On generalized absolute Cesàro summability factors

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Abstract

In this paper, a known theorem dealing with $|C, \alpha, \gamma; \delta|_k$ summability factors has been generalized for $|C, \alpha, \beta, \gamma; \delta|_k$ summability factors. Some results have also been obtained.

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1 Introduction

A sequence (b_n) of positive numbers is said to be quasi-monotone if $n\Delta b_n \geq -\rho b_n$ for some $\rho > 0$ and is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha, \beta}$ and $t_n^{\alpha, \beta}$ the nth Cesàro means of order (α, β), with $\alpha + \beta > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [2]),

$$u_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v, \tag{1}$$

$$t_n^{\alpha, \beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{2}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad \alpha + \beta > -1, \qquad A_0^{\alpha+\beta} = 1 \quad \text{and} \quad A_{-n}^{\alpha+\beta} = 0 \quad \text{for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $|C, \alpha, \beta|_k$, $k \geq 1$ and $\alpha + \beta > -1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^{\alpha, \beta} - u_{n-1}^{\alpha, \beta}|^k < \infty. \tag{4}$$

Since $t_n^{\alpha, \beta} = n(u_n^{\alpha, \beta} - u_{n-1}^{\alpha, \beta})$ (see [3]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha, \beta}|^k < \infty. \tag{5}$$

The series $\sum a_n$ is said to be summable $|C, \alpha, \beta, \gamma; \delta|_k$, $k \geq 1$, $\alpha + \beta > -1$, $\delta \geq 0$ and γ is a real number, if (see [4])

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k + k - 1)} \left| u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \right|^k = \sum_{n=1}^{\infty} n^{\gamma(\delta k + k - 1) - k} \left| t_n^{\alpha,\beta} \right|^k < \infty. \tag{6}$$

If we take $\beta = 0$, then $|C, \alpha, \beta, \gamma; \delta|_k$ summability reduces to $|C, \alpha, \gamma; \delta|_k$ summability (see [5]).

2 Known result

In [6], we have proved the following theorem dealing with $|C, \alpha, \gamma; \delta|_k$ summability factors of infinite series.

Theorem A *Let $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, and γ be a real number such that $-\gamma(\delta k + k - 1) + (\alpha + 1)k > 1$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $|\Delta\lambda_n| \leq |B_n|$, $\lambda_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} n\delta_n \log n < \infty$ and $\sum_{n=1}^{\infty} nB_n \log n$ is convergent. If the sequence (w_n^α) defined by (see [7])*

$$w_n^\alpha = \left| t_n^\alpha \right|, \quad \alpha = 1, \tag{7}$$

$$w_n^\alpha = \max_{1 \leq v \leq n} \left| t_v^\alpha \right|, \quad 0 < \alpha < 1, \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k + k - 1) - k} \left(w_n^\alpha \right)^k = O(\log m) \quad as\ m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ is summable $|C, \alpha, \gamma; \delta|_k$.

3 The main result

The aim of this paper is to generalize Theorem A for $|C, \alpha, \beta, \gamma; \delta|_k$ summability. We shall prove the following theorem.

Theorem *Let $k \geq 1$, $0 \leq \delta < \alpha \leq 1$, and γ be a real number such that $(\alpha + \beta + 1 - \gamma(\delta + 1))k > 1$, and let there be sequences (B_n) and (λ_n) such that the conditions of Theorem A are satisfied. If the sequence $(w_n^{\alpha,\beta})$ defined by*

$$w_n^{\alpha,\beta} = \left| t_n^{\alpha,\beta} \right|, \quad \alpha = 1, \beta > -1, \tag{10}$$

$$w_n^{\alpha,\beta} = \max_{1 \leq v \leq n} \left| t_v^{\alpha,\beta} \right|, \quad 0 < \alpha < 1, \beta > -1, \tag{11}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\gamma(\delta k + k - 1) - k} \left(w_n^{\alpha,\beta} \right)^k = O(\log m) \quad as\ m \to \infty, \tag{12}$$

then the series $\sum a_n\lambda_n$ is summable $|C, \alpha, \beta, \gamma; \delta|_k$. It should be noted that if we take $\beta = 0$, then we get Theorem A.

We need the following lemmas for the proof of our theorem.

Lemma 1 ([8]) *Under the conditions on* (B_n), *as taken in the statement of the theorem, we have the following*:

$$nB_n \log n = O(1), \tag{13}$$

$$\sum_{n=1}^{\infty} n \log n |\Delta B_n| < \infty. \tag{14}$$

Lemma 2 ([9]) *If* $0 < \alpha \leq 1$, $\beta > -1$, *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \right|. \tag{15}$$

4 Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(na_n \lambda_n)$. Then by (2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Firstly applying Abel's transformation and then using Lemma 2, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\left| T_n^{\alpha,\beta} \right| \leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} |\Delta \lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \right|$$

$$\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} w_v^{\alpha,\beta} |\Delta \lambda_v| + |\lambda_n| w_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad \text{say}$$

since

$$\left| T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \right|^k \leq 2^k \left(\left| T_{n,1}^{\alpha,\beta} \right|^k + \left| T_{n,2}^{\alpha,\beta} \right|^k \right). \tag{16}$$

In order to complete the proof of the theorem, by (6), it is sufficient to show that for $r = 1, 2$,

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k + k - 1) - k} \left| T_{n,r}^{\alpha,\beta} \right|^k < \infty.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\sum_{n=2}^{m+1} n^{\gamma(\delta k + k - 1) - k} \left| T_{n,1}^{\alpha,\beta} \right|^k$$

$$\leq \sum_{n=2}^{m+1} n^{\gamma(\delta k + k - 1) - k} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} w_v^{\alpha,\beta} \Delta \lambda_v \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} |\Delta \lambda_v| \left(w_v^{\alpha,\beta}\right)^k \right\} \left\{ \sum_{v=1}^{n-1} |\Delta \lambda_v| \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} |B_v| \left(w_v^{\alpha,\beta}\right)^k \right\} \left\{ \sum_{v=1}^{n-1} |B_v| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} |B_v| \left(w_v^{\alpha,\beta}\right)^k \sum_{n=v+1}^{m+1} \frac{1}{n^{(\alpha+\beta+1-\gamma(\delta+1))k}}$$

$$= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} |B_v| \left(w_v^{\alpha,\beta}\right)^k \int_{v}^{\infty} \frac{dx}{x^{(\alpha+\beta+1-\gamma(\delta+1))k}}$$

$$= O(1) \sum_{v=1}^{m} |B_v| v^{\gamma(\delta k+k-1)-k+1} \left(w_v^{\alpha,\beta}\right)^k$$

$$= O(1) \sum_{v=1}^{m} v |B_v| v^{\gamma(\delta k+k-1)-k} \left(w_v^{\alpha,\beta}\right)^k$$

$$= O(1) \sum_{v=1}^{m-1} \left| \Delta \left(v |B_v| \right) \right| \sum_{p=1}^{v} p^{\gamma(\delta k+k-1)-k} \left(w_p^{\alpha,\beta}\right)^k + O(1) m |B_m| \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)-k} \left(w_v^{\alpha,\beta}\right)^k$$

$$= O(1) \sum_{v=1}^{m-1} \left| \Delta \left(v |B_v| \right) \right| \log v + O(1) m |B_m| \log m$$

$$= O(1) \sum_{v=1}^{m-1} v |\Delta B_v| \log v + O(1) \sum_{v=1}^{m-1} |B_{v+1}| \log v + O(1) m |B_m| \log m$$

$$= O(1) \quad \text{as } m \to \infty,$$

in view of the hypotheses of the theorem and Lemma 1. Similarly, we have that

$$\sum_{n=2}^{m+1} n^{\gamma(\delta k+k-1)-k} \left| T_{n,2}^{\alpha,\beta} \right|^k = O(1) \sum_{n=1}^{m} |\lambda_n| n^{\gamma(\delta k+k-1)-k} \left(w_n^{\alpha,\beta}\right)^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| \sum_{v=1}^{n} v^{\gamma(\delta k+k-1)-k} \left(w_v^{\alpha,\beta}\right)^k$$

$$+ O(1) |\lambda_m| \sum_{v=1}^{m} v^{\gamma(\delta k+k-1)-k} \left(w_v^{\alpha,\beta}\right)^k$$

$$= O(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| \log n + O(1) |\lambda_m| \log m$$

$$= O(1) \sum_{n=1}^{m-1} |B_n| \log n + O(1) |\lambda_m| \log m$$

$$= O(1) \quad \text{as } m \to \infty,$$

by virtue of the hypotheses of the theorem and Lemma 1. Therefore, by (6), we get that for $r = 1, 2$,

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k+k-1)-k} \left| T_{n,r}^{\alpha,\beta} \right|^k < \infty.$$

This completes the proof of the theorem. If we take $\delta = 0$ and $\gamma = 1$, then we get a result for $|C, \alpha, \beta|_k$ summability factors. Also, if we take $\beta = 0$, $\delta = 0$, and $\alpha = 1$, then we get a result for $|C, 1|_k$ summability.

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