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Some convergence properties for weighted sums of pairwise NQD sequences

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Abstract

Some properties for pairwise NQD sequences are discussed. Some strong convergence results for weighted sums of pairwise NQD sequences are obtained, which generalize the corresponding ones of Wang *et al.* (Bull. Korean Math. Soc. 48(5):923-938, 2011) for negatively orthant dependent sequences.

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1 Introduction

Definition 1.1 Two random variables X and Y are said to be negatively quadrant dependent (NQD) if for any $x, y \in R$,

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y). \tag{1.1}$$

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be pairwise NQD if X_i and X_j are NQD for all $i, j \in \mathbf{N}$ and $i \ne j$.

The concept of pairwise NQD was introduced by Lehmann [1] and many applications have been found. See, for example, Matula [2], Wang *et al.* [3], Wu [4], Li and Wang [5], Gan [6], Huang *et al.* [7], Chen [8], and so forth. Obviously, the sequence of pairwise NQD random variables is a family of very wide scope, which contains a pairwise independent random variable sequence. Many known types of negative dependence such as negative upper (lower) orthant dependence and negative association (see Joag-Dev and Proschan [9]) have been developed on the basis of this notion. Among them the negatively associated class is the most important and special case of a pairwise NQD sequence. So, it is very significant to study probabilistic properties of this wider pairwise NQD class which contains negatively orthant dependent (NOD) random variables as special cases. The main purpose of this paper is to study strong convergence results for weighted sums of pairwise NQD random variables, which generalize the previous known results for negatively associated random variables and negatively orthant dependent random variables, such as those of Wang *et al.* [10, 11].

Throughout the paper, denote $X^+ \doteq \max(0, X)$, $X^- \doteq \max(0, -X)$. C denotes a positive constant which may be different in various places. Let $a_n \ll b_n$ denote that there exists a

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constant $C > 0$ such that $a_n \leq Cb_n$ for sufficiently large n. The main results of this paper depend on the following lemmas:

Lemma 1.1 (Lehmann [1]) *Let X and Y be NQD, then*
(i) $EXY \leq EXEY$;
(ii) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$, *for any* $x, y \in R$;
(iii) *If f and g are both nondecreasing* (*or nonincreasing*) *functions, then $f(X)$ and $g(Y)$ are NQD.*

Lemma 1.2 (Matula [2]) *Let $\{A_n, n \geq 1\}$ be a sequence of events defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.*
(i) *If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n, i.o.) = 0$;*
(ii) *If $P(A_k A_m) \leq P(A_k)P(A_m)$, $k \neq m$ and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n, i.o.) = 1$.*

Lemma 1.3 (Wu [4]) *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables. If $\sum_{n=1}^{\infty} \log^2 n \operatorname{Var}(X_n) < \infty$, then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.*

2 Properties for pairwise NQD random variables

In this section, we will provide some properties for pairwise NQD random variables.

Property 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables, for any $\varepsilon > 0$, $X_n \to 0$ a.s. $\Leftrightarrow \sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$.*

Proof '$\Leftarrow$' If $\sum\limits_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$, then $X_n \to 0$ a.s. follows immediately from the Borel-Cantelli lemma.

'$\Rightarrow$' Let $X_n \to 0$ a.s., we can see that $X_n^+ \to 0$ a.s., $X_n^- \to 0$ a.s. Denote

$$E_n(1) = \left(X_n^+ > \varepsilon/2\right), \qquad E_n(2) = \left(X_n^- > \varepsilon/2\right),$$

it follows that $P(E_n(j), \text{i.o.}) = 0$, $j = 1, 2$. By Lemma 1.1(iii), we can see that $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ are both pairwise NQD. By Lemma 1.1(ii) and Lemma 1.2(ii), we have $\sum_{n=1}^{\infty} P(E_n(j)) < \infty$, $j = 1, 2$. Therefore, $\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) \leq \sum_{n=1}^{\infty} P(X_n^+ > \varepsilon/2) + \sum_{n=1}^{\infty} P(X_n^- > \varepsilon/2) < \infty$. □

Property 2.2 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise NQD random variables and $\{x_n, n \geq 1\}$ be a sequence of positive numbers. Denote $E_i = (X_i > x_i)$, $F_i = (X_i < -x_i)$ and $G_i = (|X_i| > x_i)$ for $i = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} P(G_n) = \infty$ implies $P(G_n, i.o.) = 1$.*

Proof It is easily seen that $G_i = E_i + F_i$, $E_i \cap F_i = \emptyset$ and $P(G_i) = P(E_i) + P(F_i)$ for each $i \geq 1$. Thus, $\sum_{n=1}^{\infty} P(G_n) = \infty$ implies that $\sum_{n=1}^{\infty} P(E_n) = \infty$, or $\sum_{n=1}^{\infty} P(F_n) = \infty$. By Lemma 1.2(ii), we have $P(E_n, \text{i.o.}) = 1$ or $P(F_n, \text{i.o.}) = 1$, which implies that $P(G_n, \text{i.o.}) = 1$. □

Property 2.3 *Under the conditions of Property* 2.2, *$\sum_{n=1}^{\infty} P(G_n) = \infty \Leftrightarrow P(G_n, i.o.) = 1$.*

3 Strong convergence properties for weighted sums of pairwise NQD random variables

In this section, we will provide some sufficient conditions to prove the strong convergence for weighted sums of pairwise NQD random variables.

Theorem 3.1 *Let $1 < \alpha < 2$ and $\{X_n, n \geq 1\}$ be a sequence of mean zero pairwise NQD random variables with identical distribution*

$$P(|X_1| > x) = \begin{cases} L(x)x^{-\alpha}, & x \geq 1, \\ 1, & x < 1, \end{cases} \tag{3.1}$$

where $L(x)$ is a slowly varying function at infinity and $L(cx)/L(x) \to 1$ as $x \to \infty$ for all $c > 0$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive constants satisfying $0 < b_n \uparrow \infty$. Denote $c_1 = b_1/a_1$ and $c_n = b_n/(a_n \log n)$ for $n \geq 2$. Assume that

$$\sum_{n=1}^{\infty} P(|X_n| > c_n) < \infty, \tag{3.2}$$

then

$$\frac{1}{b_n}\sum_{k=1}^{n} a_k X_k \to 0 \quad a.s. \tag{3.3}$$

Proof By the Borel-Cantelli lemma and Kronecker's lemma, it is easily seen that (3.2) implies that

$$\sum_{k=1}^{n} a_k X_k I(|X_k| > c_k) = o(b_n) \quad \text{a.s.} \tag{3.4}$$

Denote

$$Y_k = -c_k I(X_k < -c_k) + X_k I(|X_k| \leq c_k) + c_k I(X_k > c_k), \quad k \geq 1,$$

then $\{Y_k, k \geq 1\}$ is still pairwise NQD from Lemma 1.1(iii). Since

$$\begin{aligned} \sum_{k=1}^{n} a_k X_k &= \sum_{k=1}^{n} a_k (Y_k - EY_k) + \sum_{k=1}^{n} a_k EY_k + \sum_{k=1}^{n} a_k c_k \big(I(X_k < -c_k) - I(X_k > c_k)\big) \\ &\quad + \sum_{k=1}^{n} a_k X_k I(|X_k| > c_k), \end{aligned} \tag{3.5}$$

in order to show $\frac{1}{b_n}\sum_{k=1}^{n} a_k X_k \to 0$ a.s., we only need to show that the first three terms above are $o(b_n)$ or $o(b_n)$ a.s.

By C_r inequality, Theorem 1b in Feller [12, p.281] and (3.2), we can get

$$\begin{aligned} &\sum_{k=1}^{\infty} \log^2 k \operatorname{Var}\left(\frac{a_k Y_k}{b_k}\right) \\ &\quad \leq C \sum_{k=1}^{\infty} c_k^{-2} E\big[c_k^2 I(|X_k| > c_k) + X_k^2 I(|X_k| \leq c_k)\big] \\ &\quad \leq O(1) + C \sum_{k=1}^{\infty} c_k^{-2} \int_0^{c_k} tP(|X_k| > t)\, dt \end{aligned}$$

$$
\begin{aligned}
&\le O(1) + C\sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha} \\
&\le O(1) + C\sum_{k=1}^{\infty} P\big(|X_k| > c_k\big) \\
&< \infty.
\end{aligned}
$$

By Lemma 1.3 and Kronecker's lemma, we have $\sum_{k=1}^{n} a_k(Y_k - EY_k) = o(b_n)$ a.s.

By (3.2) again,

$$
\begin{aligned}
\sum_{k=1}^{\infty} E\left|\frac{a_k(\log k)c_k(I(X_k < -c_k) - I(X_k > c_k))}{b_k}\right| &\le \sum_{k=1}^{\infty} E\big(I(X_k < -c_k) + I(X_k > c_k)\big) \\
&= \sum_{k=1}^{\infty} P\big(|X_k| > c_k\big) < \infty,
\end{aligned}
$$

which implies that

$$
\sum_{k=1}^{\infty} \frac{a_k c_k(I(X_k < -c_k) - I(X_k > c_k))}{b_k} \text{ converges} \quad \text{a.s.}
$$

By Kronecker's lemma, it follows that

$$
\sum_{k=1}^{n} a_k c_k\big(I(X_k < -c_k) - I(X_k > c_k)\big) = o(b_n) \quad \text{a.s.}
$$

By Theorem 1a in Feller [12, p.281] and (3.2) again, we have

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left|\frac{a_k(\log k)EY_k}{b_k}\right| &\le \sum_{k=1}^{\infty} c_k^{-1}\big[c_k P\big(|X_k| > c_k\big) + E|X_k| I\big(|X_k| > c_k\big)\big] \\
&= 2\sum_{k=1}^{\infty} P\big(|X_k| > c_k\big) + \sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} P\big(|X_k| > t\big)\,dt \\
&\le O(1) + C\sum_{k=1}^{\infty} c_k^{-1}\int_{c_k}^{\infty} L(t)t^{-\alpha}\,dt \\
&\le O(1) + C\sum_{k=1}^{\infty} L(c_k)c_k^{-\alpha} \\
&\le O(1) + C\sum_{k=1}^{\infty} P\big(|X_k| > c_k\big) \\
&< \infty,
\end{aligned}
$$

which implies that

$$
\sum_{k=1}^{\infty} \frac{a_k EY_k}{b_k} \text{ converges.}
$$

By Kronecker's lemma, it follows that

$$\sum_{k=1}^{n} a_k EY_k = o(b_n).$$

Hence, the desired result (3.3) follows from the statements above immediately. □

Theorem 3.2 *Let* $1 \le r < 2$, $\{X_n, n \ge 1\}$ *be a sequence of pairwise NQD random variables with identical distribution and* $\{a_n, n \ge 1\}$ *be sequences of positive numbers with* $A_n \doteq \sum_{j=1}^{n} a_j \uparrow \infty$. *Denote* $c_1 = 1$ *and* $c_n = A_n/(a_n \log n)$ *for* $n \ge 2$. *Assume that*

$$EX_1 = 0, \qquad E|X_1|^r < \infty, \tag{3.6}$$

$$N(n) \doteq \operatorname{Card}\{i : c_i \le n\} \ll n^r, \quad n \ge 1, \tag{3.7}$$

then

$$A_n^{-1} \sum_{i=1}^{n} a_i X_i \to 0 \quad a.s., n \to \infty. \tag{3.8}$$

Proof Let $N(0) = 0$ and

$$Y_n = -c_n I(X_n < -c_n) + X_n I\big(|X_n| \le c_n\big) + c_n I(X_n > c_n), \quad n \ge 1.$$

By (3.7), it is easily seen that $c_n \to \infty$ as $n \to \infty$ (otherwise, there exist infinite subscripts i and some n_0 such that $c_i \le n_0$, hence, $N(n_0) = \infty$, which is contrary to $N(n_0) \ll n_0^r$ from (3.7)). It follows by (3.6) and (3.7) that

$$\begin{aligned}
\sum_{i=1}^{\infty} P(X_i \ne Y_i) &= \sum_{i=1}^{\infty} P\big(|X_i| > c_i\big) = \sum_{j=1}^{\infty} \sum_{c_i \le j < c_i + 1} P\big(|X_i| > c_i\big) \\
&\le \sum_{j=1}^{\infty} \sum_{j-1 < c_i \le j} P\big(|X_1| > j-1\big) \\
&= \sum_{j=1}^{\infty} \big(N(j) - N(j-1)\big) P\big(|X_1| > j-1\big) \\
&= \sum_{j=1}^{\infty} \big(N(j) - N(j-1)\big) \sum_{l=j}^{\infty} P\big(l-1 < |X_1| \le l\big) \\
&= \sum_{l=1}^{\infty} \sum_{j=1}^{l} \big(N(j) - N(j-1)\big) P\big(l-1 < |X_1| \le l\big) \\
&\ll \sum_{l=1}^{\infty} l^r P\big(l-1 < |X_1| \le l\big) \\
&\ll E|X_1|^r < \infty.
\end{aligned}$$

By the inequality above and Borel-Cantelli lemma, we can see that $P(X_i \neq Y_i, \text{i.o.}) = 0$. Therefore, in order to prove (3.8), we only need to prove

$$A_n^{-1}\sum_{i=1}^{n} a_i Y_i \to 0 \quad \text{a.s.}, n \to \infty. \tag{3.9}$$

By (3.6) and (3.7) again,

$$\begin{aligned}
\sum_{i=1}^{\infty} \log^2 i \operatorname{Var}\left(\frac{a_i Y_i}{A_i}\right) &\le \sum_{i=1}^{\infty} c_i^{-2} EY_i^2 \\
&\le C\sum_{i=1}^{\infty} P\big(|X_i| > c_i\big) + C\sum_{i=1}^{\infty} c_i^{-2} EX_1^2 I\big(|X_1| \le c_i\big) \\
&\le C + C\sum_{j=1}^{\infty}\sum_{j-1<c_i\le j} c_i^{-2} EX_1^2 I\big(|X_1| \le c_i\big) \\
&\le C + C\sum_{j=1}^{\infty}\sum_{j-1<c_i\le j} c_i^{-2} EX_1^2 I\big(|X_1| \le j\big) \\
&\ll C + \sum_{j=2}^{\infty}\big(N(j) - N(j-1)\big)(j-1)^{-2}\sum_{k=1}^{j} EX_1^2 I\big(k-1 < |X_1| \le k\big) \\
&\ll C + \sum_{k=2}^{\infty}\sum_{j=k}^{\infty}\big(N(j) - N(j-1)\big)(j-1)^{-2} EX_1^2 I\big(k-1 < |X_1| \le k\big) \\
&\ll C + \sum_{k=2}^{\infty}\sum_{j=k}^{\infty} N(j)\big((j-1)^{-2} - j^{-2}\big) EX_1^2 I\big(k-1 < |X_1| \le k\big) \\
&\ll C + \sum_{k=2}^{\infty}\sum_{j=k}^{\infty} j^{r-3} EX_1^2 I\big(k-1 < |X_1| \le k\big) \\
&\ll C + \sum_{k=2}^{\infty} k^{r-2} E|X_1|^r k^{2-r} I\big(k-1 < |X_1| \le k\big) \\
&= C + \sum_{k=2}^{\infty} E|X_1|^r I\big(k-1 < |X_1| \le k\big) \\
&\ll C + E|X_1|^r < \infty.
\end{aligned}$$

Therefore, by the inequality above, Lemma 1.3 and Kronecker's lemma, we have

$$A_n^{-1}\sum_{i=1}^{n} a_i (Y_i - EY_i) \to 0, \quad \text{a.s.} \tag{3.10}$$

In order to prove (3.9), it suffices to prove that

$$A_n^{-1}\sum_{i=1}^{n} a_i EY_i \to 0, \quad n \to \infty. \tag{3.11}$$

It is easily seen that $E|X_1|^r < \infty$ for $1 \le r < 2$ implies that $E|X_1| < \infty$, thus

$$\lim_{i\to\infty} c_i P(|X_i| > c_i) = 0. \tag{3.12}$$

By the Lebesgue dominated convergence theorem and $EX_1 = 0$, we have

$$EX_i I(|X_i| \le c_i) = EX_1 I(|X_1| \le c_i) \to EX_1 = 0, \quad i \to \infty.$$

Therefore,

$$|EY_i| \le c_i P(|X_i| > c_i) + \left|EX_i I(|X_i| \le c_i)\right| \to 0, \quad \text{as } i \to \infty, \tag{3.13}$$

which implies (3.11) by Toeplitz's lemma. The proof is completed. □

Remark 3.1 In Theorem 3.2, the condition $A_n = \sum_{j=1}^n a_j \uparrow \infty$ can be relaxed to $0 < A_n \uparrow \infty$ when $1 < r < 2$. It suffices to prove (3.11). In fact, it follows by (3.6) and (3.7) that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left|\frac{a_k(\log k)EY_k}{A_k}\right| \\
&\quad \le \sum_{k=1}^{\infty} c_k^{-1}\left[c_k P(|X_k| > c_k) + E|X_k|I(|X_k| > c_k)\right] \\
&\quad = \sum_{k=1}^{\infty} P(|X_k| > c_k) + \sum_{k=1}^{\infty} c_k^{-1} E|X_k|I(|X_k| > c_k) \\
&\quad \le C + \sum_{k=1}^{\infty} \sum_{c_k \le j < c_k+1} c_k^{-1} E|X_1|I(|X_1| > c_k) \\
&\quad \le C + \sum_{j=1}^{\infty} \sum_{j-1 < c_k \le j} c_k^{-1} E|X_1|I(|X_1| > j-1) \\
&\quad \le C + C\sum_{j=2}^{\infty} \big(N(j) - N(j-1)\big)(j-1)^{-1} \sum_{k=j-1}^{\infty} E|X_1|I(k < |X_1| \le k+1) \\
&\quad \le C + C\sum_{k=1}^{\infty} \sum_{j=2}^{k+1} N(j)\big((j-1)^{-1} - j^{-1}\big) E|X_1|I(k < |X_1| \le k+1) \\
&\quad \le C + C\sum_{k=1}^{\infty} \sum_{j=2}^{k+1} j^{r-2} E|X_1|I(k < |X_1| \le k+1) \\
&\quad \le C + C\sum_{k=1}^{\infty} k^{r-1} E|X_1|I(k < |X_1| \le k+1) \\
&\quad \le C + C\sum_{k=1}^{\infty} E|X_1|^r I(k < |X_1| \le k+1) \\
&\quad \le C + CE|X_1|^r < \infty.
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

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