intervals in $\mathbb{R}$, with $0 < \mu(B_k) \leq 1$. Then there exists a positive constant M depending on φ, i_k, and B_k, $1 \leq k \leq n$, such that*

$$|c_j| \leq \frac{M}{\min_{1\leq k\leq n}\left\{\mu(E\cap B_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E\cap B_k)}{\mu(E)}\right)\right\}}\|P\|_{\phi,E}, \quad 0 \leq j \leq N, \tag{6}$$

for all $P(x) = \sum_{j=0}^{N} c_j x^j$, $E \subset \bigcup_{k=1}^{n} B_k$ with $\mu(E \cap B_k) > 0$, $1 \leq k \leq n$.

Proof. In the following proof, the constant M can be different in each occurrence. Let $P(x) = \sum_{j=0}^{N} c_j x^j \in \Pi^N$, and let $E \subset \bigcup_{k=1}^{n} B_k$ be a measurable set with $\mu(E \cap B_k) > 0$, $1 \leq k \leq n$. By Lemma 2.3 for $C = B_k$, there exist n intervals $F_k = [a_k - r_k, a_k + r_k] \subset B_k$, $1 \leq k \leq n$, such that $\mu(F_k) \geq \mu(E \cap B_k)/2N$ and $\|P\|_{\phi,F_k} \leq 2N\|P\|_{\phi,E\cap B_k}$. From Lemma 2.2, there exists a positive constant M depending on p, i_k, and B_k, $1 \leq k \leq n$, such that for all j, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, it verifies

$$\left|P^{(j)}(a_k)\right| \leq \frac{M}{\mu(F_k)^j}\|P\|_{\phi,F_k} \leq \frac{M}{\mu(F_k)^{i_k-1}}\|P\|_{\phi,F_k} \leq \frac{M}{\mu(E\cap B_k)^{i_k-1}}\|P\|_{\phi,E\cap B_k}. \tag{7}$$

From (7) and (2), there is a constant M such that

$$\left|P^{(j)}(a_k)\right| \leq \frac{M}{\mu(E\cap B_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E\cap B_k)}{\mu(E)}\right)}\|P\|_{\phi,E}$$

for $0 \le j \le i_k - 1$, $1 \le k \le n$. So

$$\left|P^{(j)}(a_k)\right| \le \frac{M}{\min_{1\le s\le n}\left\{\mu(E\cap B_s)^{i_s-1}\phi^{-1}\left(\frac{\mu(E\cap B_s)}{\mu(E)}\right)\right\}}\|P\|_{\phi,E},$$

for $0 \le j \le i_k - 1$, $1 \le k \le n$. From the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on Π^N,

$$\|P\|_1 = \max_{1\le k\le n}\sup_{a_k\in B_k}\max_{0\le j\le i_k-1}\left|P^{(j)}(a_k)\right| \quad \text{and} \quad \|P\|_2 = \max_{0\le j\le N}\left|c_j\right|,$$

we obtain (6).

4 Best local approximation

In this section, we introduce a concept of balanced neighborhood in L^φ and we prove the existence of the best local approximation using the neighborhoods E_k, $1 \le k \le n$, mentioned in the Section 1.

It is easy to see that $E_k = x_k + \mu(E_k)A_k$, where A_k is a measurable set with measure 1. Henceforward, we assume the sets A_k are uniformly bounded.

For each $\alpha \in \mathbb{R}$ and k, $1 \le k \le n$, we denote

$$\mathcal{A}_k(\alpha) := \frac{\mu(E_k)^\alpha}{\phi^{-1}\left(\frac{\mu(E)}{\mu(E_k)}\right)}.$$

We assume the following condition, which allows us that $\mathcal{A}_k(\alpha)$ can be compared with each other as functions of α when $\mu(E) \to 0$.

For any nonnegative integers α and β, and any pair j, k, $1 \le j$, $k \le n$,

$$\text{either } \mathcal{A}_k(\alpha) = O(\mathcal{A}_j(\beta)) \text{ or } \mathcal{A}_j(\beta) = o(\mathcal{A}_k(\alpha)), \quad \text{as } \mu(E) \to 0. \tag{8}$$

Let $<i_k>$ be an ordered n-tuple of nonnegative integers. We say that $\mathcal{A}_j(i_j)$ is a *maximal* element of $<\mathcal{A}_k(i_k)>$ if $\mathcal{A}_k(i_k) = O(\mathcal{A}_j(i_j))$ for all $1 \le k \le n$. We denote it by

$$\mathcal{A}_j(i_j) = \max\left\{\mathcal{A}_k(i_k)\right\}.$$

Observe that $\sum_{k=1}^n \mathcal{A}_k(i_k) = O(\max\{\mathcal{A}_k(i_k)\})$..

Definition 4.1. *An n-tuple* $<i_k>$*of nonnegative integers is balanced if*

$$\sum_{k=1}^n \mathcal{A}_k(i_k) = o\left(\min_{1\le k\le n}\left\{\mu(E_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E_k)}{\mu(E)}\right)\right\}\right).$$

In this case, we say that $\sum_{k=1}^n i_k$ *is a balanced integer, and* $<E_k>$*are balanced neighborhoods.*

Lemma 4.2. *To each balanced integer there corresponds exactly one balanced n-tuple.*

Proof. Let $<i_k>$ be a balanced n-tuple. If $<i'_k>$ is distinct from $<i_k>$ and $\sum_{k=1}^n i_k = \sum_{k=1}^n i'_k$, there exist indices j and s such that $i_j \ge i'_j + 1$ and $i'_s \ge i_s + 1$. From definition of balanced neighborhood, we have

$$\mathcal{A} := \sum_{k=1}^n \mathcal{A}_k(i_k) = o\left(\mu(E_j)^{i_j-1}\phi^{-1}\left(\frac{\mu(E_j)}{\mu(E)}\right)\right).$$

In addition, by (1) we get $\mu(E_j)^{i_j-1}\phi^{-1}\left(\frac{\mu(E_j)}{\mu(E)}\right) \le \mu(E_j)^{i'_j}\phi^{-1}\left(\frac{\mu(E_j)}{\mu(E)}\right) \le K\phi^{-1}(1)\mathcal{A}_j(i'_j)$.

So, $\mathcal{A} = o\left(\sum_{k=1}^{n}\mathcal{A}_k(i'_k)\right)$. Again, by (1) we get

$$\frac{\sum_{k=1}^{n}\mathcal{A}_k(i'_k)}{\mu(E_s)^{i'_s-1}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right)} \ge \frac{\sum_{k=1}^{n}\mathcal{A}_k(i'_k)}{\mu(E_s)^{i_s}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right)} \ge \frac{\sum_{k=1}^{n}\mathcal{A}_k(i'_k)}{K\phi^{-1}(1)\mathcal{A}_s(i_s)} \to \infty.$$

Then $< i'_k >$ cannot be balanced.

The following lemma allows us to state an algorithm to compute all the balanced integers greater than a given balanced integer.

Lemma 4.3. *Let* $<i_k>$ *and* $< i'_k >$ *be two balanced n-tuples with* $\sum_{k=1}^{n} i_k < \sum_{k=1}^{n} i'_k$. *Let* $A = A(< i_k >) := \{j : \mathcal{A}_j(i_j) = \max\{\mathcal{A}_k(i_k)\}\}$ *and* $B = B(<i_k>) := \{1,2,...,n\} \setminus A$. *Then*

(a) for $j \in A\, i'_j \ge i_j + 1$.

(b) for $j \in A\;\; i'_j \ge i_j$.

Proof. (a) Suppose $i'_j \le i_j$ for some $j \in A$. For any $l \in B$, from (8) we get $\mathcal{A}_l(i_l) = o(\mathcal{A}_j(i_j))$. Assume now $i'_l \ge i_l + 1$ for some $l \in B$. By (1), there exists a constant $M > 0$ such that

$$\frac{\mathcal{A}_j(i'_j)}{\mu(E_l)^{i'_l-1}\phi^{-1}\left(\frac{\mu(E_l)}{\mu(E)}\right)} \ge \frac{\mathcal{A}_j(i_j)}{\mu(E_l)^{i_l}\phi^{-1}\left(\frac{\mu(E_l)}{\mu(E)}\right)} \ge \frac{\mathcal{A}_j(i_j)}{M\mathcal{A}_l(i_l)} \to \infty,$$

as $\mu(E) \to 0$. Thus $< i'_k >$ cannot be balanced, a contradiction. Therefore, either $B = \emptyset$ or $i'_l \le i_l$, for all $l \in B$. On the other hand, since $\sum_{k=1}^{n} i_k < \sum_{k=1}^{n} i'_k$, there is $s \in A$ such that $i'_s \ge i_s + 1$. According to (1) and the definition of A we obtain

$$\frac{\mathcal{A}_j(i'_j)}{\mu(E_s)^{i'_s-1}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right)} \ge \frac{\mathcal{A}_j(i_j)}{\mu(E_s)^{i_s}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right)} \ge \frac{\mathcal{A}_j(i_j)}{M\mathcal{A}_s(i_s)} \ge M',$$

as $\mu(E) \to 0$, for some constant $M' > 0$. Therefore, $< i'_k >$ cannot be balanced.

(b) Suppose $i'_j < i_j$ for some $j \in B$. From (a), (1) and the definition of balanced n-tuple, we obtain for each $l \in A$,

$$\frac{\mathcal{A}_j(i'_j)}{\mu(E_l)^{i'_l-1}\phi^{-1}\left(\frac{\mu(E_l)}{\mu(E)}\right)} \ge \frac{\mathcal{A}_j(i_j-1)}{M\mathcal{A}_l(i_l)} \ge M'\frac{\mu(E_j)^{i_j-1}\phi^{-1}\left(\frac{\mu(E_j)}{\mu(E)}\right)}{\mathcal{A}_l(i_l)} \to \infty,$$

as $\mu(E) \to 0$. Therefore $< i'_k >$ cannot be balanced.

Given a balanced integer, the above lemma gives us a necessary condition which must satisfy the next balanced integer. The following example shows that the conditions of Lemma 4.3 are not sufficient to get a balanced n-tuple.

Example 4.4. Define $\varphi(x) = x^3(1 + |\ln x|)$, $x > 0$, and $\varphi(0) = 0$. Consider two points x_1, x_2 with $\mu(E_1) = \delta^{4/3}$, $\mu(E_2) = \delta^{1/3}$, and $A_1 = A_2 = [0,1]$. The 2-tuple $< 0,1 >$ is balanced. Here, the set $A(< 0,1 >) = \{0\}$, however $< 1,1 >$ is not a balanced 2-tuple. In fact, if $<i_k> = < 0,1 >$ we obtain

$$\min_{1\le k\le 2}\left\{\mu(E_k)^{i_k-1}\left(\frac{\mu(E_k)}{\mu(E)}\right)\right\} = \min\left\{\frac{\phi^{-1}(\delta)}{\delta^{4/3}},\phi^{-1}(1)\right\} + o(1) \to \phi^{-1}(1),$$

as $\delta \to 0$. Since $\mathcal{A}_2(i_2) = o(\mathcal{A}_1(i_1))$ and $\mathcal{A}_1(i_1) = o(1)$, as $\delta \to 0$, we have

$$\frac{\sum_{k=1}^{2}\mathcal{A}_k(i_k)}{\min\limits_{1\le k\le 2}\left\{\mu(E_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E_k)}{\mu(E)}\right)\right\}} = o(1), \quad \text{as } \delta - 0.$$

So $< 0,1 >$ is a balanced 2-tuple, $A(< 0,1 >) = \{0\}$, and $< 1,1 >$ is the next 2-tuple generated by the algorithm. For $<i_k> = < 1,1 >$ we have

$$\frac{\mathcal{A}_2(i_2)}{\min\limits_{1\le k\le 2}\left\{\mu(E_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E_k)}{\mu(E)}\right)\right\}} \ge \frac{\mathcal{A}_2(i_2)}{\phi^{-1}\left(\frac{\mu(E_1)}{\mu(E)}\right)} \to \infty, \quad \text{as } \delta \to 0.$$

Thus $< 1,1 >$ is not a balanced 2-tuple.

Next, we establish an algorithm which gives all balanced n-tuples. First, we observe that $< 0 >$ is a balanced n-tuple. In fact, since φ^{-1} is a concave positive function on $\mathbb{R}_+$ with $\varphi^{-1}(0) = 0$, we have $\varphi^{-1}(x) \ge \varphi^{-1}(1)x$, for $x \le 1$. This yields

$$\frac{\mu(E_j)}{\phi^{-1}\left(\frac{\mu(E)}{\mu(E_k)}\right)\phi^{-1}\left(\frac{\mu(E_j)}{\mu(E)}\right)} \le \frac{\mu(E)}{(\phi^{-1}(1))^2}, \quad 1 \le j, k \le n.$$

Algorithm. Let v_q be a balanced integer and let $< i_k^{(vq)} >$ be the corresponding balanced n-tuple. To build the next n-tuple, $< i_k^{(vq+1)} >$, put $i_k^{(vq+1)} = i_k^{(vq)} + 1$ for $k \in A\left(< i_k^{(vq)} >\right)$ and $i_k^{(v_q+1)} = i_k^{(v_q)}$ for $k \in B\left(< i_k^{(vq)} >\right)$.

The following lemma shows that all balanced n-tuples are contained in the set of n-tuples generated by the algorithm.

Lemma 4.5. *if* $<i_k>$ *is a balanced n-tuple with* $\sum_{k=1}^{n} i_k = q$, *then the algorithm generates all the balanced n-tuple* $< i_k^* >$ *with* $\sum_{k=1}^{n} i_k^* > q$.

Proof. Suppose $< i_k^* >$ is a balanced n-tuple with $\sum_{k=1}^{n} i_k^* = m > q$, and the n-tuple $< i_k^{(m)} >$ is not balanced. Since $\sum\limits_{k=1}^{n} i_k^* = \sum\limits_{k=1}^{n} i_k^{(m)}$, there exist r and s such that $i_r^{(m)} > i_r^*$ and $i_s^* > i_s^{(m)}$. By definition of balanced integer we have

$$\mathcal{A}_r\left(i_r^{(m)} - 1\right) = O\left(\mathcal{A}_r\left(i_r^*\right)\right) = o\left(\mu(E_s)^{i_s^*-1}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right)\right), \tag{9}$$

and (1) implies $\mu(E_s)^{i_s^*-1}\phi^{-1}\left(\frac{\mu(E_s)}{\mu(E)}\right) \le K\phi^{-1}(1)\mathcal{A}_s\left(i_s^*-1\right)$. So, $\mathcal{A}_r\left(i_r^{(m)}-1\right) = o\left(\mathcal{A}_s\left(i_s^{(m)}\right)\right)$.

On the other hand, since $m > q$, Lemma 4.3 implies $i_r^* \ge i_r$, so $i_r^{(m)} > i_r$. Therefore $\mathcal{A}_r\left(i_r^{(m)}-1\right)$ is maximal in a previous step of the algorithm, i.e., there exists m', $q \le m' < m$, such that $\mathcal{A}_r\left(i_r^{(m)}-1\right)$ is maximal of $<\mathcal{A}_k\left(i_k^{(m')}\right)>$. Since the exponents $i_k^{(m)}$ are nondecreasing,

$$\mathcal{A}_s\left(i_s^{(m)}\right) = O\left(\mathcal{A}_s\left(i_s^{(m')}\right)\right) = O\left(\mathcal{A}_r\left(i_r^{(m)}-1\right)\right),$$

which contradicts (9).

Remark 4.6. If we assume the additional condition $\varphi^{-1}(x)\varphi^{-1}(1/x) \ge c > 0$ for $x > 0$, given a balanced n-tuple $<i_k>$, it is easy to see that the n-tuple $<i'_k>$ defined by $i'_k = i_k + 1$ for $k \in A\left(<i_k^{(vq)}>\right)$, and $i'_k = i_k$ for $k \in B\left(<i_k^{(vq)}>\right)$, is balanced. It give us an algorithm that generates the infinite sequences of all balanced n-tuples.

Let $PC^m(X)$ be the class of functions with derivatives up to order m - 1 and with bounded piecewise continuous m^{th} derivative on X.

Next, we prove the following auxiliary lemma.

Lemma 4.7. *Let* $<i_k>$ *be an ordered n-tuple of nonnegative integers. Suppose* $h \in PC^m(X)$, *where* $m = \max\{i_k\}$ *and* $h^{(j)}(x_k) = 0$, $0 \le j \le i_k - 1$, $1 \le k \le n$. *Then*

$$\|h\|_{\phi,E} = O\left(\max\{\mathcal{A}_k(i_k)\}\right).$$

Proof. Expanding h by the Taylor polynomial at x_k up to the order n, we obtain

$$h(x) = \sum_{k=1}^{n} h^{(i_k)}(\xi_k)\frac{(x-x_k)^{i_k}}{i_k!}\chi_{E_k}(x), \quad x \in E,$$

where ξ_k is between x and x_k. The change of variable $x - x_k = \epsilon y$, $y \in A_k$, yields

$$\|h\|_{\phi,E} = \inf\left\{\lambda > 0 : \sum_{k=1}^{n}\int_{A_k}\mu(E_k)\phi\left(\frac{\left|h^{(i_k)}(\xi_k)\right|\frac{\mu(E_k)^{i_k}\left|y^{i_k}\right|}{i_k!}}{\lambda}\right)\frac{dy}{\mu(E)} \le 1\right\}.$$

For

$$\lambda := M\sum_{j=1}^{n}\frac{\mu(E_j)^{i_j}}{\phi^{-1}\left(\frac{\mu(E)}{n\,\mu(E_j)}\right)},$$

where $M = \max_{1\le k\le n}\left\{\frac{1}{i_k!}\max_{x\in X}\left\{\left|h^{(i_k)}(x)\right|\right\}\max_{y\in A_k}\left\{|y|^{i_k}\right\}\right\}$, we obtain

$$\sum_{k=1}^{n}\int_{A_k}\mu(E_k)\phi\left(\frac{\left|h^{(i_k)}(\xi_k)\right|\frac{\mu(E_k)^{i_k}\left|y^{i_k}\right|}{i_k!}}{\lambda}\right)\frac{dy}{\mu(E)} \le 1.$$

Therefore $\|h\|_{\phi,E} = O\left(\sum_{k=1}^{n} \frac{\mu(E_k)^{i_k}}{\phi^{-1}\left(\frac{\mu(E)}{n\,\mu(E_k)}\right)}\right)$. Using the convexity of φ, we have $\frac{\phi^{-1}(x)}{n} \leq \phi^{-1}\left(\frac{x}{n}\right)$, $x \geq 0$. So, $\|h\|_{\phi,E} = O\left(\max\{\mathcal{A}_k(i_k)\}\right)$.

If a polynomial $P \in \Pi^N$, $N + 1 = \sum_{k=1}^{n} i_k$, satisfies $P^{(j)}(x_k) = f^{(j)}(x_k)$, $1 \leq j \leq i_k - 1$, $1 \leq k \leq n$, we call it *the Hermite interpolating polynomial* of the function f on $\{x_1,\ldots,x_n\}$.

Now, we are in condition to prove the main result in this Section.

Theorem 4.8. *Let* $<i_k>$ *be a balanced n-tuple and* $N + 1 = \sum_{k=1}^{n} i_k$. *If* $m = \max\{i_k\}$ *and* $f \in PC^m(X)$, *then the best local approximation of f from* Π^N *on* $\{x_1,\ldots,x_k\}$ *is the Hermite interpolating polynomial of f on* $\{x_1,\ldots,x_n\}$.

Proof Let $H \in \Pi^N$ be the Hermite interpolating polynomial and let $\{g_E\}$ be a net of best approximations of f from Π^N respect to $\|\cdot\|_{\varphi,E}$. From Lemma 4.7,

$$\left\|g_E - H\right\|_{\phi,E} = O\left(\max\{\mathcal{A}_k(i_k)\}\right).$$

Using Theorem 3.1 and the equivalence of the norms in Π^N, we get

$$\left\|g_E - H\right\|_{\infty} \leq \frac{K}{\min\limits_{1\leq k\leq n}\left\{\mu(E_k)^{i_k-1}\phi^{-1}\left(\frac{\mu(E_k)}{\mu(E)}\right)\right\}}\left\|g_E - H\right\|_{\phi,E.}$$

So, the definition of balanced n-tuple implies $g_E \to H$, as $\mu(E) \to 0$.

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Authors' contributions

The three authors participated in the preparation of all work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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